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Convolution Structures and Haar Matrices

RYSZARD SZWARC

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, and Institute of Mathematics, Wroclaw University, pl. Grunwaldzki 2/4, 50–384 Wroclaw, Poland

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We show that under certain conditions any cubic matrix $b(n, m, k)$, $n, m, k \in \mathbb{N}$, determines a locally compact topological space, a Radon measure, and an orthogonal system of continuous vanishing at infinity functions t_n such that

$$t_n t_m = \sum_{k=0}^{\infty} b(n, m, k)\, t_k.$$

The space and the system of functions t_n, $n \in \mathbb{N}$, are unique.

1. INTRODUCTION

The motivation for this work comes from the following result of R. Askey and G. Gasper [1]. Let $\mathscr{L}_n^\alpha(x)$ be the Lageurre polynomials normalized so that $\mathscr{L}_n^\alpha(0) = 0$. The functions $e^{-x}\mathscr{L}_n^\alpha$ form an orthogonal basis for the Hilbert space $L^2([0, +\infty), x^\alpha e^x\, dx)$. The product of two such functions $e^{-x}\mathscr{L}_n^\alpha$ and $e^{-x}\mathscr{L}_m^\alpha$ belongs to this Hilbert space hence there are coefficients $b(n, m, k; \alpha)$ such that

$$e^{-x}\mathscr{L}_n^\alpha e^{-x}\mathscr{L}_m^\alpha \sim \sum_{k=0}^{\infty} b(n, m, k; \alpha)\, e^{-x}\mathscr{L}_k^\alpha. \tag{1}$$

THEOREM (Askey and Gasper [1]). *Let* $\alpha \geqslant (\sqrt{17} - 5)/2$. *Then* $b(n, m, k; \alpha)$ *are nonnegative.*

It turns out that the series on the right hand side of (1) is uniformly convergent on $[0, +\infty)$ thus we have pointwise equality in (1). Evaluation of (1) at $x = 0$ gives then

$$\sum_{k=0}^{\infty} b(n, m, k; \alpha) = 1. \tag{2}$$

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Let ω_n be defined as

$$\omega_n = \left(\int_0^\infty [e^{-x} \mathcal{L}_n(x)]^2 \, x^\alpha e^x \, dx \right)^{-1}. \tag{3}$$

By multiplying both sides of (1) by $e^{-x} \mathcal{L}_k^\alpha$ and integrating against the measure $x^\alpha e^x \, dx$ we get

$$\int_0^\infty \mathcal{L}_n^\alpha(x) \, \mathcal{L}_m^\alpha(x) \, \mathcal{L}_k^\alpha(x) \, x^\alpha e^{-2x} \, dx = b(n, m, k; \alpha) \, \omega_k^{-1}. \tag{4}$$

This implies that the quantity $b(n, m, k; \alpha) \, \omega_k^{-1}$ is invariant for the permutations of the variables n, m, k.

The formula (1) and the associativity of the product

$$e^{-x} \mathcal{L}_n^\alpha(x) (e^{-x} \mathcal{L}_m^\alpha(x) \, e^{-x} \mathcal{L}_k^\alpha(x)) = (e^{-x} \mathcal{L}_n^\alpha(x) \, e^{-x} \mathcal{L}_m^\alpha(x)) \, e^{-x} \mathcal{L}_k^\alpha(x)$$

implies

$$\sum_{j=0}^{\infty} b(n, m, j; \alpha) \, b(j, k, l; \alpha) = \sum_{j=0}^{\infty} b(m, k, j; \alpha) \, b(n, j, l; \alpha). \tag{5}$$

The aim of this paper is to show that any matrix $b(n, m, k)$ with nonnegative entries satisfying (2), (5), and such that $b(n, m, k) \, \omega_k^{-1}$ is symmetric for a sequence of positive numbers ω_n, determines a locally compact topological space, a Radon measure, and an orthogonal system with respect to this measure of functions which satisfy the formula analogous to (1). The problem was already studied by Haar [4] under significantly stronger assumptions.

2. Convolutions of Sequences

Let $\ell^1(\mathbb{N})$ be the space of absolutely summable sequences of complex numbers. Let δ_n denote the sequence whose terms are all zero except for the nth one which is equal to 1. Let $b(n, m, k)$ be a matrix such that

$$\sum_{k=0}^{\infty} b(n, m, k) = 1, \tag{6}$$

$$b(n, m, k) \geqslant 0, \qquad n, m, k \in \mathbb{N}. \tag{7}$$

Let us define the operation $*$, called a convolution, according to

$$\delta_n * \delta_m = \sum_{k=0}^{\infty} b(n, m, k)\, \delta_k. \tag{8}$$

By (6) and (7) this operation can be extended linearly to the whole space $\ell^1(\mathbb{N})$.

Assume that the condition securing the associativity of the operation $*$

$$\sum_{j=0}^{\infty} b(n, m, j)\, b(j, k, l) = \sum_{j=0}^{\infty} b(m, k, j)\, b(n, j, l) \tag{9}$$

is satisfied. Assume also that there is a sequence of positive numbers ω_n such that

$$b(n, m, k)\, \omega_k^{-1} = b(m, n, k)\, \omega_k^{-1} = b(n, k, m)\, \omega_m^{-1}. \tag{10}$$

In other words the quantity $b(n, m, k)\, \omega_k^{-1}$ is symmetric. In particular this implies that $*$ is a commutative operation.

The formula (8) gives rise to a convolution operation on $\ell^1(\mathbb{N})$. Indeed, if sequences $a = \{a_n\}$ and $b = \{b_n\}$ are absolutely summable then

$$\begin{aligned} \|a * b\|_{\ell^1} &\leqslant \sum_{n, m=0}^{\infty} |a(n)\, b(m)|\; \|\delta_n * \delta_m\|_{\ell^1} \\ &= \sum_{n, m=0}^{\infty} |a(n)\, b(m)| \sum_{k=0}^{\infty} b(n, m, k) \\ &= \sum_{n, m=0}^{\infty} |a(n)\, b(m)| = \|a\|_{\ell^1}\, \|b\|_{\ell^1}. \end{aligned}$$

However, working with $\ell^1(\mathbb{N})$ has a heavy disadvantage. Namely we have no inequality

$$\|a * b\|_{\ell^p} \leqslant \|a\|_{\ell^1}\, \|b\|_{\ell^p}$$

that one would like to hold. This can be achieved by considering a weighted ℓ^1 space. It turns out that the right choice is to take $\ell^1(\mathbb{N}, \omega_n)$, that is, the space of all sequences $a = \{a_n\}$ such that

$$\|a\|_{\ell^1(\omega_n)} = \sum_{n=0}^{\infty} |a_n|\, \omega_n < +\infty.$$

Let T_n be the linear operator acting on sequences by

$$(T_n a)_m = \sum_{k=0}^{\infty} b(n, m, k)\, a_k. \tag{11}$$

Thus

$$T_n\delta_m = \sum_{k=0}^{\infty} b(n, k, m)\, \delta_k.$$

Then T_n is symmetric with respect to the inner product

$$\langle a, b \rangle_{\ell^2(\omega)} = \sum_{n=0}^{\infty} a_n \bar{b}_n \omega_n.$$

Indeed,

$$\begin{aligned}\langle T_n\delta_m, \delta_k \rangle_{\ell^2(\omega)} &= b(n, k, m)\, \omega_k = b(n, m, k)\, \omega_m \\ &= \langle \delta_m, T_n\delta_k \rangle_{\ell^2(\omega)}.\end{aligned}$$

If we show that T_n is bounded on $\ell^2(\mathbb{N}, \omega_n)$ then we conclude it is selfadjoint.

LEMMA 1. *T_n is a contraction on $\ell^p(\mathbb{N}, \omega_n)$, for every $p \geqslant 1$.*

Proof. First we show that T_n is a contraction on ℓ^∞. To this end observe that (6) and (11) imply

$$T_n\mathbf{1} = \mathbf{1},$$

where $\mathbf{1}$ denotes the constant unit sequence. Assume that $a = \{a_n\}$ is a sequence bounded by 1. Thus

$$\| T_n a \|_{\ell^\infty} \leqslant \| T_n \mathbf{1} \|_{\ell^\infty} = \| \mathbf{1} \| = 1.$$

The first inequality makes use of the positivity of the coefficients $b(n, m, k)$, which yields that the operators T_n are order preserving. Once we have made sure that T_n is a contraction on ℓ^∞ by symmetry it is such on $\ell^1(\mathbb{N}, \omega_n)$. Now by interpolation (which in this case reduces to applying Hölder's inequality twice) we can conclude that T_n are contractions on the intermediate spaces $\ell^p(\mathbb{N}, \omega_n)$, for $p \geqslant 1$.

We gave up the space $\ell^1(\mathbb{N})$ for the weighted space $\ell^1(\mathbb{N}, \omega_n)$. It is worthwhile transferring convolution operation $*$ to the new space to get an operation $\circ$. It can be done as follows.

Define the linear operator from ℓ^1 to $\ell^1(\omega)$ acting according to

$$\tilde{}: \delta_n \mapsto \tilde{\delta}_n = \omega_n^{-1} \delta_n.$$

The mapping is clearly an isometry from ℓ^1 to $\ell^1(\omega)$. Let us define the convolution $\circ$ on $\ell^1(\omega)$ by the rule

$$\tilde{\delta}_n \circ \tilde{\delta}_m = (\delta_n * \delta_m)^\sim. \tag{12}$$

Since $*$ is commutative and associative obviously the operation $\circ$ is such. Let us compute $\tilde{\delta}_n \circ \tilde{\delta}_m$ explicitly.

$$\begin{aligned}\tilde{\delta}_n \circ \tilde{\delta}_m &= (\delta_n * \delta_m)^\sim = \sum_{k=0}^{\infty} b(n, m, k)\, \tilde{\delta}_k \\ &= \sum_{k=0}^{\infty} b(n, m, k)\, \omega_k^{-1} \delta_k = \sum_{k=0}^{\infty} b(n, k, m)\, \omega_m^{-1} \delta_k \\ &= \omega_n^{-1} T_n \delta_m = T_n \tilde{\delta}_m. \end{aligned} \tag{13}$$

Thus the action of T_n on $\ell^2(\mathbb{N}, \omega_n)$ coincides with the $\circ$-convolution with $\tilde{\delta}_n$, i.e.,

$$T_n a = \tilde{\delta}_n \circ a, \qquad a = \{a_n\} \in \ell^2(\mathbb{N}, \omega_n). \tag{14}$$

Using (14) and the fact that the operation $\circ$ is commutative and associative we can conclude that the T_n commute with each other. Moreover by (13) and (14) we have the following expression for their product:

$$T_n T_m = \sum_{k=0}^{\infty} b(n, m, k)\, T_k. \tag{15}$$

The series on the right hand side of (15) is absolutely convergent with respect to the operator norm on $\ell^2(\mathbb{N}, \omega_n)$, as each T_n is a contraction and the coefficients are nonnegative and sum up to 1. Since T_n are contractions on $\ell^2(\mathbb{N}, \omega_n)$, we can show the following.

LEMMA 2.

$$\|a \circ b\|_{\ell^p(\omega_n)} \leqslant \|a\|_{\ell^1(\omega_n)} \|b\|_{\ell^p(\omega_n)}, \tag{16}$$

$$\|a \circ b\|_{\ell^\infty} \leqslant \|a\|_{\ell^p(\omega_n)} \|b\|_{\ell^q(\omega_n)}, \tag{17}$$

where $p \geqslant 1$, and $p^{-1} + q^{-1} = 1$.

Proof. Let $a = \sum_{n=0}^{\infty} a_n \tilde{\delta}_n$. Then

$$a \circ b = \sum_{n=0}^{\infty} a_n (\tilde{\delta}_n \circ b) = \sum_{n=0}^{\infty} a_n T_n b.$$

Thus by Lemma 1 we have

$$\|a\circ b\|_{\ell^p}\leqslant\sum_{n=0}^{\infty}|a_n|\,\|b\|_{\ell^p}=\|a\|_{\ell^1}\,\|b\|_{\ell^p}.$$

By the first part of the lemma we have in particular

$$\|a\circ b\|_{\ell^1}\leqslant\|a\|_{\ell^1(\omega_n)}\,\|b\|_{\ell^1},$$
$$\|a\circ b\|_{\ell^1}\leqslant\|a\|_{\ell^1}\,\|b\|_{\ell^1(\omega_n)}.$$

Then we get (17) by the multilinear interpolation theorem [2, Theorem 4.4.1, p. 96, and Theorem 5.1.2, p. 107].

We can state now the main result of the paper.

THEOREM 1. *Assume that a matrix $b(n, m, k)$ and a sequence of positive numbers ω_n satisfy the conditions* (6), (7), (9), *and* (10). *Assume also there is no nonzero sequence $a=\{a_n\}_{n=0}^{\infty}$ in $\ell^2(\mathbb{N}, \omega_n)$, such that*

$$\sum_{k=0}^{\infty} b(n, m, k)\, a_k=0, \qquad \textit{for every} \quad n, k\in\mathbb{N}. \tag{18}$$

Then there exist a locally compact topological space $\mathscr{X}$ and a Radon measure μ on $\mathscr{X}$ such that supp $\mu=\mathscr{X}$, *and a system of functions $t_n(x)$ in $L^2(\mathscr{X}, d\mu)$ satisfying the following conditions.*

(i) *$t_n(x)$ is continuous and vanishing at infinity.*

(ii) *The linear span of $t_n(x)$, $n=0, 1, 2, \ldots$, is dense in $C_0(\mathscr{X})$.*

(iii) *The $t_n(x)$ form a complete orthogonal system in $L^2(\mathscr{X}, d\mu)$.*

(iv) *$t_n(x)\, t_m(x)=\sum_{k=0}^{\infty} b(n, m, k)\, t_k(x)$, $n, m\in\mathbb{N}$.*

(v) *$|t_n(x)|\leqslant 1$, $x\in\mathscr{X}$, $n\in\mathbb{N}$.*

The space $\mathscr{X}$ and the system $\{t_n(x)\}_{n=0}^{\infty}$ are uniquely determined in the following sense. If there exist another locally compact space $\mathscr{Y}$, a Radon measure ν on it such that supp $\nu=\mathscr{Y}$, *and a system of functions $q_n(y)$ satisfying* (i), (ii), (iii), *and* (iv) (*with $\mathscr{X}$, μ, t_n replaced by $\mathscr{Y}$, ν, q_n respectively*), *then there is a homeomorphic mapping $h\colon\mathscr{X}\to\mathscr{Y}$ such that*

$$q_n(h(x))=t_n(x).$$

Proof. Let $\mathscr{A}$ be the norm closed algebra of operators acting on the Hilbert space $\ell^2(\mathbb{N}, \omega_n)$ generated by the operators T_n, $n\in\mathbb{N}$, defined by the formula (11) or (14). Since the T_n are selfadjoint on $\ell^2(\mathbb{N}, \omega_n)$ and commuting with each other $\mathscr{A}$ is a commutative C^*-algebra. Let $\mathscr{A}^1$

denote the C^*-algebra obtained from $\mathscr{A}$ by adjoining the identity operator I. In case I is already in $\mathscr{A}$ we have $\mathscr{A} = \mathscr{A}^1$. In general we have $\mathscr{A} \neq \mathscr{A}^1$, like in the example from the Introduction. We discuss this issue again at the end of the paper.

By the Gelfand–Naimark theorem [3, Theorem 7, p. 876] there is a locally compact space $\mathscr{X}$ such that $\mathscr{A}$ is isometrically isomorphic to $C_0(\mathscr{X})$ while $\mathscr{A}^1$ is isometrically isomorphic to $C_l(\mathscr{X})$, the set of all complex functions continuous on $\mathscr{X}$ and having limit at infinity. We use lowercase letters for elements in the function spaces $C_0(\mathscr{X})$ and $C_l(\mathscr{X})$, and capital letters for their corresponding operators in the algebras $\mathscr{A}$ and $\mathscr{A}^1$:

$$C_l(\mathscr{X}) \ni a(x) \leftrightarrow A \in \mathscr{A}_1.$$

Consider the linear functional on $C_l(\mathscr{X})$ given by

$$a(x) \mapsto \langle A\tilde{\delta}_k, \tilde{\delta}_l \rangle_{\ell^2(\omega)}. \tag{19}$$

Then by Riesz's representation theorem there exists a signed (when $k = l$ nonnegative) Borel measure $\mu_{k,l}$ with bounded variation and a real number $d(\tilde{\delta}_k, \tilde{\delta}_l)$, such that

$$\langle A\tilde{\delta}_k, \tilde{\delta}_l \rangle_{\ell^2(\omega)} = \int_{\mathscr{X}} a(x)\, d\mu_{k,l}(x) + d(\tilde{\delta}_k, \tilde{\delta}_l) \lim_{x \to \infty} a(x). \tag{20}$$

Let $t_n(x)$ be the function corresponding to the operator T_n. Since the algebra $\mathscr{A}$ is generated by the operators T_n the sequence of functions $t_0, t_1, t_2, \ldots$, is linearly dense in the function space $C_0(\mathscr{X})$. This shows (i) and (ii). By Lemma 1 the operators T_n are contractions on $\ell^2(\mathbb{N}, \omega_n)$, so the functions $t_n(x)$ are bounded by 1. This gives (v). By (15) we get (iv).

Now we construct a measure $d\mu(x)$ and show (iii). Let us substitute $A = BT_nT_m$ into (20). Then

$$\langle BT_nT_m\tilde{\delta}_k, \tilde{\delta}_l \rangle_{\ell^2(\omega)} = \int_{\mathscr{X}} b(x)\, t_n(x)\, t_m(x)\, d\mu_{k,l}(x), \tag{21}$$

as the $t_n(x)$ belong to $C_0(\mathscr{X})$.

On the other hand since all appearing operators commute, $T_n^* = T_n$ and $T_n\tilde{\delta}_m = T_m\tilde{\delta}_n$ (see (13)), we get

$$\begin{aligned}&\langle BT_nT_m\tilde{\delta}_k, \tilde{\delta}_l \rangle_{\ell^2(\omega)} \\ &\quad = \langle BT_kT_l\tilde{\delta}_n, \tilde{\delta}_m \rangle_{\ell^2(\omega)} = \int_{\mathscr{X}} b(x)\, t_k(x)\, t_l(x)\, d\mu_{n,m}(x).\end{aligned} \tag{22}$$

As $b(x)$ is an arbitrary function from $C_l(\mathscr{X})$, the formulas (21) and (22) imply

$$t_n(x)\, t_m(x)\, d\mu_{k,l}(x) = t_k(x)\, t_l(x)\, d\mu_{n,m}(x). \tag{23}$$

Similarly, substituting $A = BT_n$ to (20) we get

$$t_n(x)\, d\mu_{k,l}(x) = t_k(x)\, d\mu_{n,l}(x). \tag{24}$$

Let E_k be the set of zeros of the function $t_k(x)$. Then by (24) we have

$$\int_{E_k} t_n(x)\, d\mu_{k,k}(x) = \int_{E_k} t_k(x)\, d\mu_{n,k}(x) = 0,$$

for every n. Since the $t_n(x)$, $n = 0, 1, 2, ...,$ form a linearly dense subset of $C_0(\mathscr{X})$, we can conclude that

$$\mu_{k,k}(E_k) = 0,$$

for every natural k. We remind the reader that $d\mu_{k,k}(x)$ is an ordinary, i.e., nonnegative measure on $\mathscr{X}$. Moreover by the linear density of $t_n(x)$, $n \in \mathbb{N}$, the intersection of all sets E_k is empty. Set $n = m$ and $k = l$ in (23). Then

$$t_k^2(x)\, d\mu_{n,n}(x) = t_n^2(x)\, d\mu_{k,k}(x). \tag{25}$$

Let us define a nonnegative measure $d\mu(x)$ on $\mathscr{X}$ in the following way. Let E be a Borel subset of $\mathscr{X}$ such that $E \cap E_k = \emptyset$, for some index k. In this case let

$$\mu(E) = \mu_k(E) = \int_E t_k^{-2}(x)\, d\mu_{k,k}(x). \tag{26}$$

Clearly the formula (26) defines a nonnegative Radon measure μ_k on the complement E_k^c of the set E_k. We show that μ_k coincides with μ_l on the intersection $E_k^c \cap E_l^c$. Indeed, let E be a Borel set in $\mathscr{X}$ such that $E \cap E_k = \emptyset$ and $E \cap E_l = \emptyset$. Then by (25)

$$t_n^{-2}(x)\, d\mu_{n,n}(x) = t_k^{-2}(x)\, d\mu_{k,k}(x), \qquad x \in E,$$

so $\mu_k(E) = \mu_l(E)$. Hence the family of measures (E_k^c, μ_k) is compatible on $\bigcup_{k=0}^{\infty} E_k^c = \mathscr{X}$. Since the intersection of the E_k's is empty, the formula (26) defines a Radon measure μ on $\mathscr{X}$.

The formula (26) and the fact that $d\mu_k(x)$ vanishes on E_k imply

$$d\mu_{k,k}(x) = t_k^2(x)\, d\mu(x). \tag{27}$$

This gives that the $t_k(x)$ are square-integrable on $\mathscr{X}$ with respect to μ, and by (20) applied to $A = I$ we have

$$\omega_n^{-1} = \langle \tilde{\delta}_n, \tilde{\delta}_n \rangle_{\ell^2(\omega)} = \int_{\mathscr{X}} t_n^2(x)\, d\mu(x) + d(\tilde{\delta}_n, \tilde{\delta}_n).$$

LEMMA 3. $d\mu_{k,l}(x) = t_k(x)\, t_l(x)\, d\mu(x)$.

Proof. The formula can be shown by the polar identity and an extension of (27). However, we have chosen a different explanation.

Observe first that $d\mu_{k,l}(x)$ is absolutely continuous with respect to the measure $d\mu_{k,k}(x) + d\mu_{l,l}(x)$. Indeed, let $a(x)$ be an arbitrary nonnegative function from $C_0(\mathscr{X})$. Then the corresonding operator A in $\mathscr{A}$ is positive definite and

$$\begin{aligned}\left|\int_{\mathscr{X}} a(x)\, d\mu_{k,l}(x)\right| &= |\langle A\tilde{\delta}_k, \tilde{\delta}_l \rangle_{\ell^2(\omega)}| \leqslant \sqrt{\langle A\tilde{\delta}_k, \tilde{\delta}_k \rangle_{\ell^2(\omega)}}\,\sqrt{\langle A\tilde{\delta}_l, \tilde{\delta}_l \rangle_{\ell^2(\omega)}} \\ &= \left(\int_{\mathscr{X}} a(x)\, d\mu_{k,k}(x)\right)^{1/2} \left(\int_{\mathscr{X}} a(x)\, d\mu_{l,l}(x)\right)^{1/2} \\ &\leqslant \tfrac{1}{2} \int_{\mathscr{X}} a(x)(d\mu_{k,k}(x) + d\mu_{l,l}(x)).\end{aligned}$$

Let now $a(x)$ be a real function from $C_0(\mathscr{X})$, and let $a(x) = a_+(x) - a_-(x)$ be its Hahn decomposition as a difference of two nonnegative functions. Then $a_\pm(x) \in C_0(\mathscr{X})$ and

$$\begin{aligned}\left|\int_{\mathscr{X}} a(x)\, d\mu_{k,l}(x)\right| &\leqslant \left|\int_{\mathscr{X}} a_+(x)\, d\mu_{k,l}(x)\right| + \left|\int_{\mathscr{X}} a_-(x)\, d\mu_{k,l}(x)\right| \\ &\leqslant \int_{\mathscr{X}} a_+(x)(d\mu_{k,k}(x) + d\mu_{l,l}(x)) \\ &\quad + \int_{\mathscr{X}} a_-(x)(d\mu_{k,k}(x) + d\mu_{l,l}(x)) \\ &= \int_{\mathscr{X}} |a(x)|\, (d\mu_{k,k}(x) + d\mu_{l,l}(x)).\end{aligned}$$

Therefore $\mu_{k,l} \prec \mu_{k,k} + \mu_{l,l}$. This implies in particular that

$$d\mu_{k,l}(x) = 0, \qquad x \in E_k \cap E_l.$$

By (24) and (27) we have

$$t_l(x)\, d\mu_{k,l}(x) = t_k\, d\mu_{l,l}(x) = t_k(x)\, t_l^2(x)\, d\mu(x).$$

Similarly we get

$$t_k(x)\, d\mu_{k,l}(x) = t_l\, d\mu_{k,k}(x) = t_l(x)\, t_k^2(x)\, d\mu(x).$$

These formulas yield

$$d\mu_{k,l}(x) = t_k(x)\, t_l(x)\, d\mu(x), \qquad x \notin E_k \cap E_l.$$

Since both appearing measures vanish on $E_k \cap E_l$ the lemma follows.

Let us get back to the proof of Theorem 1. Using (20) with $A = I$

$$\omega_k^{-1} \delta_k^l = \langle \tilde{\delta}_k, \tilde{\delta}_l \rangle_{\ell^2(\omega)} = \int_{\mathcal{X}} d\mu_{k,l}(x) + d(\tilde{\delta}_k, \tilde{\delta}_l)$$

$$= \int_{\mathcal{X}} t_k(x)\, t_l(x)\, d\mu(x) + d(\tilde{\delta}_k, \tilde{\delta}_l). \tag{28}$$

Let $\mathcal{F}$ denote the subspace of $\ell^2(\omega_n)$ consisting of finite linear combinations of $\tilde{\delta}_n$, $n \in \mathbb{N}$. Let us also extend linearly the function $d(\tilde{\delta}_k, \tilde{\delta}_l)$ to a hermitian form on $\mathcal{F} \times \mathcal{F}$. If $F, G \in \mathcal{F}$, then by Lemma 3 and by (20) we have

$$\langle F, G \rangle_{\ell^2(\omega)} = \int_{\mathcal{X}} f(x)\, \overline{g(x)}\, d\mu(x) + d(F, G). \tag{29}$$

$d(F, F)$ is nonnegative and we also have

$$\langle F, F \rangle_{\ell^2(\omega)} \geqslant d(F, F) \geqslant 0.$$

Thus $d(F, G)$ is a bounded positive definite quadratic form on $\ell^2(\omega_n)$. As such it can be represented by a positive definite linear operator D, bounded on $\ell^2(\omega_n)$, as

$$d(F, G) = \langle DF, G \rangle_{\ell^2(\omega_n)}. \tag{30}$$

As $t_n(x)$ tends to zero at infinity we have

$$\langle T_n F, G \rangle_{\ell^2(\omega)} = \int_{\mathcal{X}} t_n(x) f(x)\, \overline{g(x)}\, d\mu(x) + d(F, G) \lim_{x \to \infty} t_n(x)$$

$$= \int_{\mathcal{X}} t_n(x) f(x)\, \overline{g(x)}\, d\mu(x). \tag{31}$$

On the other hand by (13), (14), and (iv) we have

$$\begin{aligned}\langle T_n\tilde{\delta}_k, \tilde{\delta}_l\rangle_{\ell^2(\omega)} &= \sum_{m=0}^{\infty} b(n, k, m)\langle\tilde{\delta}_m, \tilde{\delta}_l\rangle \\ &= \sum_{m=0}^{\infty} b(n, k, m)\int_{\mathscr{X}} t_m(x)\, t_l(x)\, d\mu(x) \\ &\quad + \sum_{m=0}^{\infty} b(n, k, m)\, d(\tilde{\delta}_m, \tilde{\delta}_l) \\ &= \int_{\mathscr{X}} t_n(x)\, t_k(x)\, t_l(x)\, d\mu(x) + d(T_n\tilde{\delta}_k, \tilde{\delta}_l). \end{aligned} \tag{32}$$

Actually we have to make sure that the change of the order of integration and infinite summation in the above calculations is justified. It suffices to show that the series

$$\sum_{m=0}^{\infty} b(n, k, m)\, t_m(x)$$

is convergent in L^2 mean. To this end substitute $F = G$ into (29) to get

$$\begin{aligned}\langle F, F\rangle_{\ell^2(\omega)} &= \int_{\mathscr{X}} \|f(x)\|^2\, d\mu(x) + d(F, F) \\ &\geqslant \int_{\mathscr{X}} \|f(x)\|^2\, d\mu(x). \end{aligned} \tag{33}$$

Now since the series

$$\sum_{m=0}^{\infty} b(n, k, m)\, \tilde{\delta}_m$$

is convergent in $\ell^2(\omega_n)$, by (33) the series $\sum_{m=0}^{\infty} b(n, k, m)\, t_m(x)$ is convergent in $L^2(\mathscr{X}, d\mu)$.

The fact that we could perform infinite summation under the d-sign follows from the boundedness of the hermitian form $d(F, G)$ on $\ell^2(\omega_n)$.

By linearity (32) implies

$$\langle T_nF, G\rangle_{\ell^2(\omega)} = \int_{\mathscr{X}} t_n(x) f(x)\, \overline{g(x)}\, d\mu(x) + d(T_nF, G). \tag{34}$$

Combining (31) and (34) gives

$$d(T_nF, G) = 0, \qquad n = 0, 1, 2, ...,$$

for all $F, G \in \ell^2(\omega_n)$. Using (30) we get

$$\langle DT_n F, G \rangle_{\ell^2(\omega_n)} = 0, \qquad n = 0, 1, 2, ..., \tag{35}$$

for all $F, G \in \ell^2(\omega_n)$. Thus $DT_n = 0$. Since both T_n and D are selfadjoint we get $T_n D = 0$, for every natural number n. By assumptions (18) the intersection of the kernels of the operators T_n is trivial. Therefore $D = 0$. Hence by (28) we obtain

$$\omega_k^{-1} \delta_k^l = \int_{\mathscr{X}} t_k(x)\, t_l(x)\, d\mu(x). \tag{36}$$

This shows (iv).

As for completeness, let $\mathscr{M}$ be the closed subspace of the Hilbert space $L^2(\mathscr{X}, d\mu(x))$ spanned by the functions $t_n(x)$. Since $t_n(x)\, t_m(x) = \sum b(n, m, k)\, t_k(x)$ the series being convergent in $L^2(\mathscr{X}, d\mu(x))$ (see the comments following (33)), the product $t_n(x)\, t_m(x)$ belongs to $\mathscr{M}$. As any function from $C_0(\mathscr{X})$ can be uniformly approximated by linear combinations of the functions $t_m(x)$, we get $t_n(x)\, C_0(\mathscr{X}) \subseteq \mathscr{M}$. Consequently as $\mathscr{M}$ is spanned by $t_n(x)$, $n \in \mathbb{N}$, we obtain that $\mathscr{M}$ is invariant for multiplication with functions from $C_0(\mathscr{X})$, i.e.,

$$C_0(\mathscr{X})\, \mathscr{M} \subseteq \mathscr{M}.$$

We are going to show that $\mathscr{M} = L^2(\mathscr{X}, d\mu(x))$. Let $f(x)$ be a square-integrable function orthogonal to $\mathscr{M}$. Fix a compact set $\mathscr{K} \subset \mathscr{X}$, and a real continuous function $g(x)$ vanishing outside $\mathscr{K}$. Since $g\mathscr{M} \subset \mathscr{M}$, the function $f(x)\, g(x)$ is orthogonal to $\mathscr{M}$. For the purpose of this proof let the subscript $\mathscr{K}$ denote the restriction of considered objects to the subset $\mathscr{K}$. Thus $f(x)\, g(x)$ is orthogonal to $t_{n,\mathscr{K}}(x)$, in the Hilbert space $L^2(\mathscr{K}, d\mu_{\mathscr{K}}(x))$. But since the measure $d\mu_{\mathscr{K}}(x)$ is finite, $f(x)\, g(x)$ is orthogonal to the closure of the linear span of $t_{n,\mathscr{K}}(x)$ with respect to uniform convergence topology. As the $t_n(x)$ form a linearly dense subset of $C_0(\mathscr{X})$, the functions $t_{n,\mathscr{K}}(x)$ do so of $C(\mathscr{K})$. Hence $f(x)\, g(x)$ is orthogonal to $C(\mathscr{K})$. But again the fact that $d\mu_{\mathscr{K}}(x)$ is finite implies that $C(\mathscr{K})$ is dense in $L(\mathscr{K}, d\mu_{\mathscr{K}}(x))$. Thus $f(x)\, g(x) = 0$. Since $g(x)$ was an arbitrary function with compact support we can conclude that $f(x) = 0$. This shows the completeness of the orthogonal system $\{t_n(x)\}_{n=0}^{\infty}$.

The only thing left to be proved is the uniqueness. Assume that there is a locally compact space $\mathscr{Y}$, a Radon measure $d\nu(y)$ on $\mathscr{Y}$ whose support is equal to $\mathscr{Y}$, and a complete orthogonal system of continuous and vanishing at infinity functions $q_n(y)$ satisfying

$$q_n(y)\, q_m(y) = \sum_{k=0}^{\infty} b(n, m, k)\, q_k(y). \tag{37}$$

LEMMA 4. *Let real-valued functions $q_n(y)$ be orthogonal in $L^2(\mathscr{Y}, d\nu)$. Let (6) and (7) be satisfied and*

$$q_n(y)\, q_m(y) = \sum_{k=0}^{\infty} b(n, m, k)\, q_k(y) \tag{38}$$

(the equality should be understood as equality in the Hilbert space $L^2(\mathscr{Y}, d\nu)$). Then for every n

$$|q_n(y)| \leqslant 1, \qquad \text{a.e. in } \mathscr{Y}.$$

Proof. Let $\mathscr{G}$ be a linear transformation from the space $\ell^2(\mathbb{N}, \omega)$ into the function space $L^2(\mathscr{Y}, d\nu)$, defined by the rule

$$\mathscr{G}: \tilde{\delta}_n \mapsto q_n(y). \tag{39}$$

Observe that $\mathscr{G}$ is a constant multiple of an isometry. In view of orthogonality relations it suffices to show that $\mathscr{G}$ preserves the length of $\tilde{\delta}_n$.

$$\|\mathscr{G}\tilde{\delta}_k\|_{L^2}^2 = \int_{\mathscr{Y}} q_k^2(y)\, d\nu(y),$$

$$\|\tilde{\delta}_k\|_{\ell^2}^2 = \omega_k^{-1}.$$

We show that

$$c \int_{\mathscr{Y}} q_k^2(y)\, d\nu(y) = \omega_k^{-1},$$

for some positive constant c. Multiplying (38) by $q_k(y)$ and integrating with respect to $d\nu(y)$ yields the quantity

$$b(n, m, k) \int_{\mathscr{Y}} q_k^2(y)\, d\nu(y)$$

is symmetric. By assumption (10) the quantity $b(n, m, k)\, \omega_k^{-1}$ is also symmetric. Playing around a little bit with this property we can show that in fact we have

$$c \int_{\mathscr{Y}} q_k^2(y)\, d\nu(y) = \omega_k^{-1},$$

with a positive coefficient c. Actually we can set $c = 1$, by changing the measure $d\nu(y)$. So adjusting $d\nu(y)$ if necessary we can get

$$\int_{\mathscr{Y}} q_k^2(y)\, d\nu(y) = \omega_k^{-1},$$

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which means $\mathscr{G}$ is an isometry. By (13) and (38) we have

$$\mathscr{G}(\tilde{\delta}_n \circ \tilde{\delta}_m) = q_n(y)\, q_m(y).$$

Again by (13) we get

$$\mathscr{G} T_n a = \mathscr{G}(\tilde{\delta}_n \circ a) = q_n(y)\, \mathscr{G}(a). \tag{40}$$

Using the fact that T_n is a contraction on $\ell^2(\mathbb{N}, \omega_n)$ we obtain

$$\| q_n(y)\, \mathscr{G}\, a \|_{L^2} = \| \mathscr{G} T_n a \|_{L^2} \leqslant \| a \|_{\ell^2(\omega_n)}.$$

Put $a = (\tilde{\delta}_n)^{\circ N}$. Then $\mathscr{G} a = q_n(y)^N$, and

$$\| q_n(y)^{N+1} \|_{L^2} = \| q_n(y)\, \mathscr{G}(\tilde{\delta}_n)^{\circ N} \|_{L^2} \leqslant \| q_n(y)^N \|_{L^2}.$$

This implies the integrals

$$\int_{\mathscr{Y}} (q_n(y))^{2N}\, d\nu(y)$$

are uniformly bounded. Therefore the functions $q_n(y)$ are bounded by 1 almost everywhere on the support of $d\nu(y)$. This completes the proof of the lemma.

Let M_{q_n} denote the linear operator acting on $L^2(\mathscr{Y}, d\nu)$ by multiplication with the function $q_n(y)$. By (40) we have

$$\mathscr{G} T_n = M_{q_n} \mathscr{G}. \tag{41}$$

Since $\mathscr{G}$ is an isometry and the functions $q_n(y)$ form a linearly dense subset of $C_0(\mathscr{Y})$, the correspondence

$$T_n \mapsto q_n(y)$$

extends linearly to an isometric isomorphism from the algebra of operators $\mathscr{A}$, generated by T_n, $n = 0, 1, 2, \ldots$, to the algebra $C_0(\mathscr{Y})$. As the functions $t_n(x)$ share the properties of $q_n(y)$ we have that the mapping

$$T_n \mapsto t_n(x)$$

also extends to an isometric isomorphism from $\mathscr{A}$ onto $C_0(\mathscr{X})$. Finally we can conclude that the correspondence

$$q_n(y) \mapsto t_n(x)$$

induces isometric isomorphism between $C_0(\mathscr{Y})$ and $C_0(\mathscr{X})$. By [3, Theorem 26, p. 278] there is a homeomorphic mapping $h:\mathscr{X}\to\mathscr{Y}$, such that

$$q_n(h(x)) = t_n(x), \qquad x\in\mathscr{X}. \tag{42}$$

By (42) the functions $t_n(x)$ are orthogonal on $\mathscr{X}$ also with respect to the measure $d\nu(h(x))$. Hence if the measures ν and μ are finite (which is not always the case) then $d\nu(h(x)) = d\mu(x)$. In general we don't have this property due for instance to nonuniqueness of solutions of the moment problem.

The proof of Theorem 1 is complete.

Remark 1. It is worthwhile observing that the condition that there is no nonzero sequence $a = \{a_n\}_{n=0}^{\infty}$ in $\ell^2(\mathbb{N}, \omega_n)$, such that

$$\sum_{k=0}^{\infty} b(n, m, k)\, a_k = 0, \qquad \text{for every} \quad n, m \in \mathbb{N}, \tag{43}$$

is also necessary for the existence of an orthogonal system satisfying the conclusions (iii) and (iv) of Theorem 1.

Indeed, assume that there is a sequence of orthogonal to each other functions $q_n(y)$ in a Hilbert space $L^2(\mathscr{Y}, d\mu)$ such that

$$q_n(y)\, q_m(y) = \sum_{k=0}^{+\infty} b(n, m, k)\, q_k(y).$$

Let us adopt the notations and the results of the last part of the proof of Theorem 1. Let $\mathscr{G}$ be the transform from $\ell^2(\mathbb{N}, \omega_n)$ to $L^2(\mathscr{Y}, d\mu)$ given by

$$\mathscr{G}: \tilde{\delta}_n \mapsto q_n(y).$$

Then $\mathscr{G}$ is an isometry (up to a constant multiple, cf. the last part of the proof of Theorem 1). Assume also that there is a sequence $a = \{a_n\}_{n=0}^{\infty}$ in $\ell^2(\mathbb{N}, \omega_n)$ such that (43) is satisfied. This implies that

$$T_n a = 0 \qquad \text{for every} \quad n = 0, 1, 2, \ldots .$$

Thus by (40) we have

$$0 = \mathscr{G}(T_n a) = q_n(y)\, \mathscr{G}(a).$$

This means that $\mathscr{G}(a)$ is orthogonal to all the functions $q_n(y)$. But since

$$a = \sum_{n=0}^{+\infty} a_n \tilde{\delta}_n$$

we have

$$\mathscr{G}(a) = \sum_{n=0}^{+\infty} a_n q_n(y).$$

Hence $a_n = 0$ for every $n = 0, 1, 2, \ldots$, i.e., $a = 0$.

Remark 2. Cubic matrices $b(n, m, k)$ satisfying (6), (7), (9), and (10) were studied by Haar [4] (see also [5, pp. 467–471]). However, he assumed that the identity operator I on $\ell^2(\mathbb{N}, \omega_n)$ can be represented as

$$\sum_{n=0} c_n \omega_n T_n = I, \qquad \sum_{n=0} |c_n|^2 \omega_n < \infty. \tag{44}$$

Based on the theorem of von Neumann on a family of commuting selfadjoint operators [6, pp. 401–404], Haar was able to construct a finite measure $d\mu(x)$ on $[-1, 1]$, and a sequence of orthogonal with respect to $d\mu(x)$ bounded measurable functions $t_n(x)$ satisfying $t_n(x)\ t_m(x) = \sum_{k=0}^{\infty} b(n, m, k)\ t_k(x)$.

The condition (44) was meant to replace the existence of the identity operator among the operators T_n. In examples arising in the theory of orthogonal polynomials we always have $T_0 = I$, see [7]. Roughly the condition (44) means that the identity operator is in the closure of the linear span of T_n with respect to the weak operator topology. It also implies that the function constantly 1 can be represented as $1 = \sum_{n=0}^{\infty} c_n \omega_n t_n(x)$, which in turn yields that 1 is square integrable with respect to $d\mu(x)$, i.e., the measure $d\mu(x)$ is finite. In view of the example from the Introduction we cannot afford the condition (44) in general.

Our condition (18) is much weaker than (44). Indeed, assume that (44) is satisfied and there is a sequence $a \in \ell^2(\omega_n)$ such that (18) holds. Therefore $T_n a = 0$, for every n. Let $c = \{c_n\}_{n=0}^{\infty}$. Then by (44), $c \in \ell^2(\omega_n)$ and by (17), $c \circ a \in \ell^{\infty}$. Moreover by (14)

$$c \circ a = \left(\sum_{n=0}^{\infty} c_n \omega_n \tilde{\delta}_n \right) \circ a = \sum_{n=0}^{\infty} c_n \omega_n T_n a = 0.$$

However, by (44)

$$c \circ a = \left(\sum_{n=0}^{\infty} c_n \omega_n T_n \right) a = Ia = a.$$

Hence $a = 0$.

Concerning the example from the Introduction, our assumption (18) is clearly satisfied, since the kernel of each T_n is trivial. For example, the operator T_0 corresponds via (31) to multiplication by e^{-x} in the Hilbert space $L^2(\mathbb{R}, x^{\alpha} e^x\, dx)$.

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