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Journal of Computational and Applied Mathematics 199 (2007) 89–94

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Nonnegative linearization for little q-Laguerre polynomials and Faber basis

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Received 30 November 2004; received in revised form 13 March 2005

Abstract

The support of the orthogonality measure of so-called little q-Laguerre polynomials $\{l_n(.;a|q)\}_{n=0}^{\infty}$, $0<q<1$, $0<a<q^{-1}$, is given by $S_q=\{1,q,q^2,\ldots\}\cup\{0\}$. Based on a method of Młotkowski and Szwarc we deduce a parameter set which admits nonnegative linearization. Moreover, we use this result to prove that little q-Laguerre polynomials constitute a so-called Faber basis in $C(S_q)$.

Keywords: Orthogonal polynomials; Nonnegative linearization; Little q-Laguerre polynomials; Faber basis

1. Introduction

Let S denote an infinite compact subset of $\mathbb{R}$. A sequence of functions $\{\varphi_n\}_{n=0}^{\infty}$ in $C(S)$, the set of real-valued continuous functions on S, is called a basis of $C(S)$ if every $f\in C(S)$ has a unique representation

$$f=\sum_{k=0}^{\infty}\lambda_k\varphi_k, \tag{1}$$

with coordinates λ_k. In 1914, Faber [5] proved that there is no basis in $C([a,b])$ which consists of algebraic polynomials $\{P_n\}_{n=0}^{\infty}$ with deg $P_n=n$. One advantage of such a basis, which we call a Faber basis of $C(S)$, is that the nth partial sums of a representation (1) are converging towards f with the same order of magnitude as the elements of best approximation in $\mathscr{P}_n$ do, where $\mathscr{P}_n$ denotes the set of real algebraic polynomials with degree less or equal n, see [11, 19, Theorem 19.1].

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[1] Partially supported by KBN (Poland) under grant 2 P03A 028 25 by Research Training Network "Harmonic Analysis and Related Problems" Contract HPRN-CT-2001-00273 and by DFG fellowship 436 POL 17/1/04.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.03.095

In [8,9] we have investigated the case $S = S_q$, where

$$S_q = \{1, q, q^2, \ldots\} \cup \{0\}, \tag{2}$$

$0 < q < 1$. Besides a so-called Lagrange basis the little q-Jacobi polynomials, which are orthogonal on S_q, have been proven to constitute a Faber basis in $C(S_q)$.

Orthogonal polynomial sequences $\{P_n\}_{n=0}^{\infty}$ with respect to a probability measure π on S are of special interest, because a representation (1) is based on the Fourier coefficients given by

$$\hat{f}(k) = \int_S f(x) P_k(x)\, \mathrm{d}\pi(x), \quad k \in \mathbb{N}_0, \tag{3}$$

of $f \in C(S)$.

The linearization coefficients $g(i, j, k)$ for a orthogonal polynomial sequence are defined by

$$P_i P_j = \sum_{k=0}^{\infty} g(i, j, k) P_k = \sum_{k=|i-j|}^{i+j} g(i, j, k) P_k, \quad i, j \in \mathbb{N}_0, \tag{4}$$

where $g(i, j, |i-j|), g(i, j, i+j) \neq 0$. The nonnegativity of the linearization coefficients has many useful consequences. For instance, it is sufficient for a special boundedness property. Namely, for $x_0 = \sup S$ or $x_0 = \inf S$ we have

$$\max_{x \in S} |P_n(x)| = P_n(x_0) \quad \text{for all } n \in \mathbb{N}_0, \tag{5}$$

see for instance [10, p. 166(17); 9].

Here, we use a recent result of Młotkowski and Szwarc to prove nonnegative linearization for a certain parameter set of so-called little q-Laguerre polynomials. Finally, we check that the resulting boundedness property also implies the sequence of little q-Laguerre polynomials constitutes a Faber basis. The given proof goes along the lines of the one given in [8], see also [9].

2. Little q-Laguerre polynomials and nonnegative linearization

For parameters $0 < q < 1$, $0 < a < q^{-1}$ the sequence $\{l_n(x; a|q)\}_{n=0}^{\infty}$ of little q-Laguerre polynomials is defined by the three term recurrence relation

$$-x l_n(x; a|q) = A_n l_{n+1}(x; a|q) - (A_n + C_n) l_n(x; a|q) + C_n l_{n-1}(x; a|q), \quad n \geqslant 0, \tag{6}$$

with

$$A_n = q^n (1 - a q^{n+1}), \tag{7}$$

$$C_n = a q^n (1 - q^n), \tag{8}$$

where $l_{-1}(x; a|q) = 0$ and $l_0(x; a|q) = 1$. They are normalized by $l_n(0; a|q) = 1$ and they fulfill the orthogonalization relation

$$\sum_{k=0}^{\infty} \frac{(aq)^k}{(q; q)_k} l_m(q^k; a|q) l_n(q^k; a|q) = \frac{(aq)^n (q; q)_n}{(aq; q)_\infty (aq; q)_n} \delta_{n,m}, \tag{9}$$

where $(c; q)_k = (1-c)(1-cq)\cdots(1-cq^{k-1})$ and $(c; q)_\infty = \prod_{k=0}^{\infty}(1-cq^k)$, see [6]. We use a criterion of Młotkowski and Szwarc to deduce a set of parameters which guarantees nonnegativity of the linearization coefficients $g(i, j, k)$. The criterion given in [7] fits especially for orthogonality measures supported by a sequence of numbers accumulating at one point. Let us recall this result.

Theorem 1 (*Młotkowski and Szwarc*). *Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of polynomials with $P_0 = 1$ and $P_{-1} = 0$ satisfying the three term recurrence relation*

$$xP_n(x) = \alpha_n P_{n+1} + \beta_n P_n(x) + \gamma_n P_{n-1}(x). \tag{10}$$

If the sequence $\{\beta_n\}_{n=0}^{\infty}$ is increasing and the sequence $\{v_n\}_{n=0}^{\infty}$ with

$$v_n = \frac{\alpha_n \gamma_{n+1}}{(\beta_{n+2} - \beta_{n+1})(\beta_{n+1} - \beta_n)} \tag{11}$$

is a chain sequence, then the linearization coefficients are nonnegative.

Note that a sequence $\{u_n\}_{n=0}^{\infty}$ is called a chain sequence if there exists a sequence of numbers $\{g_n\}_{n=0}^{\infty}$, $0 \leqslant g_n \leqslant 1$, satisfying $u_n = (1 - g_n)g_{n+1}$. We gain the following result.

Theorem 2. *If the parameters a and q with respect to the sequence of little q-Laguerre polynomials $\{l_n(.; a|q)\}_{n=0}^{\infty}$ satisfy*

$$\frac{4a}{(1-q)^2[1 + aq(2-q)]^2} \leqslant 1, \tag{12}$$

then the linearization coefficients are nonnegative.

Proof. For to apply the previous theorem we write (6) as

$$(1 - x)l_n(x; a|q) = \alpha_n l_{n+1}(x; a|q) + \beta_n l_n(x; a|q) + \gamma_n l_{n-1}(x; a|q) \tag{13}$$

with

$$\alpha_n = A_n, \tag{14}$$

$$\beta_n = (1 - A_n - C_n), \tag{15}$$

$$\gamma_n = C_n. \tag{16}$$

By the transformation $y = 1 - x$ we get

$$yP_n(y) = \alpha_n P_{n+1}(y) + \beta_n P_n(y) + \gamma_n P_{n-1}(y) \tag{17}$$

with $P_n(y) = l_n(1 - y; a|q)$. Such transformation does not influence the linearization coefficients. It is easy to check that a necessary and sufficient condition for $\{\beta_n\}_{n=0}^{\infty}$ to be an increasing sequence is

$$a < \frac{1}{q(2+q)}. \tag{18}$$

Since the constant sequence $\frac{1}{4}, \frac{1}{4}, \ldots$ is a chain sequence, by Wall's comparison test for chain sequences [3, Theorem 5.7] a sufficient condition for $\{v_n\}_{n=0}^{\infty}$ to be a chain sequence is $v_n \leqslant \frac{1}{4}$. A simple computation yields

$$v_n = \frac{a(1-q^n)(1-aq^{n+1})}{(1-q)^2\{1 + a[1 - q^n(1+q)^2]\}\{1 + a[1 - q^{n+1}(1+q)^2]\}}, \tag{19}$$

which implies

$$v_n \leqslant \frac{a}{(1-q)^2[1 + aq(2-q)]^2} \quad \text{for all } n \in \mathbb{N}_0. \tag{20}$$

Hence a sufficient condition for $\{v_n\}_{n=0}^{\infty}$ to be a chain sequence is

$$\frac{4a}{(1-q)^2[1 + aq(2-q)]^2} \leqslant 1. \tag{21}$$

It remains to prove that (21) implies (18), but there are elementary arguments. For instance, if $\frac{2}{3} \leqslant q < 1$ then (21) implies $a \leqslant \frac{1}{3}$ which yields (18). In case of $0 < q < \frac{2}{3}$ we get by (21) that $a \leqslant \frac{9}{16}$ and hence (18) is also fulfilled. □

We should mention that there are parameters q and a, which admit negative linearization coefficients. For instance $g(1,1,1)<0$ if and only if $A_1+C_1>A_0$, which is equivalent to

$$q(1-aq^2)+aq(1-q)>(1-aq). \tag{22}$$

The last inequality holds for a close to q^{-1}.

Problem. Determine the range of parameters q and a for little q-Laguerre polynomials, for which nonnegative product linearization holds.

Before we take advantage of our result to prove an approximation theoretic consequence let us make a remark on combinatorics and special functions. Even and Gillis [4] gave the quantity

$$(-1)^{n_1+\cdots+n_k}\int_0^\infty \mathrm{e}^{-x}\prod_{i=1}^{k} L_{n_i}^{(0)}(x)\,\mathrm{d}x, \tag{23}$$

where $L_n^{(\alpha)}$, $\alpha>-1$, denote the classical Laguerre polynomials, a combinatorial interpretation. Namely (23) is the number of possible derangements of a sequence composed of n_1 objects of type 1, n_2 objects of type 2, …, n_k objects of type k. In such a way they have shown the nonnegativity of (23) and as a simple consequence they have proven the nonnegativity of the linearization coefficients of $\{(-1)^n L_n^{(0)}\}_{n=0}^\infty$. This property was reproved by Askey and Ismail [2] using more analytical methods for $\alpha>-1$. They also gave a combinatorial interpretation of

$$\frac{(-1)^{n_1+\cdots+n_k}}{\Gamma(\alpha+1)}\int_0^\infty \mathrm{e}^{-x}x^\alpha\prod_{i=1}^{k} L_{n_i}^{(\alpha)}(x)\,\mathrm{d}x, \tag{24}$$

in case of $\alpha=0,1,2,\ldots$. Our result concerning the q-analogues of classical Laguerre polynomials is achieved only by means of analytical methods and is without any combinatorial interpretation until now. So it would be of interest if there is a connection with combinatorics, too. The reader is invited to check our results also from this point of view.

3. Little q-Laguerre polynomials and Faber basis

Now we use the fact that nonnegative linearization yields the boundedness property (5) for to prove that certain little q-Laguerre polynomials constitute a Faber basis in $C(S_q)$.

Theorem 3. *If the parameters a and q with respect to the sequence of little q-Laguerre polynomials* $\{l_n(.;a|q)\}_{n=0}^\infty$ *satisfy*

$$\frac{4a}{(1-q)^2[1+aq(2-q)]^2}\leqslant 1, \tag{25}$$

then $\{l_n(.;a|q)\}_{n=0}^\infty$ *constitutes a Faber basis in* $C(S_q)$.

Proof. Let π denote the orthogonality measure. We have

$$\pi(\{q^k\})=\frac{(aq)^k}{(q;q)_k}=\frac{(aq)^k}{(1-q)(1-q^2)\cdots(1-q^k)},\quad k\in\mathbb{N}_0, \tag{26}$$

and $\pi(\{0\})=0$. The corresponding orthonormal polynomials are given by

$$p_n(.;a|q)=\sqrt{\frac{(aq;q)_\infty(aq;q)_n}{(aq)^n(q;q)_n}}\;l_n(.;a/q). \tag{27}$$

Let $K_n(x, y)$ denote the kernel

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x; a|q) p_k(y; a|q). \tag{28}$$

For proving that the sequence $\{l_n(.; a|q)\}_{n=0}^{\infty}$ constitutes a Faber basis in $C(S_q)$ it is necessary and sufficient to show

$$\sup_{x \in S_q} \int_{S_q} |K_n(x, y)| \, \mathrm{d}\pi(y) \leqslant C \quad \text{for all } n \in \mathbb{N}_0, \tag{29}$$

see for instance [9]. For this purpose we split the integration domain into two parts $[0, q^n]$ and $[q^n, 1]$. Using $\max_{x \in S_q} |p_n(x; a|q)| = p_n(0; a|q)$ we deduce

$$\int_0^{q^n} |K_n(x, y)| \, \mathrm{d}\pi(y) \leqslant K_n(0, 0)\pi([0, q^n]) = \mathcal{O}((aq)^{-n})\mathcal{O}((aq)^n) = \mathcal{O}(1). \tag{30}$$

For investigating the second part we use in case of $x \neq y$ the Christoffel–Darboux formula

$$K_n(x, y) = \sqrt{A_n C_{n+1}} \frac{p_{n+1}(x; a|q) p_n(y; a|q) + p_n(x; a|q) p_{n+1}(y; a|q)}{x - y} \tag{31}$$

and $|x - y| \geqslant (1 - q)y$ for to get

$$\begin{aligned} \int_{q^n}^1 |K_n(x, y)| \, \mathrm{d}\pi(y) \leqslant & \frac{\sqrt{A_n C_{n+1}} p_{n+1}(0; a|q)}{1 - q} \int_{q^n}^1 \frac{|p_n(y; a|q)|}{y} \, \mathrm{d}\pi(y) \\ & + \frac{\sqrt{A_n C_{n+1}} p_n(0; a|q)}{1 - q} \int_{q^n}^1 \frac{|p_{n+1}(y; a|q)|}{y} \, \mathrm{d}\pi(y) + \sum_{k=0}^{n} p_k(x; a|q)^2 \pi(\{x\}). \end{aligned} \tag{32}$$

First, note that

$$\sum_{k=0}^{n} p_k(x; a|q)^2 \pi(\{x\}) \leqslant 1, \tag{33}$$

see [1, Theorem 2.5.3, p. 63]. Next, we compute

$$A_n C_{n+1} = q^n (1 - aq^{n+1}) aq^{n+1} (1 - q^{n+1}) = \mathcal{O}(q^{2n}) \tag{34}$$

and

$$p_n(0; a|q) = \mathcal{O}((aq)^{-n/2}). \tag{35}$$

By Cauchy–Schwarz inequality we get

$$\begin{aligned} \int_{q_n}^1 \frac{|p_n(y; a|q)|}{y} \, \mathrm{d}\pi(y) & \leqslant \left(\int_{q_n}^1 \frac{1}{y^2} \, \mathrm{d}\pi(y) \right)^{1/2} = \left(\sum_{k=0}^{n} \frac{(aq)^k}{(q; q)_k q^{2k}} \right)^{1/2} \\ & \leqslant \left(\frac{1}{(q; q)_\infty} \sum_{k=0}^{n} \left(\frac{a}{q} \right)^k \right)^{1/2} = \mathcal{O}\left(\left(\frac{a}{q} \right)^{n/2} \right), \end{aligned} \tag{36}$$

which completes the proof. □

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