Convolution structures associated

with orthogonal polynomials

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Abstract. We study Banach algebras associated with orthogonal polynomials via the product formula. Sufficient conditions under which the spectrum of this algebra coincides with the support of the orthogonalizing measure are given. The results apply to the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha \geq \beta$ and $\alpha+\beta+1 \geq 0$.

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Introduction.

Let $\{P_n\}_{n\geq 0}$ be polynomials orthogonal with respect to a measure μ on the real line. It is well known that the non-negativity of the coefficients $c(n,m,k)$ in the product formula

$$P_nP_m = \sum c(n,m,k)P_k$$

gives rise to a convolution structure on $\ell^1(N)$ (see [3], [5], [6], [8], [9], [10]) which makes $\ell^1(N)$ the Banach algebra. At this point the study of the maximal ideal space $\mathcal{M}$ of this algebra seems appropriate especially because $\mathcal{M}$ can be easily identified with the set $\{z \in \mathbf{C} : |P_n(z)| \leq 1, n = 0,1,\ldots\}$ or $\{z \in \mathbf{C} : \sup_n |P_n(z)| < +\infty\}$. It is always the case that supp $\mu \subset \mathcal{M}$ (Theorem 1). Our aim is to find some reasonable conditions which give the opposite inclusion thus securing supp $\mu = \mathcal{M}$. This is done in Theorem 2.

In Chapter 2, Theorem 2 and its generalization (Proposition 2) are applied to derive a maximum value principle for P_n asserting that on any ellipse with foci at -1 and 1 the polynomial P_n attains its absolute maximal value at the right end of the major axis.

Applications to the Jacobi polynomials are also given.

In the appendix we separated two proposition concerning the unilatelar shift operator on $l^p(\mathbf{N})$. These result are well-known. The proofs are given for the sake of self-containedness.

The convolution structure.

Let the polynomials P_n, $n = 0,1,2,\ldots$, satisfy the recurrence formula

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}, \tag{1}$$

where $\alpha_n, \gamma_n > 0$ for $n = 1,2,\ldots, \gamma_0 > 0$ and $\alpha_0 = 0$. By the Favard theorem there exists a measure μ such that the P_n are orthogonal with respect to μ.

We assume throughout the paper that

$$\alpha_n + \beta_n + \gamma_n = 1 \qquad \text{for} \quad n = 0,1,2,\ldots. \tag{2}$$

The latter implies that the P_n are normalized at the point $x = 1$ i.e.

$$P_n(1) = 1 \qquad \text{for} \quad n = 0,1,2,\ldots. \tag{3}$$

Besides this normalization, our other blanket assumption which we will adhere to is that in the product formula

$$P_nP_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k \tag{4}$$

the coefficients $c(n,m,k)$ are non-negative. By (3) and (4) we get

$$\sum_{k=|n-m|}^{n+m} c(n,m,k) = 1\,. \tag{5}$$

We refer to [1], [5], [6], [9], [10] for sufficient conditions under which $c(n,m,k)$ are non-negative.

The formula (4) gives rise to a convolution structure on ℓ^1, the space of absolutely summable sequences. More precisely, if δ_n denotes the sequence which is zero except at the n-th coordinate which is 1 then

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)\delta_k\,. \tag{6}$$

Then $*$ can be extended linearly to all sequences. The positivity of the coefficients $c(n,m,k)$ and (5) together imply

$$\|a*b\|_1 \le \|a\|_1 \cdot \|b\|_1 \qquad a,b \in \ell^1\,. \tag{7}$$

Indeed:

$$\|a*b\|_1 = \Big\| \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \delta_n * \delta_m \Big\|_1 \le$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |a_n||b_m| \|\delta_n * \delta_m\|_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |a_n||b_m| = \|a\|_1 + \|b\|_1\,.$$

Therefore $(\ell^1, *)$ becomes a Banach algebra. The aim of this paper is to identify its maximal ideal space.

Let φ be a linear multiplicative functional on $(\ell^1,*)$. The algebra is generated by the single element δ_1 (cf. (6)), so φ is determined by $\varphi(\delta_1)$. Since P_1 is linear $\varphi(\delta_1) = P_1(z)$ for a complex number z. Next combining (4) and (6) leads to $\varphi(\delta_n) = P_n(z)$ for $n = 0,1,2,\ldots$. Now in a Banach algebra every multiplicative functional is continuous and its norm does not exceed 1. Hence the sequence $\{P_n(z)\}_{n\ge 0}$ is bounded; moreover $\sup_n |P_n(z)| \le 1$. Thus the maximal ideal space of $(\ell^1,*)$ can be identified with

$$\mathcal{M} = \{z \in \mathbf{C} : \sup_n |P_n(z)| < +\infty\} = \{z \in \mathbf{C} : \sup_n |P_n(z)| \le 1\}. \tag{8}$$

We intend to examine the relation between $\mathcal{M}$ and supp μ, the support of the orthogonalizing measure.

Let $\omega_n = (\int P_n^2 d\mu)^{-1}$ for $n = 0,1,2,\ldots$. Then the quantity

$$c(n,m,k)\omega_k^{-1} = \int P_n P_m P_k d\mu \tag{9}$$

is invariant under permutation of the variables n, m, k. In particular we have:

$$c(n,m,k)\omega_k^{-1} = c(n,k,m)\omega_m^{-1}\,. \tag{10}$$

Define linear operators T_n acting on sequences $\{a_k\}_{k\geq 0}$ by

$$(T_n a)(m) = \sum_k c(n,m,k)a_k\,. \tag{11}$$

We will examine the operators T_n on the weight spaces $\ell^p(\omega) = \{a = \{a_n\}_{n\geq 0} : \sum_{n=0}^{\infty} |a_n|^2\omega_n < +\infty\}$. We rewrite (11) as follows

$$(T_n a)(m) = \sum_k c(n,m,k)\omega_k^{-1} a_k \omega_k\,.$$

Therefore the matrix of T_n with respect to the weight ω is

$$t_n(m,k) = c(n,m,k)\omega_k^{-1}\,. \tag{12}$$

This matrix is symmetric so T_n is selfadjoint on $\ell^2(\omega)$.

Proposition 1. *The operators T_n, $n = 0, 1, 2, \ldots,$ are contractions on $\ell^p(\omega)$.*

Proof. Combining (5), (10) and (12) gives

$$\sum_k t_n(m,k)\omega_k = \sum_k c(n,m,k) = 1$$

$$\sum_m t_n(m,k)\omega_m = \sum_m c(n,k,m) = 1\,.$$

The conclusion follows now from the Schur theorem which is stated below.

Theorem (Schur, [7] Thm. 5.2, p. 22). *Let A be a linear operator acting on $\ell^p(\omega)$ by*

$$(Aa)(m) = \sum_{k=0}^{\infty} a(m,k)a_k\omega_k\,.$$

Assume

$$\sum_{k=0}^{\infty} |a(m,k)|\omega_k \leq 1$$

$$\sum_{m=0}^{\infty} |a(m,k)|\omega_m \leq 1\,.$$

Then A is a contraction on $\ell^p(\omega)$ for $1 \le p \le +\infty$.

Let us introduce a transform from the space of all sequences $\{a_n\}_{n\ge 0}$ into the space of functions on the real line by

$$\hat{}: a = \{a_n\}_{n\ge 0} \mapsto \sum_{n=0}^{\infty} a_n \omega_n P_n \,. \tag{13}$$

We will also write $\hat{a} = \sum_{n=0}^{\infty} a_n \omega_n P_n$. This transform is an isometry from $\ell^2(\omega)$ onto $L^2(\mu)$. Indeed, by the definition of ω we have

$$\begin{aligned} \|\{a_n\}\|^2_{\ell^2(\omega)} &= \sum_{n=0}^{\infty} |a_n|^2 \omega_n \\ &= \Big\| \sum_{n=0}^{\infty} a_n \omega_n P_n \Big\|^2_{L^2(\mu)} = \|\hat{a}\|^2_{L^2(\mu)} \,. \end{aligned}$$

Let $\widehat{T_n}$ denote the operator associated with T_n acting on the transforms of sequences. Then by (4) and (9)

$$\begin{aligned} \widehat{T_n a} &= \sum_k (Ta)(k) \omega_k P_k = \sum_k \sum_m c(n,k,m) a_m \omega_k P_k \\ &= \sum_m a_m \sum_k c(n,k,m) \omega_k P_k = \sum_m a_m \omega_m \sum_k c(n,m,k) P_k \\ &= \sum_m a_m \omega_m P_m P_n = P_n \cdot \hat{a} \,. \end{aligned} \tag{14}$$

We are now in a position to derive the following.

Theorem 1. *If $x \in$ supp μ then $|P_n(x)| \le 1$ for $n = 0, 1, 2, \ldots$.*

Proof. Let $a \in \ell^2(\omega)$. Then

$$\|T_n a\|_{\ell^2(\omega)} = \|\widehat{T_n a}\|_{L^2(\mu)} = \|P_n \hat{a}\|_{L^2(\mu)} \,.$$

On the other hand by the earlier Proposition the following holds

$$\|T_n a\|_{\ell^2(\omega)} \le \|a\|_{\ell^2(\omega)} = \|\hat{a}\|_{L^2(\mu)} \,.$$

Thus we have $\|P_n \hat{a}\|_{L^2(\mu)} \le \|\hat{a}\|_{L^2(\mu)}$, i.e. the linear operator $M_{P_n} : L^2(\mu) \to L^2(\mu)$ whose action is to multiply by P_n is a contraction. It is well known that $\|M_{P_n}\| = \sup\{|P_n(x)| : x \in \text{supp } \mu\}$. Therefore $\sup\{|P_n(x)| : x \in \text{supp } \mu\} \le 1$.

Consider the linear operator L acting on sequences as

$$(La)(n) = \gamma_n a_{n+1} + \beta_n a_n + \alpha_n a_{n-1} \,. \tag{15}$$

The coefficients γ_n, α_n and ω_n are interdependent. By (1) we have $\langle xP_n, P_{n+1}\rangle_{L^2(\mu)} = \gamma_n \omega_{n+1}^{-1}$. On the other hand $\langle xP_n, P_{n+1}\rangle_{L^2(\mu)} = \langle P_n, xP_n\rangle_{L^2(\mu)} = \alpha_{n+1}\omega_n^{-1}$. Therefore

$$\frac{\omega_{n+1}}{\omega_n} = \frac{\gamma_n}{\alpha_{n+1}}. \tag{16}$$

Using (1) and (15) gives

$$\widehat{La} = x\hat{a} \,. \tag{17}$$

Comparing (14) and (17) we conclude

$$T_n = P_n(L) \,. \tag{18}$$

The formula (17) has one important consequence. The spectrum of the operator L on $\ell^2(\omega)$ coincides with that of multiplication by x on $L^2(\mu)$. The latter is nothing other than supp μ.

Lemma. *Let $\beta_n \to 0$, $\alpha_n \to \frac{1}{2}$, $\gamma_n \to \frac{1}{2}$ as n tends to infinity. Then supp $\mu \subset [2\beta_0 - 1, 1]$ and $[-1, 1] \subset$ supp μ .*

Proof. $P_1 = \gamma_0^{-1}(x - \beta_0)$ by (1); thus $T_1 = \gamma_0^{-1}(L - \beta_0 I)$. Hence $L = \gamma_0 T_1 + \beta_0 I$. By the earlier Proposition T_1 is a contraction on $\ell^2(\omega)$ so $\sigma(T_1) \subset [-1, 1]$. Therefore, since $\gamma_0 + \beta_0 + \alpha_0 = 1$ and $\alpha_0 = 0$

$$\sigma(L) \subset [-\gamma_0 + \beta_0, \gamma_0 + \beta_0] = [-1 + 2\beta_0, 1].$$

This proves the first part of the conclusion.

The second part of the conclusion follows from the Blumenthal theorem (see [4], Ch.IV.4). For readers convenience we give a proof of it based on the Fredholm theory.

It is rather inconvenient to deal with operators acting on the weight space $\ell^2(\omega)$. We therefore find a similar operator acting on the usual $\ell^2(\mathbf{N})$ space and examine its spectrum.

Let $\tilde{L}$ be a linear operator acting on $\ell^2(\mathbf{N})$ as

$$(\tilde{L}a)(n) = \lambda_n a_{n+1} + \beta_n a_n + \lambda_{n-1} a_{n-1} \tag{19}$$

where $\lambda_n = (\alpha_{n+1}\gamma_n)^{1/2}$. Then $\tilde{L}$ is similar to L. The isometry Φ

$$\ell^2(\omega) \ni \delta_n \overset{\Phi}{\mapsto} \omega_n^{1/2}\delta_n \in \ell^2$$

intertwines L and $\tilde{L}$, i.e. $\Phi \circ L = \tilde{L} \circ \Phi$. Hence their spectra coincide. Observe that by the assumptions $\lambda_n = (\alpha_{n+1}\gamma_n)^{1/2} \to \frac{1}{2}$ as n tends to infinity.

Let U be the operator given by

$$(Ua)(n) = \frac{1}{2}a_{n+1} + \frac{1}{2}a_{n-1}\,. \tag{20}$$

It is well known that the spectrum of U on ℓ^2 coincides with $[-1, 1]$ (see Appendix). Moreover the difference

$$(\tilde{L} - U)a(n) = (\lambda_n - \frac{1}{2})a_{n+1} + \beta_n a_n + (\lambda_{n-1} - \frac{1}{2})a_{n-1}$$

is a compact operator because $\lambda_n \to \frac{1}{2}$ and $\beta_n \to 0$. Hence by the Weyl theorem the continuous spectra of $\tilde{L}$ and U coincide. Thus $[-1, 1] \subset \sigma_{\ell^2}(\tilde{L}) = \sigma_{\ell^2(\omega)}(L) = \text{supp } \mu$. Furthermore by this same theorem any number x in supp $\mu \backslash [-1, 1]$ is an eigenvalue of the operator L.

Now we can state the main result of the paper.

Theorem 2 *Let $\{P_n\}_{n \geq 0}$ be polynomials orthogonal with respect to a measure μ on the real line. Assume that (1), (2) hold and the linearization coefficients in (4) are non-negative. If $\alpha_n \to \frac{1}{2}, \gamma_n \to \frac{1}{2}$ and $\beta_n \to 0$ as $n \to +\infty$ then the following condition are equivalent for every $z \in \mathbf{C}$:*

(i) $\sup_{n \geq 0} |P_n(z)| < +\infty$,

(ii) $\sup_{n \geq 0} |P_n(z)| = 1$,

(iii) $z \in \text{supp } \mu$.

Proof. Let $\overline{L}$ be a linear operator acting on ℓ^1 by

$$\overline{L}\delta_n = \gamma_n \delta_{n+1} + \beta_n \delta_n + \alpha_n \delta_{n-1}. \tag{21}$$

By (1) and (6) we have

$$P_1(\overline{L})\delta_n = \delta_1 * \delta_n.$$

Thus the operator $\overline{L}$ belongs to the convolution algebra generated by δ_1, namely $\overline{L} = \gamma_0 \delta_1 + \beta_0 \delta_0$. Moreover, if φ is a multiplicative functional such that $\varphi(\delta_1) = P_1(z)$, then $\varphi(\overline{L}) = z$. This means that the set $\{z \in \mathbf{C} : |P_n(z)| \leq 1, n = 0, 1, 2, \cdots\}$ coincides with the spectrum of the operator $\overline{L}$ on the space ℓ^1. Let U be the operator acting on ℓ^1 defined by (20). Then $\sigma_{\ell^1}(U) = [-1, 1]$ (see Appendix). Again as in the proof of our Lemma the difference $\overline{L} - U$ is a compact operator on ℓ^1. By the Weyl theorem $\sigma_{\ell^1}(\overline{L}) = [-1, 1] \dot{\cup} D$, where D is a countable set consisting of the eigenvalues of $\overline{L}$. We already know that $[-1, 1]$ is contained in supp μ. It remains to show that $D \subset \text{supp } \mu$ as well.

Let $z \in D$. Then $\overline{L}a = za$ for a nonzero sequence $a \in \ell^1$. By (21)

$$(\overline{L}a)(n) = \alpha_{n+1}a_{n+1} + \beta_n a_n + \gamma_{n-1}a_{n-1} = za_n.$$

Applying (16) gives

$$\gamma_n(\omega_{n+1}^{-1}a_{n+1}) + \beta_n(\omega_n^{-1}a_n) + \alpha_n(\omega_{n-1}^{-1}a_{n-1}) = z(\omega_n^{-1}a_n).$$

The above formula can be rewritten as (cf (1))

$$\gamma_n b_{n+1} + \beta_n b_n + \alpha_n b_{n-1} = zb_n, \tag{22}$$

where $b_n = \omega_n^{-1}a_n$. Moreover the sequence $b = \{b_n\}_{n\geq 0}$ belongs to $\ell^1(\omega)$ since $a = \{a_n\}_{n\geq 0}$ is in ℓ^1. Furthermore, by virtue of (1) and (22) we have $b_n = P_n(z)b_0$. Since $z \in D$ the sequence $P_n(z)$ is bounded by 1. Thus $|b_n| \leq |b_0|$ and $\{b_n\}_{n\geq 0}$ is a bounded sequence from $\ell^1(\omega)$. This implies that $\{b_n\}_{n\geq 0}$ belongs also to $\ell^2(\omega)$. Now (22) is equivalent to $Lb = zb$ (see (15)), so z is an eigenvalue of the operator L with an eigenvector from $\ell^2(\omega)$. Thus $z \in \sigma_{\ell^2}(L) = \operatorname{supp}\mu$. This completes the proof of Theorem 2.

Example Consider the Jacobi polynomials $R_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$. They are orthogonal with respect to the measure $d\mu(x) = (1-x)_+^{\alpha}(1+x)_+^{\beta}dx$. When normalized at the point $x = 1$ they satisfy the recurrence formula (1) with

$$\begin{aligned}
\gamma_n &= \frac{2(n+\alpha+\beta+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},\\
\beta_n &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},\\
\alpha_n &= \frac{2n(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}.
\end{aligned} \tag{23}$$

We have $\alpha_n + \beta_n + \gamma_n = 1$ (as $R_n^{(\alpha,\beta)}$ are normalized at $x = 1$); also $\alpha_n \to \frac{1}{2}, \gamma_n \to \frac{1}{2}$ and $\beta_n \to 0$. If $\alpha \geq \beta$ and $\alpha + \beta + 1 \geq 0$ then by Gasper's theorem ([5], [6], [10]) the linearization coefficients are non-negative. Hence all the assumptions of Theorem 2 hold. Therefore the maximal ideal space of the convolution algebra associated with $R_n^{(\alpha,\beta)}$ can be identified with $[-1, 1]$. The multiplicative functionals on $(\ell^1, *)$ are then given by

$$\ell^1 \ni \{a_n\}_{n\geq 0} \mapsto \sum_{n=0}^{\infty} a_n R_n(x) \quad x \in [-1, 1].$$

Applying Gelfand's theorem gives the following.

Proposition 1. *Let* $\alpha \geq \beta > -1$ *and* $\alpha+\beta+1 \geq 0$. *If* $\sum_{n=0}^{\infty} |a_n| < +\infty$ *and* $\sum_{n=0}^{\infty} a_n R_n(x) \neq 0$ *for* $x \in [-1,1]$, *then* $(\sum_{n=0}^{\infty} a_n R_n)^{-1} = \sum_{n=0}^{\infty} b_n R_n$, *where* $\sum_{n=0}^{\infty} |b_n| < +\infty$.

A maximum value principle.

Let us examine the set $\{z \in \mathbf{C} : |P_n(z)| \leq 1, n = 0, 1, 2, \cdots\}$ with the same assumptions as in Theorem 2 except that we assume $\alpha_n \to \alpha, \gamma_n \to \gamma$, $\beta_n \to \beta$ as n tends to infinity and $\alpha \neq \gamma$. First of all analyzing the proof of Theorem 1 gives

$$\operatorname{supp} \mu = [-(\alpha\gamma)^{1/2}, (\alpha\gamma)^{1/2}] \cup D,$$

where D is a countable set contained in $(-\infty, 1]$ consisting of the eigenvalues of $\tilde{L}$ on the space ℓ^2. Next, if U is the linear operator on ℓ^1 defined by

$$U\delta_m = \gamma\delta_{n+1} + \alpha\delta_{n-1},$$

then $\overline{L} - U$ (see (21)) is a compact operator. Thus the continuous spectra of $\overline{L}$ and U coincide, the latter being the ellipse $E = \{z \in \mathbf{C} : |z - 2(\alpha\gamma)^{1/2}| + |z + 2(\alpha\gamma)^{1/2}| \leq 2\}$ (see Appendix). Hence $\sigma_{\ell^1}(\overline{L}) = E \cup D'$, where D' is a countable set consisting of the eigenvalues of $\overline{L}$ on ℓ^1. As in the proof of Theorem 2 we can show that $D' \subset \operatorname{supp} \mu$. In particular the following holds.

Proposition 2. *Let the polynomials* $\{P_n\}_{n\geq 0}$ *satisfy all the assumptions of Theorem 2 except that* $\alpha_n \to \alpha, \;\; \gamma_n \to \gamma, \;\; \beta_n \to 0$ *as* n *tends to infinity. Then for each* $n = 0, 1, 2, \cdots$ *the maximal absolute value of* P_n *on the ellipse*

$$E = \{z \in \mathbf{C} : |z - 2(\alpha\gamma)^{1/2}| + |z + 2(\alpha\gamma)^{1/2}| \leq 2\}$$

is attained at $z = 1$ *and is equal to* 1.

Proposition 2 implies a maximum value principle for orthogonal polynomials satisfying the assumptions of Theorem 2.

Suppose that $\{P_n\}_{n\geq 0}$ satisfy the assumptions of Theorem 2. Assume also that the sequences $\{\gamma_n\}$ and $\{\gamma_n - \alpha_n\}$ are decreasing. Fix a number $a > 1$. We are going to show that the sequence $\frac{P_{n+1}(a)}{P_n(a)}$ is increasing and converges to $a + \sqrt{a^2 - 1}$. Let $c_n = \frac{P_{n+1}(a)}{P_n(a)}$. Then

$$c_0 = P_1(a) = \frac{a - \beta_0}{\gamma_0} > \frac{1 - \beta_0}{\gamma_0} = 1.$$

Assume that $c_n \geq c_{n-1} > 1$. We will show $c_{n+1} \geq c_n$. For a contradiction suppose that $c_{n+1} < c_n$. Then by substituting a in (1) and dividing by $P_n(a)$ we obtain

$$a = \gamma_n c_n + \beta_n \frac{\alpha_n}{c_{n-1}} \geq \gamma_n c_n + \beta_n + \frac{\alpha_n}{c_n}$$

and

$$a = \gamma_{n+1} c_{n+1} + \beta_{n+1} + \frac{\alpha_{n+1}}{c_n} < \gamma_{n+1} c_n + \beta_{n+1} + \frac{\alpha_{n+1}}{c_n}.$$

Therefore

$$\gamma_{n+1} c_n + \beta_{n+1} + \frac{\alpha_{n+1}}{c_n} > \gamma_n c_n + \beta_n + \frac{\alpha_n}{c_n}.$$

Multiplying both sides by c_n and using (2) yields

$$(\gamma_{n+1} - \gamma_n)(c_n^2 - c_n) - (\alpha_{n+1} - \alpha_n)(c_n - 1) > 0$$

Since by assumption $c_n - 1 > 0$

$$(\gamma_{n+1} - \gamma_n)c_n - (\alpha_{n+1} - \alpha_n) > 0.$$

As $\{\gamma_n\}$ is decreasing and $c_n > 1$ we have

$$(\gamma_{n+1} - \alpha_{n+1}) = (\gamma_n - \alpha_n) - (\gamma_{n+1} - \gamma_n) - (\alpha_{n+1} - \alpha_n) \geq (\gamma_{n+1} - \gamma_n)c_n - (\alpha_{n+1} - \alpha_n) > 0.$$

This gives a contradiction since $\{\gamma_n - \alpha_n\}$ is decreasing. Hence the sequence $c_n = \frac{P_{n+1}(a)}{P_n(a)}$ must be increasing.

The formula

$$a = \gamma_n c_n + \beta_n + \frac{\alpha_n}{c_{n-1}}$$

implies that $\{c_n\}$ is bounded because $\gamma_n \searrow \frac{1}{2}$ and $\beta_n \to 0$. Thus $\{c_n\}$ converges to a limit $c > 1$. Taking the limits on the right hand side we obtain

$$a = \frac{1}{2}(c + \frac{1}{c}).$$

Hence $c = a + \sqrt{a^2 - 1}$.

Let us introduce the renormalized polynomials $P_n^{(a)}$ by the formula

$$P_n^{(a)}(x) = \frac{1}{P_n(a)} P_n(ax). \tag{24}$$

Then the $P_n^{(a)}$ satisfy the recurrence relation

$$xP_n^{(a)} = \gamma_n^{(a)} P_{n+1}^{(a)} + \beta_n^{(a)} P_n^{(a)} + \alpha_n^{(a)} P_{n-1}^{(x)},$$

where

$$\gamma_n^{(u)} = a^{-1}\gamma_n c_n, \;\; \beta_n^{(a)} = a^{-1}\beta_n, \;\; \alpha_n^{(a)} = a^{-1}\alpha_n c_{n-1}^{-1}.$$

Observe that by (24), $P_n^{(a)}(1) = 1$. Thus $\gamma_n^{(a)} + \beta_n^{(a)} + \alpha_n^{(a)} = 1$ for $n = 0, 1, 2, \cdots$. Hence $\{P_n^{(a)}\}$ satisfy all the assumptions of Proposition 2. In particular $\alpha_n^{(a)} \to \frac{a-\sqrt{a^2-1}}{2a}$ and $\gamma_n^{(a)} \to \frac{a+\sqrt{a^2-1}}{2a}$. By Proposition 2 the maximal absolute value of $P_n^{(a)}$ on the ellipse $\{z \in \mathbf{C} : |z - a^{-1}| + |z + a^{-1}| \le 2\}$ is 1 and it is attained at $z = 1$. Using (24) and rescaling $P_n^{(a)}$ yields that the maximal value of P_n on the ellipse $\{z \in \mathbf{C}|z - 1| + |z + 1| \le 2a\}$ is attained at $z = a$. Collecting all the above we have the following.

Theorem 3. *Let the orthogonal polynomials $\{P_n\}_{n\ge0}$ satisfy the assumptions of Theorem 2. Let the sequences $\{\gamma_n\}_{n\ge0}$ and $\{\gamma_n - \alpha_n\}_{n\ge0}$ be decreasing. Thus on any ellipse with the foci at -1 and 1 the maximum absolute value of $P_n, \;\; n = 0, 1, 2, \cdots,$ is attained on the right end of the major half-axis.*

Example. Consider again the Jacobi polynomials $R_n^{(\alpha,\beta)}$ normalized at $x = 1$. If $\alpha \ge \beta$ and $\alpha + \beta + 1 \ge 0$ then by (23) $\{\gamma_n\}$ is a decreasing sequence, while $\{\alpha_n\}$ is increasing. Thus the assumptions of Theorem 3 are fulfilled. So the conclusion of Theorem 3 holds in this case.

Remark. Let $\{T_n\}_{n\ge0}$ be the Tchebyshev polynomials of the first kind. They are the special case of the Jacobi polynomials ($\alpha = \beta = -\frac{1}{2}$). In particular the Tchebyshev polynomials satisfy the conclusion of Theorem 3. This can be verified directly using the formula $T_n(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^n + z^{-n})$ or by applying Theorem 3. Furthermore, if the polynomials P_n can be expressed as linear combinations of the $T_n s$ with non-negative coefficients then the $P_n s$ satisfy the maximum principle introduced in Theorem 3. In a forthcoming paper we will show that if the polynomials P_n satisfy $xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}, \;\; \alpha_n + \gamma_n \le 1, \;\; \alpha_n \le \frac{1}{2}$ and $\beta_n \le 0$ then the coefficients $a(n, m)$ in $P_n = \sum_{m=0}^{n} a(n, m)T_m$ are non-negative.

Appendix.

Proposition A. *Let U be a linear operator acting on $\ell^p(\mathbf{N}), \;\; 1 \le p \le \infty$, by*

$$U\delta_n = \frac{1}{2}(\delta_{n-1} + \delta_{n+1}).$$

Then the spectrum $\sigma(U)$ can be identified with the closed interval $[-1,1]$.

Proof. Let S be the unilateral shift defined by $S\delta_n = \delta_{n+1}$. Then S^* is given by $S^*\delta_n = \delta_{n-1}$ for $n \geq 1$ and $S^*\delta_0 = 0$. Moreover $U = \frac{1}{2}(S + S^*)$ and $S^*S = I$. Both S and S^* are contractions on ℓ^p, so their spectra are contained in the closed unit disc $\{z \in \mathbf{C} : |z| \leq 1\}$. On the other hand, any sequence $\{z^n\}_{n\geq 0}$ is an eigenvector for S^* corresponding to the eigenvalue z if $|z| < 1$. Thus $\sigma_{\ell^p}(S^*) = \{z \in \mathbf{C} : |z| \leq 1\}$.

Let α be complex number outside the interval $[= 1, 1]$. There exists a unique complex number z such that $|z| < 1$ and $\alpha = \frac{1}{2}(z + z^{-1})$. Therefore

$$\alpha I - U = \frac{1}{2}(z + z^{-1})I - \frac{1}{2}(S + S^*) = \frac{1}{2z}(I - zS^*)(I - zS). \qquad (A1)$$

The operator on the right hand side is invertible because the operator norms of zS and zS^* are strictly less than 1. This means that $\alpha I - U$ is an invertible operator. Hence α does not belong to the spectrum of U, i.e. $\sigma_{\ell^p}(U) \subset [-1, 1]$.

In order to complete the proof we will show that the interval $[-1,1]$ is contained in the spectrum of U. Let $\alpha \in [-1, 1]$. Then $\alpha = \frac{1}{2}(e^{-it} + e^{it})$ for some $t \in \mathbf{R}$. By (A1) it suffices to show that $I - e^{it}S^*$ is not surjective. First note that $I - e^{it}S^*$ is injective. Indeed, let $(I - e^{it}S^*)a = 0$ for $a \in \ell^p$. Then $a_n - e^{it}a_{n+1} = 0$ for $n = 0, 1, 2, \cdots$. This implies that $|a_{n+1}| = |a_n|$ for $n = 0, 1, 2, \cdots$. Thus $\{a_n\} \in \ell^p$ only if $a_n = 0$ for all n. So $I - e^{it}S^*$ is injective and non-invertible, as $\sigma_{\ell^p}(S^*) = \{z \in \mathbf{C} : |z| \leq 1\}$. Thus it cannot be surjective.

Proposition B. *Let U be a linear operator acting on $\ell^1(\mathbf{N})$ by*

$$U\delta_n = \alpha\delta_{n+1} + \gamma\delta_{n-1},$$

where $\alpha, \gamma > 0$, $\alpha \neq \gamma$ and $\alpha + \gamma = 1$. Then the spectrum $\sigma(U)$ coincides with the elipse E

$$E = \{z \in \mathbf{C} : |z - 2\sqrt{\alpha\gamma}| + |z + 2\sqrt{\alpha\gamma}| \leq 2\}.$$

Proof. We adopt the notation from the proof of Proposition A. Thus we have $U = \alpha S + \gamma S^*$. We consider the case $\gamma > \alpha$. The complementary case can be treated similarly. One can observe that the ellipse E is the holomorphic image of the annulus $\{w \in \mathbf{C} : \gamma^{-1} \leq |w| \leq (\alpha\gamma)^{-1/2}\}$ under the mapping $w \mapsto w^{-1} + \alpha\gamma w$, while the punctured disc $\{w \in \mathbf{C} : 0 < |w| < \gamma^{-1}\}$ is mapped onto $\mathbf{C}\backslash E$.

Let $0 \neq z \in \mathbf{C}$. Then $z = w^{-1} + \alpha\gamma w$ for some w satisfying $|w| \leq (\alpha\gamma)^{-1/2}$. We have

$$zI - U = (w^{-1} + \alpha\gamma w)I - (\alpha S + \gamma S^*) = w^{-1}(I - \gamma w S^*)(I - \alpha w S). \qquad (A2)$$

If $z \notin E$ then $|w| < \gamma^{-1}$. Thus $\|\gamma w S^*\| = \gamma|w| < 1$ and $\|\alpha w S\| = \alpha|w| < \alpha\gamma^{-1} < 1$. Therefore by $(A2)$ $zI - U$ is invertible, so $z \notin \sigma(U)$. In case that $z \in E$ we have $\gamma^{-1} \leq |w| \leq (\alpha\gamma)^{-1/2}$. Hence $\|\alpha w S\| = \alpha|w| < (\alpha\gamma^{-1})^{1/2} < 1$. Consequently $I - \alpha w S$ is invertible. But $I - \gamma w S^*$ is non-invertible as $(\gamma w)^{-1}$ is in the unit disc which coincides with $\sigma(S^*)$. Thus by (A2) $zI - U$ is non-invertible, i.e. $z \in \sigma(U)$. This completes the proof.

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