Canad. Math. Bull.Vol. **35** (4), 1992 pp. 548–556

CONNECTION COEFFICIENTS OF ORTHOGONAL POLYNOMIALS

RYSZARD SZWARC

ABSTRACT. Let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be polynomials orthogonal with respect to different distributions. Conditions are given which imply that the coefficients in the expansion of P_n in terms of $Q_0, Q_1, \ldots, Q_n$ are non-negative.

1. **Introduction.** Let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be polynomials orthogonal with respect to different measures. We are concerned with finding conditions under which the constants $a(n,m)$ in the expansion

$$P_n = \sum_{m=0}^{n} a(n,m)Q_m, \quad n = 0,1,2,\ldots, \tag{1}$$

are all non-negative.

There are several results in this subject. Askey [1] gives conditions in terms of the recurrence formulas which P_n and Q_n satisfy. Trench [3], [4] imposes conditions on the measures associated with $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$. Also, Askey [2] shows how and where the problem arises.

In the present work we study the recurrence formulas and corresponding difference operators to derive non-negativity of the coefficients $a(n,m)$ in (1). The proofs of the results go via the maximum principle for a discrete boundary value problem. Our Theorem 1 is closely related to the Askey result [1], however it does not imply it. For that reason we give an alternative proof of Askey's theorem via the corresponding hyperbolic boundary value problem.

We also give conditions under which $a(n,n) > 0$ and $a(n,m) \leq 0$ for $0 \leq m \leq n-1$. We point out Theorem 2 which admits applications to the Gegenbauer polynomials.

2. **The results.**

THEOREM 1. *Let the polynomials P_n and $Q_n, n = 0,1,2,\ldots,$ satisfy*

$$\begin{aligned} xP_n &= \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}, \\ xQ_n &= \gamma'_n Q_{n+1} + \beta'_n Q_n + \alpha'_n Q_{n-1}. \end{aligned} \tag{2}$$

The paper was completed while the author was visiting the Department of Mathematics, UW-Madison, during the 1990-91 academic year.

Received by the editors November 19, 1990.

AMS subject classification: 39A70.

Key words and phrases: orthogonal polynomials, recurrence formula.

Assume that

(i) $\alpha'_m \geq \alpha_n$ *for* $m \leq n$,

(ii) $\beta'_m \geq \beta_n$ *for* $m \leq n$,

(iii) $\alpha'_m + \gamma'_m \geq \alpha_n + \gamma_n$ *for* $m \leq n$,

(iv) $\gamma'_m \geq \alpha_n$ *for* $m < n$.

Then the connection coefficients $a(n,m)$ *in the formula*

$$P_n = \sum_{m=0}^{n} a(n,m) Q_m \tag{3}$$

are non-negative.

PROOF. After an appropriate renormalization (see [5], (6)) the polynomials P_n and Q_n (we do not introduce new symbols for the renormalized polynomials) satisfy

$$\begin{aligned} xP_n &= \alpha_{n+1} P_{n+1} + \beta_n P_n + \gamma_{n-1} P_{n-1} \\ xQ_n &= \alpha'_{n+1} Q_{n+1} + \beta'_n Q_n + \gamma'_{n-1} Q_{n-1}. \end{aligned}$$

Obviously the renormalization does not affect the conclusion of the theorem. Put $\alpha'_0 = \alpha'_{-1} = \alpha'_{-2} = \cdots = 0$ and $\gamma'_{-1} = \gamma'_{-2} = \cdots = 0$. Consider the linear operators L and L' acting on sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=-\infty}^{\infty}$ respectively, according to

$$\begin{aligned} La_n &= \alpha_{n+1} a_{n+1} + \beta_n a_n + \gamma_{n-1} a_{n-1}, n = 0, 1, 2, \ldots; \\ L'b_n &= \alpha'_{n+1} b_{n+1} + \beta'_n b_n + \gamma'_{n-1} b_{n-1}, n = 0, \pm 1, \pm 2, \ldots; \end{aligned} \tag{4}$$

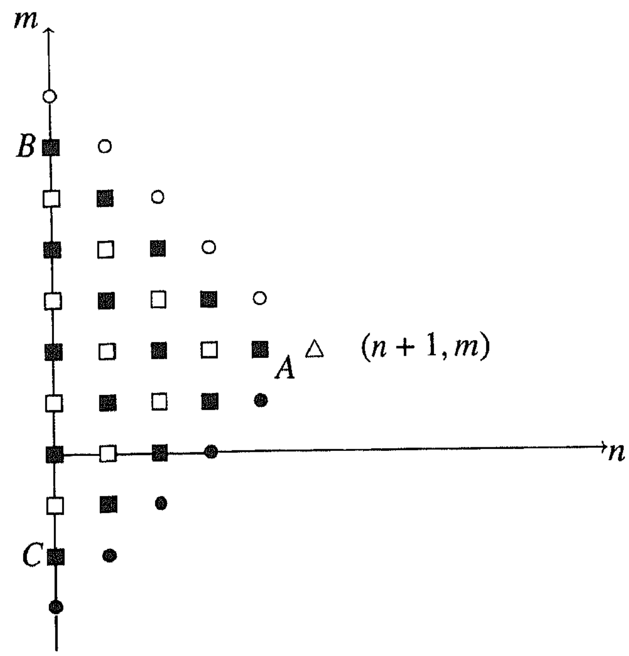
Finally, let L_n and L'_m denote the operators acting on the matrices $u(n,m)$, $n \in \mathbf{N}$, $m \in \mathbf{Z}$, as L and L' but with respect to n or m treating the other variable as a constant.

Let us consider the following boundary value problem.

$$\mathbf{N} \times \mathbf{Z} \ni (n,m) \longmapsto u(n,m) \in \mathbf{C}, \tag{5}$$
$$\begin{cases} (L_n - L'_m) u = 0 \\ u(n,n) \geq 0 \; u(n,m) = 0 \quad \text{for } n < m, \text{ and for } m < 0. \end{cases}$$

PROPOSITION 1. *If the assumptions of Theorem 1 are satisfied then* $u(n,m) \geq 0$ *for* $0 \leq m \leq n$.

PROOF. Assume that the result is false. Let $(n+1, m)$ be a point in the domain $\{(s,t) : s \geq t\}$ with the least first coordinate for which $u(n+1,m) < 0$. Thus $u(s,t)$ is non-negative if $s \leq n$. Consider the triangle ΔABC with the vertices $A(n,m)$, $B(0, n+m)$ and $C(0, m-n)$ (*cf.* [5], the proof of Theorem 3).



Let us split the lattice points in ΔABC in two subsets (see [5], the proof of Theorem 2): Ω_1 consisting of the points (k,ℓ) such that $k-\ell \equiv n-m \bmod 2$ and the rest $-\Omega_2$. We marked by ■ the points of Ω_1 and by □ those of Ω_2. Let Ω_3 denote the set of lattice points on the line connecting the points $(0,n+m+1)$ and $(n+1,m)$ and let Ω_4 denote those on the line connecting $(0,n-m-1)$ and $(n+1,m)$. The points of Ω_3 and Ω_4 are marked by ○ and ● respectively, so the vertex $(n+1,m)$ is excluded from both Ω_3 and Ω_4.

Suppose $(L_n - L'_m)u = 0$. In particular we have $\sum_{(x,y)\in\Omega_1}(L_n - L'_m)u(x,y) = 0$. Now we calculate the terms according to (1) so every term will involve the values of the function u at the immediate neighbors of (x,y). Let us tidy up the sum so as to get the expression of the form $\sum_{(s,t)} c_{s,t}u(s,t)$. First of all observe that $c_{s,t} = 0$ unless $(x,t) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \{(n+1,m)\}$.

$$0 = \sum_{(x,y)\in\Omega_1} (L_n - L'_m)u(x,y) = \sum_{i=1}^{4} \sum_{(s,t)\in\Omega_i} c_{s,t}u(s,t) + c_{n+1,m}u(n+1,m) \tag{6}$$

The coefficients $c_{s,t}$ can be easily computed from (4). Indeed, we have

1) $(s,t) \in \Omega_1$, $c_{s,t} = \beta_s - \beta'_t$,
2) $(s,t) \in \Omega_2$, $c_{s,t} = (\alpha_s + \gamma_s) - (\alpha'_t + \gamma'_t)$,
3) $(s,t) \in \Omega_3$, $c_{s,t} = \alpha_s - \alpha'_t$,
4) $(s,t) \in \Omega_4$, $c_{s,t} = \alpha_s - \gamma'_t$,
5) $c_{n+1,m} = \alpha_{n+1}$.

We can restrict ourselves to $0 \le t \le s$ because otherwise $u(s,t) = 0$. By our assumptions all the coefficients $c_{s,t}$ for $0 \le t \le s$ are non-positive while $c_{n+1,m}$ is positive. Since $u(s,t) \ge 0$ for $(s,t) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ and $u(n+1,m) < 0$ the sum in (6) is strictly negative. This gives a contradiction.

Let us get back to the proof of Theorem 1. By (1) we have

$$\langle P_n, Q_m\rangle_{L^2(\nu)} = \int P_n Q_m \, d\nu = a(n,m) \int Q_m^2 \, d\nu \tag{7}$$

Define the function $u(n,m)$ on $\mathbf{N} \times \mathbf{Z}$ by

$$u(n,m) = \begin{cases} \int P_n Q_m \, d\nu & \text{for } m \geq 0 \\ 0 & \text{for } m < 0 \end{cases} \tag{8}$$

Thus u satisfies the assumptions of Proposition 1. Indeed, by definition $u(n,m) = 0$ for $m < 0$ and $u(n,m) = 0$ for $n < m$ because Q_m is orthogonal to the polynomials of the lower degree. Moreover if $0 \leq m \leq n$ then

$$\begin{aligned}(L_n u)(n,m) &= \langle LP_n, Q_m\rangle_{L^2(\nu)} = \langle xP_n, Q_m\rangle_{L^2(\nu)} \\ &= \langle P_n, xQ_m\rangle_{L^2(\nu)} = \langle P_n, L'Q_m\rangle_{L^2(\nu)} = (L'_m u)(n,m).\end{aligned}$$

Finally since $u(n,m) = 0$ and $\alpha_m = \beta_m = \gamma_m = 0$ for $m < 0$ we get $(L_n u)(n,m) = L'_m u(n,m) = 0$ for $m < 0$. Therefore, by Proposition 1 $u(n,m) \geq 0$ for $0 \leq m \leq n$. Now the conclusion follows from (7) and (8).

The next corollary is a particular case of Theorem 1.

COROLLARY 1. *Assume the orthogonal polynomials P_n satisfy $xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}$ and $\alpha_n \leq \frac{1}{2}$, $\alpha_n + \gamma_n \leq 1$, $\beta_n \leq 0$. Then P_ns can be represented as linear combinations of the Tchebyshev polynomials with non-negative coefficients.*

COROLLARY 2. *Under assumptions of Corollary 1, for any $n = 0, 1, 2, \ldots$ and any ellipse with foci at 1 and -1 the maximal absolute value of P_n is attained at the right end of the major axis.*

EXAMPLE. Let $P_n^{(\alpha)}$ be the Gegenbauer polynomials with $\alpha \geq -\frac{1}{2}$. They satisfy the three-term recurrence formula

$$xP_n^{(\alpha)} = \frac{n+2\alpha+1}{2n+2\alpha+1} P_{n+1}^{(\alpha)} + \frac{n}{2n+2\alpha+1} P_{n-1}^{(\alpha)}.$$

By Corollary 1 we have

$$P_n^{(\alpha)} = \sum_{m=0}^{n} a(n,m) T_m,$$

with $a(n,m) \geq 0$, $0 \leq m \leq n$, where the T_n are the Tchebyshev polynomials (here: $T_n = P_n^{(-1/2)}$). The connection coefficients for Gegenbauer polynomials we know explicitly due to Gegenbauer himself (see [2], p. 59).

In the case of symmetric measures Theorem 1 can be slightly sharpened. Namely, we have the following.

COROLLARY 2. *Let the polynomials P_n and Q_n satisfy*

$$\begin{aligned} xP_n &= \gamma_n P_{n+1} + \alpha_n P_{n-1} \\ xQ_n &= \gamma'_n Q_{n+1} + \alpha'_n Q_{n-1} \end{aligned} \tag{9}$$

Assume that

(i) $\alpha'_{2m} \geq \alpha_{2n}$ and $\alpha'_{2m+1} \geq \alpha_{2n+1}$ for $0 < m \leq m$,
(ii) $\alpha'_{2m} + \gamma'_{2m} \geq \alpha_{2n} + \gamma_{2n}$ and $\alpha'_{2m+1} + \gamma'_{2m+1} \geq \alpha_{2n+1} + \gamma_{2n+1}$ for $m \leq n$,

(iii) $\gamma'_{2m} \geq \alpha_{2n}$ *and* $\gamma'_{2m+1} \geq \alpha_{2n+1}$ *for* $m < n$.
Then the connection coefficients $a(n,m)$ *in (1) are non-negative.*

PROOF. Observe that (9) implies the polynomials P_n and Q_n are even or odd according to the indices n and m. Thus by (8) $u(s,t) = 0$ whenever $s+t$ is an odd number. Hence analyzing the proof of Proposition 1 we see that it suffices to consider the coefficients $c_{s,t}$ only when $s-t$ is an even number. Therefore Corollary 2 holds.

COROLLARY 3. *Let the orthogonal polynomials* P_n *and* Q_n *satisfy the recurrence formula (1). Assume that*

(i) $\alpha_1 \geq \alpha'_1 \geq \alpha_2 \geq \alpha'_2 \geq \cdots$,
(ii) $\beta_0 \geq \beta'_0 \geq \beta_1 \geq \beta'_1 \geq \cdots$,
(iii) $\alpha_0 + \gamma_0 \geq \alpha'_0 + \gamma'_0 \geq \alpha_1 + \gamma_1 \geq \alpha'_1 + \gamma'_1 \geq \cdots$,
(iv) $\gamma'_m \geq \alpha_n$ *for* $m < n$.

Then the connection coefficients $a(n,m)$ *in the formula*

$$P_n = \sum_{m=0}^{n} a(n,m) Q_n$$

satisfy $a(n,n) > 0$ *and* $a(n,m) \leq 0$ *for* $0 \leq m < n$.

PROOF. Let us expand the polynomials P_n in terms of the Q_ms. Then

$$P_n = a(n,n)Q_n + \sum_{m=0}^{n-1} a(n,m)Q_m$$

Obviously $a(n,n) > 0$ as P_n and Q_n have positive leading coefficients. If we show that $a(n,m) \leq 0$ for $0 \leq m \leq n-1$ the proof will be complete. As before an appropriate maximum principle secures the condition $a(n,m) \leq 0$.

PROPOSITION 2. *Let the assumptions of Corollary 2 be satisfied. If* u *is a solution to the boundary value problem (2), then* $u(n,m) \geq 0$ *for* $0 \leq m < n$.

PROOF. We can follow the lines of the proof of Proposition 1. Then the expression in (6) will be strictly negative if only $c_{s,t} \geq 0$ for $0 \leq t < s$ and $c_{s,s} \leq 0$. And so they are; it suffices to scan the listing of the values $c_{s,t}$ in the proof of Proposition 1.

REMARK. Obviously if $P_n = a(n,n)Q_n + \sum_{m=0}^{n-1} a(n,m)Q_m$, where $a(n,m) > 0$ and $a(n,m) \leq 0$, then $Q_n = \sum_{m=0}^{n} b(n,m)P_m$ with the positive coefficients $b(n,m)$.

We are keen on giving a proof of Askey's theorem via the maximum principle. This theorem is stated for the monic polynomials (*i.e.* the leading coefficient is 1). This means the polynomial P_n and Q_n satisfy

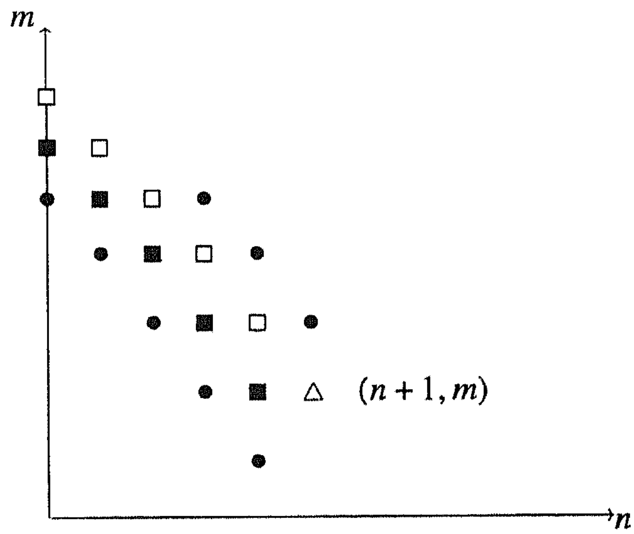
$$xP_n = P_{n+1} + \beta_n P_n + \lambda_{n-1}^2 P_{n-1},$$
$$xQ_n = Q_{n+1} + \beta'_n Q_n + \lambda'^2_{n-1} Q_{n-1}.$$

Obviously the positivity of the connection coefficients does not depend on normalization, so we can normalize the polynomials P_n and Q_n arbitrarily. For our purpose the orthonormalization of the P_ns and Q_ns is convenient. Therefore we let the polynomials P_n and Q_n satisfy

$$(10)\qquad \begin{aligned} xP_n &= \lambda_n P_{n+1} + \beta_n P_n + \lambda_{n-1} P_{n-1}, \\ xQ_n &= \lambda'_n Q_{n+1} + \beta'_n Q_n + \lambda'_{n-1} Q_{n-1}. \end{aligned}$$

THEOREM 2 (ASKEY [1]). *Let the polynomials P_n and Q_n satisfy (10). Assume $\lambda'_n \geq \lambda_n$ and $\beta'_n \geq \beta_n$ for $0 \leq m \leq n$. Then the connection coefficients $a(n,m)$ are non-negative.*

PROOF. As in the proof of Theorem 1 it suffices to prove that if $u(n,m)$ is a solution of the problem (2), then $u(n,m) \geq 0$ for $0 \leq m \leq n$. For a contradiction, let $(n+1,m)$ be a point such that $u(s,t) \geq 0$ for $0 \leq t \leq s \leq n$ and $u(n+1,m) < 0$. Let Ω_1 denote the lattice points on the line connecting $(0, m+n)$ and (n,m), Ω_2 the points on the line connecting $(0, m+n+1)$ and $(n, m+1)$, and Ω_3 the points on the line which connects $(0, m+n-1)$ and $(n, m-1)$.



Then similarly to (3) we have

$$(11)\qquad 0 = \sum_{(x,y)\in\Omega_1} (L_n - L'_m)u(x,y) = \sum_{i=1}^{3} \sum_{(s,t)\in\Omega_i} c(s,t)u(s,t) + c_{n+1,m}u(n+1,m),$$

where the coefficients $c_{s,t}$ are computed as follows.

1) $(s,t) \in \Omega_1$, $c_{s,t} = \beta_s - \beta'_t$,
2) $(s,t) \in \Omega_2$, $c_{s,t} = \lambda_{s-1} - \lambda'_{t-1}$,
3) $(s,t) \in \Omega_3$, $c_{s,t} = \lambda_s - \lambda'_t$,
4) $c_{n+1,m} = \lambda_n$.

We can restrict our attention to $0 \leq t \leq s$ as u vanishes otherwise. Then by the assumptions the coefficients $c_{s,t}$ are less than or equal to 0, while $c_{n+1,m}$ is positive. Thus the sum (11) is strictly negative as all its terms are non-positive and $c_{n+1,m}u(n+1,m) < 0$. This leads to contradiction.

Analogously to Corollary 2 we can derive the following.

COROLLARY 3. *Let the orthogonal polynomials P_n and Q_n satisfy the recurrence formula (10). Assume that $\lambda_0 \geq \lambda'_0 \geq \lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots$ and $\beta_0 \geq \beta'_0 \geq \beta_1 \geq \beta'_1 \geq \beta_2 \geq \beta'_2 \geq \cdots$. Then the connection coefficients $a(n,m)$ in*

$$P_n = \sum_{m=0}^{n} a(n,m) Q_m$$

satisfy $a(n,n) > 0$ and $a(n,m) \leq 0$ for $0 \leq m < n$.

EXAMPLE. Consider a decreasing sequence of positive numbers λ_n. Assume the polynomials P_n and $\tilde{P}_n$ satisfy

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1} P_{n-1},$$
$$x\tilde{P}_n = \lambda_{n+1}\tilde{P}_{n+1} + \lambda_n \tilde{P}_{n-1} \text{for } n = 0, 1, 2, \ldots .$$

Then Corollary 3 implies that

$$\tilde{P}_n = \sum_{m=0}^{n} a(n,m) P_m,$$

where $a(n,n) > 0$ and $a(n,m) \leq 0$ for $0 \leq m < n$.

One of the disadvantages of the previous results is that they apply in very special cases and do not cover known properties of, for example, the Jacobi polynomials. This disadvantage has its origin in the fact that our boundary condition is put on the diagonal $\{(n,n) : n \in N\}$ unlike in [5], where we had the boundary condition on the n-axis. However sometimes it is possible to secure the boundary condition on the n-axis. For example when P_n are orthogonal with respect to μ and Q_n are such with respect to ν and we can somehow (not referring to the maximum principle) determine the signs of $\int P_n \, d\nu$.

THEOREM 3. *Let P_n and Q_n be the polynomials orthogonal with respect to the measures μ and ν respectively. Assume that $\int P_n \, d\nu < 0$ for $n = 1, 2, \ldots$ and let*

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1},$$
$$xQ_n = \gamma'_n Q_{n+1} + \beta'_n Q_n + \alpha'_n Q_{n-1}.$$

Assume that

(i) $\alpha_n \geq \alpha'_m$ *for* $n > m$,
(ii) $\alpha_n + \gamma_n \geq \alpha'_m + \gamma'_m$ *for* $n > m$,
(iii) $\beta_n \geq \beta'_m$ *for* $n > m$,
(iv) $\gamma_n \geq \alpha'_m$ *for* $n > m$.

Then the coefficients $a(n,m)$ in the formula

$$P_n = a(n,n) Q_n - \sum_{m=0} a(n,m) Q_m$$

are non-negative.

The result follows easily from [5], Theorem 1. The condition $\int P_n\,d\nu < 0$ is equivalent to $a(n,0) < 0$ for $n > 0$. Of course there remains a problem of recognizing when we have $\int P_n\,d\nu < 0$.

Let ν be absolutely continuous with respect to μ, *i.e.* $d\nu(x) = h(x)\,d\nu(x)$. Assume $h(x) = h_0 - \sum_{n=1}^{\infty} h_n x^n$, where $h_0, h_1, h_2, \ldots$ are non-negative, the series being convergent uniformly on the support of μ. If also $\beta_n \geq 0$ (α_n and γ_n are always assumed to be non-negative), then we have

$$\int P_n\,d\nu = \int P_n(x)h(x)\,d\nu(x) = -\sum_{n=1}^{\infty} h_n \int P_n(x)x^n\,d\mu(x) \leq 0$$

for $n \geq 1$.

EXAMPLE. Let $P_n^{(\alpha)}, P_n^{(\alpha')}$ be the Gegenbauer polynomials corresponding to the measures $d\mu(x) = (1-x^2)_+^{\alpha}\,dx$ and $d\nu(x) = (1-x^2)_+^{\alpha'}dx$. Assume $0 < \alpha' - \alpha < 1$, $\alpha, \alpha' \geq -\frac{1}{2}$. Then $d\nu(x) = (1-x^2)_+^{\alpha'-\alpha}\,d\mu(x)$ and $(1-x^2)_+^{\alpha'-\alpha} = 1 - \sum_{n=1}^{\infty} h_n x^{2n}$, where $h_n > 0$. By the above remarks we have $\int P_n\,d\nu < 0$ for $n \geq 1$. Next observe that the coefficients of the recurrence relation for the Gegenbauer polynomials satisfy the assumptions of Theorem 3. Indeed, let $\beta_n = \beta'_n = 0$ and

$$\alpha_n = \frac{n}{2n+2\alpha+1}, \quad \alpha'_n = \frac{n}{2n+2\alpha'+1},$$
$$\gamma_n = 1-\alpha_n, \quad \gamma'_n = 1-\alpha'_n$$

Then

$$\alpha'_1 \leq \alpha'_2 \leq \cdots \leq \alpha'_n \leq \alpha_n,$$
$$\alpha_n + \gamma_n = \alpha'_m + \gamma'_m = 1,$$
$$\alpha'_n \leq \alpha_n \leq \gamma_n.$$

Thus by Theorem 3 we have

$$P_n^{(\alpha)} = a(n,n)P_n^{(\alpha')} - \sum_{m=0}^{n-1} a(n,m)P_m^{(\alpha')},$$

where $a(n,m) \geq 0$. Thus

$$P_n^{(\alpha')} = \sum_{m=0}^{n} b(n,m)P_m^{(\alpha)},$$

where $b(n,m) \geq 0$, and $0 < \alpha' - \alpha < 1$, $\alpha, \alpha' \geq -\frac{1}{2}$. By transitivity this holds for any $\alpha' > \alpha \geq -\frac{1}{2}$.

REFERENCES

1. R. Askey, *Orthogonal expansions with positive coefficients*, Proc. Amer. Math. Soc. **26**(1965), 1191–1194.
2. ———, *Orthogonal Polynomials and Special Functions*, Regional Conference Series in Applied Mathematics **21**, SIAM, Philadelphia, Pensylwania, 1975.
3. W. F. Trench, *Nonnegative and alternating expansions of one set of orthogonal polynomials in terms of another*, SIAM J. Math. Anal. **4**(1973), 111–115.
4. ———, *Proof of a conjecture of Askey on orthogonal expansions with positive coefficients*, Bull. Amer. Math. Soc. **81**(1975), 954–955.
5. R. Szwarc, *Orthogonal polynomials and a discrete boundary value problem I*, SIAM J. Math. Anal. (4) **23**(1992).
6. M. W. Wilson, *Nonnegative expansions of polynomials*, Proc. Amer. Math. Soc. **24**(1970), 100–102.

Institute of Mathematics
University of Wroclaw
pl. Grunwaldzki 2/4
50-384 Wroclaw
Poland

Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin 53706
U.S.A.