Nonlinear integral inequalities of the Volterra type

Ryszard Szwarc

Department of Mathematics
University of Wisconsin-Madison
Madison, WI 53706, USA

and

Institute of Mathematics
Wrocław University
pl.Grunwaldzki 2/4
50-384 Wrocław, Poland

Introduction. We are studying the integral inequality

$$u(x) \leq \int_0^x a(x-y)\, \psi(u(y))\, dy,$$

where all appearing functions are defined and increasing on the right half-axis and take the value zero at zero. We are interested in determining when the inequality admits solutions $u(x)$ nonvanishing in a neighbourhood of zero. It is well-known that if $\psi(x)$ is the identity function then no such solution exists. This due to the fact that the operator defined by the integral on the right hand side of the equation is linear and compact. So if we are interested in nontrivial solutions it is natural to admit that $\psi(x) \geq x$ at least in a neighbourhood of zero. One of the typical examples is the power function $\psi(x) = x^\alpha$, where $\alpha < 1$. This situation was explored in [2]. The functions $a(x)$, that admit nonzero solutions where characterized by Bushell in [1]. For general approach to the problem we refer to [2], [3] and [4].

In the present paper we give sufficient conditions for nonexistence of nontrivial solutions. We prove that no matter how large $\psi(x)$ is we can always find an appropriate $a(x)$ such, that nontrivial solutions are nonexistent. In particular when $\psi(x) = \sqrt{x}$ and $a(x) = \exp[-\exp(1/x)]$, then we have no such solutions.

We give also sufficient conditions for existence of nontrivial solutions. In particular we show that such solutions exist for

$$\psi(x) = \sqrt{x} \quad \text{and} \quad a(x) = \exp[-\exp(1/x^{\alpha})], \quad \text{where} \quad 0 < \alpha < 1.$$

The results.

We will deal with the integral inequality

$$u(x) \leq \int_0^x a(x-y)\psi\left(u(y)\right) dy. \tag{1}$$

The functions $a(x)$ and $\psi(x)$ are defined for $x \geq 0$, and are assumed to be smooth, strictly increasing and satisfying $\psi(0) = a(0) = 0$. We are looking for solutions $u(x)$, which are nonnegative and do not vanish identically in a neighborhood of 0. We can assume that $u(x)$ is strictly increasing without loss of generality. Indeed, if $u(x)$ is a solution of (1), $u(x) \geq 0$, and $u(x)$ does not vanish identically in a neighborhood of 0, then the function represented by the right hand side of (1) satisfies (1) and is strictly increasing. This is due to the fact that the operator T defined as

$$T\,u(x) = \int_0^x a(x-y)\psi\left(u(y)\right) dy. \tag{2}$$

is monotonic.

Moreover the following simple comparison result holds.

Proposition 1 *Let $\psi_1(x)$ and $\psi_2(x)$ be nonnegative functions defined on $[0, +\infty)$ and satisfying $\psi_1(x) \leq \psi_2(x)$. Then if the inequality (1) is unsolvable with $\psi(x) = \psi_2(x)$ it is such with $\psi(x) = \psi_1(x)$.*

Now we give a condition under which the inequality (1) has no solutions.

Proposition 2 *Let $\phi(x)$ be the inverse function for $\psi(x)$. Assume that there is a positive constant C such that*

$$\frac{x - a^{-1} \circ \phi \circ a\,(x)}{x^2} \leq C. \tag{3}$$

Then the inequality (1) does not admit any solution $u(x)$, which is nonnegative and does not vanish identically in a neighborhood of 0.

Proof. Assume that $u(x)$ satisfies (1). We then have the following elementary estimate.

$$u(x) \leq x\, a(x)\, \psi(u(x)).$$

Hence $u(x) = o(a(x))$, when $x \to 0^+$. Consequently there is a function $f(x)$, such that

$$u(x) = a(f(x)), \;\text{ and }\; 0 \leq f(x) \leq x. \tag{4}$$

We can assume that $\phi(x) \leq x$, replacing $\phi(x)$ with $\min\{\phi(x), x\}$ if necessary. Observe that the modified function satisfies the assumption (3) and by Proposition 1 it suffices to show the conclusion of Proposition 2 for this new function. So if $\phi(x) \leq x$, then $\psi(x) \geq x$, This implies that there is a function $g(x)$ such that

$$\psi\left(u(x - g(x))\right) = u(x). \tag{5}$$

The equations (4) and (5) imply that

$$a^{-1} \circ \phi \circ a(f(x)) = f(x - g(x)). \tag{6}$$

The integral in (1) can be then split as follows

$$\int_0^x a(x-y)\psi\left(u(y)\right) dy = \int_0^x a(y)\psi\left(u(x-y)\right) dy =$$

$$\int_0^{f(x)} a(y)\psi\left(u(x-y)\right) dy + \int_{f(x)}^{g(x)} a(y)\psi\left(u(x-y)\right) dy +$$

$$\int_{g(x)}^x a(y)\psi\left(u(x-y)\right) dy.$$

The integral over $[0, f(x)]$ can be majorized by $a(f(x))f(x)\psi(u(x))$ which by virtue of (4) is $o(u(x))$, when x tends to 0. The integral over $[g(x), x]$ is also $o(u(x))$, because it can be majorized by $\psi\left(u(x - g(x))\right)(x - g(x))a(x)$, which in turn can be majorized by $xu(x)a(x)$. We are going to prove that $f(x_n) \geq g(x_n)$ for a sequence x_n tending to 0. It will imply that

$$Tu\,(x_n) = o(u(x_n)),$$

when n tends to infinity.

First of all we can assume that

$$f(x) \leq x^2.$$

This can be achieved as follows. Let the function $u(x)$ satisfy the inequality (1), and $u(x) = a(f(x))$. Then by [4], Proposition 1, the function $u(x^2/(x^2+1))$ also satisfies (1). Moreover

$$u(x^2/(x^2+1)) = a(\tilde{f}(x)),$$

where $\tilde{f}(x) = f(x^2/(x^2+1)) \leq x^2$.

The function $f(x)/x$ cannot be decreasing as it approaches 0 when x tends to 0. Thus there is a sequence x_n such that $\lim_n x_n = 0$ and

$$\frac{f(x_n)}{x_n} = \max_{0 \leq x \leq x_n} \frac{f(x)}{x}.$$

We will show that $g(x_n) \leq f(x_n)$. By (3) and (6) we have that

$$\frac{f(x) - f(x - g(x))}{f(x)^2} \leq C,$$

for some constant C. By the choice of the sequence x_n we get

$$\frac{f(x_n) - f(x_n - g(x_n))}{f(x_n)^2} \geq \frac{x_n f(x_n) - (x_n - g(x_n)) f(x_n)}{x_n f(x_n)^2} = \frac{g(x_n)}{x_n f(x_n)}.$$

Thus we arrive at

$$g(x_n) \leq C x_n f(x_n),$$

which shows that $g(x_n) \leq f(x_n)$, for n large enough. As it was shown before this implies $Tu\,(x_n) = o(u(x_n))$, which contradicts (1).

Example 1. Let $\psi(x) = \sqrt{x}$ and

$$a(x) = \exp\left(-\exp(x^{-1})\right).$$

Then $\psi(x)$ and $a(x)$ satisfy the assumptions of Proposition 2. Indeed, since

$$a^{-1} \circ \phi \circ a(x) = \frac{x}{1 + x \log 2},$$

then

$$x - a^{-1} \circ \phi \circ a(x) = \frac{x^2 \log 2}{1 + x \log 2} \leq x^2.$$

Therefore the inequality (1) has no nontrivial solutions $u(x)$.

Proposition 3 *Let $\psi(x)$ be an increasing function on $[0, +\infty)$, such that $\psi(0) = 0$. There exists an increasing positive function $a(x)$ such that the inequality (1) has no solution $u(x)$ positive in a neighborhood of zero.*

Proof. We are going to find a function $\psi(x)$ which satisfies (3). We can assume that $\psi(x) > x$, replacing $\psi(x)$ by $\max\{\psi(x), 2x\}$, if necessary, and observing that by Proposition 1 nonsolvability of (1) with this new function implies that with $\psi(x)$. If so we have $\phi(x) < x$. Let y_n be sequence defined recursively by

$$y_{n+1} = \phi(y_n), \quad y_0 = 1.$$

Then y_n is strictly decreasing and convergent to zero. Let x_n be the sequence defined by

$$x_{n+1} = x_n - x_n^2, \quad x_0 = \frac{1}{2}.$$

Then x_n is also strictly decreasing and convergent to zero. First we define the function $a(x)$ on the terms of the sequence $\{x_n\}_{n=0}^{\infty}$ as

$$a(x_n) = y_n.$$

Then $a(x)$ can be extended (for example linearly) to a continuous nonnegative function strictly increasing on the half-axis $[0, \infty)$. We claim that $a(x)$ satisfies (3). Let $0 < x < \frac{1}{2}$. Then there is n such, that

$$x_{n+1} \leq x < x_n.$$

Thus, observing that $a^{-1} \circ \phi \circ a(x_n) = x_{n+1}$ we get

$$\begin{aligned} \frac{x - a^{-1} \circ \phi \circ a(x)}{x^2} &\leq \frac{x_n - a^{-1} \circ \phi \circ a(x_{n+1})}{x_{n+1}^2} \\ &= \frac{x_n - x_{n+2}}{x_{n+1}^2} = \frac{x_n - x_{n+1} + x_{n+1} - x_{n+2}}{x_{n+1}^2} \\ &= 1 + \frac{x_n^2}{x_{n+1}^2} = 1 + \frac{1}{(1 - x_n)^2} \leq 1 + \frac{1}{(1 - x_0)^2} = 5. \end{aligned}$$

Thus we can apply Proposition 2. Observe that (1) has no nonvanishing solution for every $\tilde{a}(x) \leq a(x)$. For example we can take

$$\tilde{a}(x) = \int_0^x \frac{a(y)}{1 + y} \, dy.$$

Therefore $a(x)$ can be taken to be smooth.

Proposition 2 can be strenghtened using Corollary 2 from [4]. Namely the following holds.

Theorem 1 *Let* $\phi(x)$ *denote the inverse function for* $\psi(x)$*. Assume that* $\lim_{t\to\infty}(\psi(t)/t) = +\infty$ *and*

$$\frac{x}{x - a^{-1}\circ\phi\circ a(x)} \geq h(x), \tag{7}$$

where $h(x)$ *is a positive function satisfying:*

(i) $h(x)$ *is nonintegrable about* 0*, and* $h(x) \leq x^{-1}$,

(ii) Let $H(x)$ *be the antiderivative of* $h(x)$*. The function* $H^{-1}(\log x)$ *is convex in a neighborhood of zero and has second derivative bounded.*

Then inequality (1) has no nontrivial solutions.

Proof. We will construct a family of functions $u_\epsilon(x)$ such, that

$$Tu_\epsilon(x) \leq u_\epsilon(x), \quad \lim_{\epsilon\to 0^+} u_\epsilon(x) = 0, \tag{8}$$

for x in a neighborhood of zero. The conclusion will then follow from Corollary 2 , [4] and the following lemma.

Lemma 1 *Let* $\lim_{t\to\infty}(\psi(t)/t) = +\infty$*. If the inequality (1) has an increasing solution not vanishing in a neighborhood of zero then there is such solution of the equation*

$$u(x) = Tu(x) = \int_0^x a(x-y)\psi(u(y))\,dy.$$

Proof. If $u(x)$ is an increasing function satisfying the inequality (1), then

$$u(x) \leq \psi(u(x))A(x), \quad \text{where } A(x) = \int_0^x a(t)\,dt.$$

Thus

$$\frac{u(x)}{\psi(u(x))} \leq A(x).$$

Fix $x_0 > 0$. Since $\lim_{t\to\infty}(\psi(t)/t) = +\infty$, there is a constant c_{x_0} such that $0 \leq u(x) \leq c_{x_0}$ for $0 \leq x \leq x_0$. Observe that if $u(x)$ satisfies (1) then also $T^n u(x)$ do so for every natural n. Hence we have

$$T^n u(x) \leq c_{x_0}.$$

The sequence of functions $T^n u$ is thus increasing and bounded from above on $[0, x_0]$, hence it converges to a limit $\tilde{u}$, on $[0, x_0]$. The function $\tilde{u}$ satisfies the equation for $0 \leq x \leq x_0$ and does not vanish about 0 as $u(x) \leq \tilde{u}(x)$. Since x_0 is arbitrary the function $\tilde{u}x$ can be defined for every $x > 0$.

Let us return to the proof of Theorem 1. For $1 \geq \epsilon > 0$ let $f(x)$ denote the function defined as

$$f(x) = H^{-1}(\log x). \tag{9}$$

Set

$$f_\epsilon(x) = f(\epsilon x).$$

By assumptions $f_\epsilon(x)$ is a convex function with bounded second derivative. We claim that if we set $u_\epsilon(x) = a\left(f_\epsilon(x)\right)$, then (8) holds. First observe that since $h(x) \leq 1/x$, we have $f_\epsilon(x) \leq f(x) \leq x$. for $0 < x \leq 1$. This is true, because

$$\begin{aligned} H(x) &= \int_{\frac{1}{2}}^{x} h(t)\, dt = -\int_{x}^{\frac{1}{2}} h(t)\, dt \\ &\geq -\int_{x}^{\frac{1}{2}} t^{-1} dt = \log x. \end{aligned}$$

Now we are going to estimate the integrals

$$\int_0^x a(x-y)\psi\left(u_\epsilon(y)\right) dy$$

following the method from the proof of Proposition 2.

$$\begin{aligned} \int_0^x a(x-y)\psi\left(u_\epsilon(y)\right) dy &= \int_0^x a(y)\psi(u(x-y))dy \\ &= \int_0^{f_\epsilon(x)} \cdots + \int_{f_\epsilon(x)}^{g_\epsilon(x)} \cdots + \int_{g_\epsilon(x)}^{x} \cdots \quad . \end{aligned}$$

The functions $f_\epsilon(x)$ and $g_\epsilon(x)$ are defined according to (4) and (6). The first and the third integrals can be majorized as follows.

$$\begin{aligned} \int_0^{f_\epsilon(x)} \cdots &\leq a\left(f_\epsilon(x)\right)\, f_\epsilon(x)\, \psi\left(u_\epsilon(x)\right) \leq u_\epsilon(x)\, x\, \psi\left(u_1(x)\right), \\ \int_{g_\epsilon(x)}^{x} \cdots &\leq \psi\left(u_\epsilon(x - g_\epsilon(x))\right)\, (x - g_\epsilon(x)\, a(x) \leq u_\epsilon(x)\, x\, a(x). \end{aligned}$$

Thus either integrals can be majorized by $Cx\,u_\epsilon(x)$ for $0 < x \le 1$, where C is a constant independent of ϵ. The proof will be complete if we show that the second integral is nonpositive for small x. Namely, we will show that $g_\epsilon(x) \le Cx\,f_\epsilon(x)$, for $0 < x \le 1$ and $0 < \epsilon \le 1$, where C is a constant independent of ϵ.

Combining (6) (where $f = f_\epsilon$ and $g = g_\epsilon$) and (7) gives

$$\frac{f_\epsilon(x)}{f_\epsilon(x) - f_\epsilon\left(x - g_\epsilon(x)\right)} \ge h\left(f_\epsilon(x)\right). \tag{10}$$

We then have

$$\begin{aligned} f_\epsilon\left(x - x\,f_\epsilon(x)\right) &= f(\epsilon x - \epsilon\,x f(\epsilon x)) \\ &= f(\epsilon x) - \epsilon x\,f(\epsilon x)\,f'(\epsilon x) + \frac{1}{2}\epsilon^2 x^2 f^2(\epsilon x)\,f''(\xi), \end{aligned}$$

where $0 < \xi < \epsilon x$. Since $f(x)$ is a convex function

$$f(\epsilon x) \le \epsilon x\,f'(\epsilon x).$$

Therefore by (6) and by definition of $f(x)$ we have

$$\begin{aligned} \frac{f_\epsilon(x) f_\epsilon'(x)}{f_\epsilon(x) - f_\epsilon\left(x - 2x f_\epsilon(x)\right)} &= \frac{\epsilon f(\epsilon x) f'(\epsilon x)}{2\epsilon x\,f(\epsilon x)\,f'(\epsilon x) - 2\epsilon^2 x^2 f^2(\epsilon x)\,f''(\xi)} \\ &= \frac{1}{2x\left(1 - \epsilon x \frac{f(\epsilon x)}{f'(\epsilon x)} f''(\xi)\right)} \\ &\le \frac{1}{2x(1 - \epsilon^2 x^2 f''(\xi)} \le \frac{1}{x}, \end{aligned}$$

for $0 \le x \le x_0$, where x_0 does not depend on ϵ.

On the other hand by (10) we have

$$\begin{aligned} \frac{f_\epsilon(x) f_\epsilon'(x)}{f_\epsilon(x) - f_\epsilon\left(x - g_\epsilon(x)\right)} &\ge f_\epsilon'(x)\,h\left(f_\epsilon(x)\right) \\ &= \left(H \circ f_\epsilon\right)'(x) = \frac{1}{x}. \end{aligned}$$

The above calculations show that

$$\frac{f_\epsilon(x) f_\epsilon'(x)}{f_\epsilon(x) - f_\epsilon\left(x - g_\epsilon(x)\right)} \ge \frac{f_\epsilon(x) f_\epsilon'(x)}{f_\epsilon(x) - f_\epsilon\left(x - 2x f_\epsilon(x)\right)}.$$

This implies

$$g_\epsilon(x) \le 2x\, f_\epsilon(x),$$

for $0 \le x \le x_0$, and for every $0 < \epsilon \le 1$. Therefore

$$g_\epsilon(x) \le f_\epsilon(x),$$

for $0 < x \le \min\{\frac{1}{2}, x_0\}$. This completes the proof of the theorem.

Example 2.
(a) Let $h(x) = x^{-1}$. Then $H(x) = \log x$ and according to (9) we have $f(x) = x$. Thus $h(x)$ satisfies the assumptions (i) and (ii) of Theorem 1.

(b) Let $h(x) = -(x \log x)^{-1}$. Then $H(x) = -\log(-\log x)$ and from (9) we get $f(x) = e^{-1/x}$. Again the assumptions (i) and (ii) are fulfilled. In particular $f(x)$ is convex in the interval $[0, \frac{1}{2}]$.

(c) Consider finally

$$h(x) = \frac{-1}{x \log x \log(-\log x)) \ldots \log(\log(\ldots(-\log x)\ldots))},$$

where the last factor in the denominator is a composition of n logarithms. Then

$$H(x) = -\log(\log(\ldots(-\log x)\ldots)).$$

By (9) we have

$$f(x) = f_1 \circ f_1 \circ \ldots \circ f_1(x) \quad (n \text{ times}),$$

where

$$f_1 = e^{-1/x}.$$

Since f_1 is convex in the interval $[0, \frac{1}{2}]$ and $f_1[0, \frac{1}{2}] \in [0, \frac{1}{2}]$, the composition of any number of exemplars of f_1 is a convex function on $[0, \frac{1}{2}]$. It is not hard to see that the second derivative of $f(x)$, as well as the higher derivatives, is bounded.

Example 3. Let $\psi(x) = \sqrt{x}$ and

$$a(x) = \exp\left(-\exp\left(\frac{-1}{x \log x}\right)\right).$$

We claim that $\phi(x)$ and $a(x)$ satisfy (7) with

$$h(x) = \frac{-1}{x\log x\ \log 2}.$$

Indeed, observe that if $a^{-1}\circ\phi\circ a(x) = y$, then

$$-y\log y = \frac{-x\log x}{1 - x\log x\log 2}.$$

Thus setting $g(x) = -x\log x$ we get

$$g(x) - g(y) = g(x)g(y)\log 2.$$

Therefore

$$\begin{aligned} x - a^{-1}\circ\phi\circ a(x) &= x - y = (g(x) - g(y))\frac{x-y}{g(x)-g(y)} \\ &= \frac{g(x)g(y)\log 2}{g'(\xi)} = \frac{xy\log x\log y\log 2}{-1-\log\xi} \\ &\le \frac{x^2\log^2 x\log 2}{-1-\log x} = -x^2\log x\frac{\log x\log 2}{1+\log x} \\ &\le -x^2\log 2\log x, \end{aligned}$$

where $y < \xi < x < e^{-2}$. Hence for $0 < x < e^{-2}$ we have

$$\frac{x}{x - a^{-1}\circ\phi\circ a(x)} \ge \frac{-1}{x\log x\log 2}$$

Combining Theorem 1 and Example 2(b) gives that the inequality (1) has no solution nonvanishing in a neighborhood of zero.

A contradiction to the condition leads to the existence of solutions nonvanishing in neighborhood of zero. Namely the following holds.

Theorem 2 *Let $\phi(x)$ be the inverse function for $\psi(x)$. Assume that $\phi(x) \le x$, and*

$$\frac{x}{x - a^{-1}\circ\phi\circ a(x)} \le h(x), \tag{11}$$

where $h(x)$ is a positive function satisfying:

(i) $h(x)$ is integrable about 0,

(ii) Let $H(x)$ be the antiderivative of $h(x)$. The function $H^{-1}(x)$ is convex in a neighborhood of zero.

Let $\Omega(x)$ be an increasing function defined on $[0, +\infty)$, such that $\Omega(0) = 0$ and $\Omega(\phi(x)) \leq \phi(\Omega(x))$. Then there exists a function $u(x)$, nonnegative and strictly increasing in a neighborhood of zero, and satisfying there

$$\int_0^x a(x-y)\psi\left(u(y)\right)dy \geq a^{-1}(\Omega(u(x)))\,\Omega(u(x))\,u(x). \tag{12}$$

Remark. Obviously this is not a direct conversion of (1). Observe that we can take $\Omega(x) = \psi \circ \psi \circ \cdots \circ \psi(x)$, ($n$ times). In this case if $a(x) \leq x^\alpha$ and $\psi(x) > x^\beta$, $\beta < 1$, then

$$a^{-1}(\Omega(u(x)))\,\Omega(u(x)) \geq u(x)^{(1+\alpha^{-1})\beta^n}.$$

The right hand side of this inequality tends to 1 when n goes to infinity. Therefore the right hand side of (12) is close to $u(x)$.

Proof. Let $\Omega(x)$ be an arbitrary function satisfying the assumptions of the theorem. Let $f(x) = H^{-1}(x^2)$. By assumptions $f(x)$ is an increasing function convex in a neighborhood of zero (composition of two convex function is again a convex function). Let $u(x)$ be defined according to

$$u(x) = \Omega^{-1}(a(f(x))). \tag{13}$$

Analogously to (5) there is $g(x)$ such that $0 \leq g(x) \leq x$, and

$$\psi\left(u(x - g(x))\right) = u(x). \tag{14}$$

By (13), (14) and by the fact that $\Omega(\phi(x)) \leq \phi(\Omega(x))$, we have (cf (6))

$$\begin{aligned} a^{-1} \circ \phi \circ a(f(x)) &\geq a^{-1} \circ \Omega \circ \phi \circ \Omega^{-1} \circ a(f(x)) \\ &= f(x - g(x)). \end{aligned} \tag{15}$$

We are going to show that $f(x)$ is infinitesimal with respect to $g(x)$ when x tend to zero. By assumptions and by (15) we get

$$\frac{f(x)}{f(x) - f(x - g(x))} \leq h(f(x)).$$

Therefore

$$\begin{aligned} 1 &\leq \frac{f(x)-f(x-g(x))}{f(x)} h(f(x)) = f'(\xi)h(f(x))\frac{g(x)}{f(x)} \\ &\leq f'(x)h(f(x))\frac{g(x)}{f(x)} = (H\circ f)'(x)\frac{g(x)}{f(x)} = 2x\frac{g(x)}{f(x)}, \end{aligned}$$

where $x-g(x)<\xi<x$, and $f'(\xi)\leq f'(x)$ by the convexity of $f(x)$. Thus for $x\leq \frac{1}{4}$ we have $2f(x)\leq g(x)$. Now we can estimate the integral as follows.

$$\begin{aligned} \int_0^x a(x-y)\psi\left(u(y)\right)dy &\geq \int_{f(x)}^{g(x)} a(x-y)\psi\left(u(y)\right)dy \\ &\geq [g(x)-f(x)]\, a(f(x))\psi(u(x-g(x))) \\ &\geq f(x)\,\Omega(u(x))\,u(x). \end{aligned}$$

By (12) $f(x)=a^{-1}(\Omega(u(x)))$. This completes the proof.

Example 4. Consider

$$\begin{aligned} \psi(x) &= x^{\beta}, \\ a(x) &= \exp\left(-\exp(x^{-\alpha})\right), \end{aligned}$$

where $0<\alpha<1$ and $0<\beta<1$. Using Theorem 2 we will show that the inequality (1) admits nonzero solutions. Let $\psi_0(x)=x^{\gamma}$, with $\gamma<\beta$. We claim that $a(x)$ and $\psi_0(x)$ satisfy the assumptions of Theorem 2 with $h(x)=c\,x^{-\alpha}$. Indeed, observe that if

$$y = a^{-1}\circ\phi_0\circ a(x),\ \ \phi_0(x)=x^{1/\gamma},$$

then

$$y = x(1-x^{\alpha}\log\gamma)^{-1/\alpha}.$$

Therefore, using for example the binomial theorem, we conclude

$$\frac{x}{x-y}\leq c\,x^{-\alpha}.$$

Since $\alpha<1$, the function $h(x)$ is integrable and $H^{-1}(x)=c'\,x^{1/(1-\alpha)}$ is convex. Let $\Omega(x)=x^{\delta}$. Then $\Omega(x)$ commutes with $\psi(x)$. Thus by Theorem 2 and by the fact that $a(x) \le x$, there exists a nonnegative strictly increasing function $u(x)$, defined in a neighborhood of zero and satisfying there

$$\int_0^x a(x-y)\psi_0(u(y))dy \ge [\Omega(u(x))]^2\, u(x).$$

Hence

$$\int_0^x a(x-y)u(y)^\gamma dy \ge u(x)^{2\delta+1}.$$

This implies that the function $v(x) = u(x)^{\gamma/\beta}$ satisfies

$$\int_0^x a(x-y)\psi(v(y))dy \ge v(x)^{(2\delta+1)\beta/\gamma}.$$

Choose δ sufficiently small so that $(2\delta+1)\beta/\gamma \le 1$. Then $v(x)$ satisfies

$$\int_0^x a(x-y)\psi(v(y))dy \ge v(x)$$

in a neighborhood of zero.

Example 5. Examples 1 and 4 can be yet strengthened. Consider

$$\begin{aligned} \psi(x) &= x^\beta, \\ a(x) &= \exp\left(-\exp(x^{-1}|\log x|^\alpha)\right), \end{aligned}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. Let (as in Example 4) $\psi_0(x) = x^\gamma$. We will show that $a(x)$ and $\psi_0(x)$ satisfy (11) with $h(x) = c\,x^{-1}|\log x|^{-\alpha}$. Let

$$y = a^{-1} \circ \phi_0 \circ a(x), \quad \phi_0(x) = x^{1/\gamma}.$$

Then $0 < y < x$ and

$$y^{-1}|\log y|^{-\alpha} = x^{-1}|\log x|^{-\alpha} - \log\gamma.$$

Let $g(x) = x|\log x|^\alpha$. Then

$$g(x) - g(y) = g(x)g(y)\log\gamma^{-1}.$$

This implies

$$\lim_{x\to 0^+} \frac{g(x)}{g(y)} = 1.$$

That in turn yields

$$\lim_{x\to 0^+} \frac{x}{y} = 1.$$

We can assume therefore that $0 < y < x/2$. Following the lines of Example 3 we get

$$\begin{aligned}
x - a^{-1}\circ\phi\circ a(x) &= x - y = (g(x) - g(y))\frac{x-y}{g(x)-g(y)} \\
&= \frac{g(x)g(y)\log\gamma^{-1}}{g'(\xi)} = \frac{x|\log x|^\alpha\, y|\log y|^\alpha \log\gamma^{-1}}{\xi|\log\xi|^\alpha - \alpha|\log\xi|^{\alpha-1}} \\
&\geq \frac{x|\log x|^\alpha\, y|\log y|^\alpha \log\gamma^{-1}}{\xi|\log\xi|^\alpha} \geq xy|\log x|^\alpha \log\gamma^{-1} \\
&\geq \frac{1}{2}\log\gamma^{-1}\, x^2|\log x|^\alpha,
\end{aligned}$$

where $0 < y < \xi < x$. Thus

$$\frac{x}{x - a^{-1}\circ\phi_0\circ a(x)} \geq \frac{2}{x|\log x|^\alpha \log\gamma^{-1}}.$$

Obviously the function $h(x) = c\,x^{-1}|\log x|^{-\alpha}$ is integrable about zero. Moreover $H(x) = c(\log x)^{1-\alpha}$ and $H^{-1}(x) = \exp(cx^{1/(1-\alpha)})$ is convex in a neighborhood of zero (c denotes a constant not necessarily the same at each ocurrence). Thus we can apply Theorem 2. The rest follows as it was shown in Example 4.

References

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