The ratio and generating function of cogrowth coefficients of finitely generated groups

Ryszard Szwarc*

Abstract

Let $G$ be a group generated by $r$ elements $g_1, g_2, \ldots, g_r$. Among the reduced words in $g_1, g_2, \ldots, g_r$ of length $n$ some, say $\gamma_n$, represent the identity element of the group $G$. It has been shown in a combinatorial way that the $2n$th root of $\gamma_{2n}$ has a limit, called the cogrowth exponent with respect to generators $g_1, g_2, \ldots, g_r$. We show by analytic methods that the numbers $\gamma_n$ vary regularly; i.e. the ratio $\frac{\gamma_{2n+2}}{\gamma_{2n}}$ is also convergent. Moreover we derive new precise information on the domain of holomorphy of $\gamma(z)$, the generating function associated with the coefficients $\gamma_n$.

Every group $G$ generated by $r$ elements can be realized as a quotient of the free group $\mathbb{F}_r$ on $r$ generators by a normal subgroup $N$ of $\mathbb{F}_r$, in such a way that the generators of the free group $\mathbb{F}_r$ are sent to the generators of the group $G$. With the set of generators of $\mathbb{F}_r$ we associate the length function of words in these generators. The cogrowth coefficients $\gamma_n = \# \{ x \in N \mid |x| = n \}$ were first introduced by Grigorchuk in [2]. The numbers $\gamma_n$ measure how big the group $G$ is when compared with $\mathbb{F}_r$. It has been shown that the quantities $2^{\sqrt{\gamma_{2n}}}$ have a limit denoted by $\gamma$, and called the growth exponent of $N$ in $\mathbb{F}_r$. Since the subgroup $N$ can have at most $2r(2r-1)^{n-1}$ elements of length $n$, the cogrowth exponent $\gamma$ can be at most $2r - 1$. The famous Grigorchuk result, proved independently by J. M. Cohen in [1], states that the group $G$ is amenable if and only if $\gamma = 2r - 1$ (see also [6], [8]).

*This work has been partially supported by KBN (Poland) under grant 2 P03A 030 09.
1991 Mathematics Subject Classification. Primary 20F05, 20E05
Key words and phrases: cogrowth of subgroups, free group, amenable groups

1
The main result of this note is that the coefficients $\gamma_{2n}$ satisfy not only the Cauchy $n$th root test but also the d’Alambert ratio test.

**Theorem 1** The ratio of two consecutive even cogrowth coefficients $\gamma_{2n+2}/\gamma_{2n}$ has a limit. Thus the ratio tends to $\gamma^2$, the square of the cogrowth exponent.

**Proof.** Let us denote by $g_1, g_2, \ldots, g_r$ the generators of $G$. Let $\mu$ be the measure equidistributed over the generators and their inverses according to the formula

$$\mu = \frac{1}{2\sqrt{q}} \sum_{i=1}^r (g_i + g_i^{-1}),$$

where $q = 2r - 1$. By an easy transformation of [6, Formula (*)] we obtain

$$\frac{z}{1 - z^2} \sum_{n=0}^\infty \gamma_n z^n = \frac{1}{2\sqrt{q}} \sum_{n=0}^\infty \mu^{*n}(c) \left( \frac{2\sqrt{q} z}{q z^2 + 1} \right)^{n+1}, \quad (1)$$

for small values of $|z|$. Let $\varrho$ denote the spectral radius of the random walk defined by $\mu$; i.e.

$$\varrho = \lim_{n \to \infty} 2^n \mu^{*2n}(c).$$

By $d\sigma(x)$ we will denote the spectral measure of this random walk. Hence

$$\mu^{*n}(c) = \int_{-\varrho}^\varrho x^n d\sigma(x). \quad (2)$$

Note that the point $\varrho$ belongs to the support of $\sigma$. Combining (1) and (2) gives

$$\frac{z}{1 - z^2} \sum_{n=0}^\infty \gamma_n z^n = \frac{1}{2\sqrt{q}} \int_{-\varrho}^\varrho \sum_{n=0}^\infty x^n \left( \frac{2\sqrt{q} z}{q z^2 + 1} \right)^{n+1} d\sigma(x)
= \frac{1}{2\sqrt{q}} \int_{-\varrho}^\varrho \frac{z}{1 - 2\sqrt{q} x z + q z^2} d\sigma(x). \quad (3)$$

By the well known formula for the generating function of the second kind Chebyshev polynomials $U_n(x)$ (see [4, (4.7.23), page 82]) where

$$U_n(t) = \frac{t^{n+1} - t^{-n-1}}{t - t^{-1}}, \quad (4)$$

2
we have

\[ \frac{1}{1 - 2\sqrt{q}xz + qz^2} = \sum_{n=0}^{\infty} U_n(x)q^{n/2}z^n. \]

Thus

\[ \frac{z^2}{1 - z^2} \sum_{n=0}^{\infty} \gamma_n z^n = z \sum_{n=0}^{\infty} q^{n/2}z^n \int_{-\varrho}^{\varrho} U_n(x)d\sigma(x). \]

Therefore for \( n \geq 2 \) we have

\[ \gamma_n = \frac{q^{n/2}}{2} \int_{-\varrho}^{\varrho} \{ U_n(x) - q^{-1}U_{n-2}(x) \} d\sigma(x). \]  \hspace{1cm} (5)

Since \( U_{2n}(-x) = U_{2n}(x) \) we get

\[ \gamma_{2n} = \frac{q^n}{2} \int_{0}^{\varrho} \{ U_{2n}(x) - q^{-1}U_{2n-2}(x) \} d\tilde{\sigma}(x), \]  \hspace{1cm} (6)

where \( \tilde{\sigma}(A) = \sigma(A) + \sigma(-A) \) for \( A \subset (0, \varrho] \) and \( \tilde{\sigma}([0]) = \sigma([0]) \). Let

\[ I_n = \int_{0}^{\varrho} \{ U_{2n}(x) - q^{-1}U_{2n-2}(x) \} d\tilde{\sigma}(x). \]

By [3, Corollary 2] we have \( \varrho > 1 \). Hence we can split the integral \( I_n \) into two integrals: the first \( I_{n,1} \) over the interval \([0, \varrho_0]\) and the second \( I_{n,2} \) over \([\varrho_0, \varrho]\), where \( \varrho_0 = (1 + \varrho)/2 \). By (4) we have \( |U_m(x)| \leq (m + 1) \) for \( x \in [0, 1] \) and

\[ |U_m(x)| \leq (m + 1)\sqrt{x^2 + 1} \]  \hspace{1cm} for \( x \geq 1 \).

Thus we get

\[ I_{n,1} \leq 2(2n + 1) \left( \varrho_0 + \sqrt{\varrho_0^2 - 1} \right)^{2n} \int_{0}^{\varrho_0} d\tilde{\sigma}(x) \leq 2(2n + 1) \left( \varrho_0 + \sqrt{\varrho_0^2 - 1} \right)^{2n}. \]  \hspace{1cm} (7)

Let’s turn to estimating the integral \( I_{n,2} \) over \([\varrho_0, \varrho]\). By (4) one can easily check that

\[ \left| U_n(x) - \frac{(x + \sqrt{x^2 + 1})^{n+1}}{2\sqrt{x^2 + 1}} \right| = o(1) \quad \text{when} \quad n \to \infty, \]
uniformly on the interval $[\varrho_0, \varrho]$. Hence
\[ \left| U_{2n}(x) - q^{-1}U_{2n-2}(x) - (x + \sqrt{x^2 - 1})^{2n-1} \left( \frac{x + \sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}} - q^{-1} \right) \right| = o(1), \]
when $n$ tends to infinity, uniformly in the interval $[\varrho_0, \varrho]$. This implies

\[ I_{n,2} \approx \tilde{I}_{n,2} = \int_{\varrho_0}^{\varrho} (x + \sqrt{x^2 - 1})^{2n} \frac{(x + \sqrt{x^2 - 1})^2 - q^{-1}}{2\sqrt{x^2 - 1}(x + \sqrt{x^2 - 1})} d\tilde{\sigma}(x). \quad (8) \]

Since the endpoint $\varrho$ belongs to the support of $\tilde{\sigma}$, we get
\[ \tilde{I}_{n,2}^{1/2n} \to \varrho + \sqrt{\varrho^2 - 1}. \quad (9) \]

By combining this with (7) and (8) we obtain
\[ I_n = I_{n,1} + I_{n,2} = \tilde{I}_{n,2}(1 + o(1)), \quad n \to \infty. \quad (10) \]

In view of (9) the integral $\tilde{I}_{n,2}$ tends to infinity. Thus by (6) and (10) we have
\[ \frac{\gamma_{2n+2}}{\gamma_{2n}} \approx q \frac{\tilde{I}_{n+1,2}}{\tilde{I}_{n,2}}. \]

Lemma 1 ([7]) Let $f(x)$ be a positive and continuous function on $[a, b]$, and $\mu$ be a finite measure on $[a, b]$. Then
\[ \lim_{n \to \infty} \frac{\int_a^b f(x)^{n+1} d\mu(x)}{\int_a^b f(x)^n d\mu(x)} = \max \{ f(x) \mid x \in \text{supp} \mu \}. \]

Applying Lemma 1 and using the fact that $\varrho$ belongs to the support of $\tilde{\sigma}$ gives
\[ \frac{\gamma_{2n+2}}{\gamma_{2n}} \to q \left\{ \varrho + \sqrt{\varrho^2 - 1} \right\}^2. \quad (11) \]

\[ \Box \]

Theorem 2 The generating function $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ can be decomposed into a sum of two functions $\gamma^{(0)}(z)$ and $\gamma^{(1)}(z)$ such that $\gamma^{(0)}(z)$ is analytic in the open disc of radius $q^{-1/2}$ (where $q = 2r - 1$), while $\gamma^{(1)}(z)$ is analytic in the whole complex plane after removing the two real intervals $[-\gamma q^{-1}, -\gamma^{-1}]$ and $[\gamma^{-1}, \gamma q^{-1}]$. Moreover, $\gamma^{(1)}$ satisfies the functional equation
\[ \frac{z\gamma^{(1)}(z)}{1 - z^2} = \frac{(q/z)\gamma^{(1)}(q/z)}{(q/z)}. \]
Proof. By (3) we have
\[
\gamma(z) = (1 - z^2) \int_{-e}^{e} \frac{1}{1 - 2\sqrt{q}zx + qz^2} d\sigma(x).
\]
Let
\[
\gamma^{(0)}(z) = (1 - z^2) \int_{-1}^{1} \frac{1}{1 - 2\sqrt{q}zx + qz^2} d\sigma(x), \\
\gamma^{(1)}(z) = (1 - z^2) \int_{1<|x|\leq \rho} \frac{1}{1 - 2\sqrt{q}zx + qz^2} d\sigma(x).
\]
For \(-1 \leq x \leq 1\) the expression \(1 - 2\sqrt{q}zx + qz^2\) vanishes only on the circle of radius \(q^{-1/2}\). Thus \(\gamma^{(0)}(z)\) has the desired property. For \(1 < |x| \leq \rho\) the expression \(1 - 2\sqrt{q}zx + qz^2\) vanishes only on the intervals
\[
\left[ -\frac{\rho + \sqrt{\rho^2 - 1}}{\sqrt{q}}, -\frac{\rho - \sqrt{\rho^2 - 1}}{\sqrt{q}} \right], \quad \left[ \frac{\rho - \sqrt{\rho^2 - 1}}{\sqrt{q}}, \frac{\rho + \sqrt{\rho^2 - 1}}{\sqrt{q}} \right].
\]
By (11) we have that \(\gamma = q^{1/2}(\rho + \sqrt{\rho^2 - 1})\). This shows that \(\gamma^{(1)}\) is analytic where it has been required.

The functional equation follows immediately from the formula
\[
\frac{z\gamma^{(1)}(z)}{1 - z^2} = \int_{1<|x|\leq \rho} \frac{1}{z^{-1} - 2\sqrt{q}x + qz} d\sigma(x).
\]

\[\square\]

Remark. Combining (6) and (10) yields
\[
\gamma_{2n} = q^n \left\{ \int_{\rho_0}^{e} (x + \sqrt{x^2 - 1})^{2n} \frac{(x + \sqrt{x^2 - 1})^2 - q^{-1}}{2\sqrt{x^2 - 1}(x + \sqrt{x^2 - 1})} d\tilde{\sigma}(x) + o(1) \right\}.
\]
We have
\[
h(\rho) := \frac{(\rho_0 + \sqrt{\rho_0^2 - 1})^2 - q^{-1}}{2\sqrt{\rho_0^2 - 1} - 1(\rho_0 + \sqrt{\rho_0^2 - 1})} \geq \frac{(x + \sqrt{x^2 - 1})^2 - q^{-1}}{2\sqrt{x^2 - 1}(x + \sqrt{x^2 - 1})}, \quad \frac{(\rho + \sqrt{\rho^2 - 1})}{\rho} x \geq x + \sqrt{x^2 - 1}.
\]
Therefore, in view of (2), we get

\[
\gamma_{2n} \leq q^n \left\{ h(\varrho_0) \frac{(\varrho + \sqrt{\varrho^2 - 1})^{2n}}{\varrho^{2n}} \int_0^\varrho x^{2n} d\tilde{\sigma}(x) + o(1) \right\}
\]

\[
= q^n h(\varrho_0) \left\{ \left( \varrho + \sqrt{\varrho^2 - 1} \right)^{2n} \frac{\mu\star(\varrho)}{\varrho^{2n}} + o(1) \right\}.
\]

Finally we obtain

\[
\frac{\gamma_{2n}}{\mu\star(\varrho)} \frac{\varrho^{2n}}{\mu\star(\varrho)} = \frac{\gamma_{2n}}{\mu\star(\varrho)} \left\{ \frac{\varrho}{\sqrt{\varrho^2 - 1}} \right\}^{2n} \leq h(\varrho_0) + o(1).
\]

We conjecture that the opposite estimate also holds; i.e. the quantity on the left hand side is bounded away from zero, by a positive constant depending only on \( \varrho \). This conjecture can be checked easily if the measure \( \sigma \) is smooth in the neighbourhood of \( \varrho \) and the density has zero of finite order at \( \varrho \).

**Acknowledgement.** For a long time I thought Theorem 1 follows from the following statement.

*If \( f(z) = \sum_{n=0}^\infty a_n z^n \) is analytic in the complex plane except the half line \([1, +\infty)\), then the ratio \( a_{n+1}/a_n \) converges to 1.*

I am grateful to Jacek Zienkiewicz from my Department for constructing a fine counterexample to this statement.

**References**


Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50–384 Wrocław, Poland
e-mail: szwarc@math.uni.wroc.pl