Absolute continuity of spectral measure for certain unbounded Jacobi matrices

Ryszard Szwarc*

Abstract

Spectral properties of unbounded symmetric Jacobi matrices are studied. Under mild assumptions on the coefficients absolute continuity of spectral measure is proved. Only operator theoretic proofs are provided. Some open problems of Ifantis are solved.

1 Introduction

Let J be a Jacobi matrix of the form

$$J = \begin{pmatrix} \beta_0 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & \beta_1 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & \beta_2 & \lambda_3 & \ddots \\ 0 & 0 & \lambda_3 & \beta_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

where $\lambda_n > 0$, for $n \geq 1$, and $\beta_n \in \mathbf{R}$, for $n \geq 0$. The matrix J gives rise to a symmetric operator on the Hilbert space $\ell^2(\mathbf{N})$ of square summable complex sequences $a = \{a_n\}_{n=0}^{\infty}$, with the domain $D(J)$ consisting of sequences with finitely many nonzero terms. This operator acts by the rule

$$(Ja)_n = \lambda_{n+1}a_{n+1} + \beta_n a_n + \lambda_n a_{n-1},$$

for $n \geq 0$, with the convention that $a_{-1} = \lambda_0 = 0$. It is well known that this operator admits selfadjoint extensions (see [1]). In case the extension is unique the operator is called essentially selfadjoint. Then there exists a unique probability measure μ on $\mathbf{R}$, with finite moments, such that

$$(J^n\delta_0, \delta_0)_{\ell^2(\mathbf{N})} = \int_{\mathbf{R}} x^n \, d\mu(x),$$

*This work was partially supported by KBN (Poland) under grant 5 P03A 034 20.

where $\delta_0 = (1, 0, 0, \ldots)$. This measure is called the spectral measure of the operator J, because it can be shown that the operator J is unitarily equivalent to the operator M_x acting on $L^2(\mathbf{R}, \mu)$ by the rule

$$M_x f(x) = x f(x).$$

This unitary equivalence is defined as follows. Let $p_n(x)$ be a system of polynomials orthonormal with respect to the inner product in $L^2(\mathbf{R}, \mu)$. Then the operator $U\delta_n = p_n$ extends to an isometry from $\ell^2(\mathbf{N})$ onto $L^2(\mathbf{R}, \mu)$, where δ_n denotes the sequence whose nth term is equal to 1, and all other terms are equal to 0. Since

$$J\delta_n = \lambda_{n+1}\delta_{n+1} + \beta_n\delta_n + \lambda_{n-1}\delta_{n-1},$$

we have

$$xp_n = \lambda_{n+1}p_{n+1} + \beta_n p_n + \lambda_{n-1}p_{n-1}.$$

In this paper we will be dealing with special unbounded Jacobi matrices such that $\lambda_n \to +\infty$ and

$$\frac{\lambda_n^2}{\beta_n\beta_{n-1}} \xrightarrow{n} \alpha.$$

It is known that if J is essentially selfadjoint and $\alpha < \frac{1}{4}$ the measure μ is discrete (see [2]). In [5] Ifantis stated a problem of studying the spectra of operators for which $\alpha > \frac{1}{4}$. In this note we are going to show that the spectra of such operators cover the whole real line and, under some mild conditions on the coefficients, the spectral measure is absolutely continuous. We will also provide an operator theoretic proof for the case $\alpha < \frac{1}{4}$, which was also one of the problems stated by Ifantis.

2 Main results

Our considerations will rely heavily on the following generalization of a result of Maté and Nevai. We will state it in a form which will be useful for our considerations. We will also provide a proof different from the one in [8], and based on ideas from [4].

Theorem 2.1 (Maté, Nevai) *Let $\Lambda_n(x)$ be a positive valued sequence whose terms depend continuously on $x \in [a, b]$. Let $a_n(x)$ be a real valued sequence of continuous functions satisfying*

$$\Lambda_{n+1}(x)a_{n+1}(x) + Ba_n(x) + \Lambda_n(x)a_{n-1}(x) = 0,$$

for $n \geq N$. Assume the sequence $\Lambda_n(x)$ has bounded variation and $\Lambda_n(x) \to \frac{1}{2}$ for $x \in [a, b]$. Let $|B| < 1$. Then there is a positive function $f(x)$ continuous on $[a, b]$ such that

$$a_n^2(x) - a_{n-1}(x)a_{n+1}(x) \xrightarrow{n} f(x)$$

uniformly for $x \in [a, b]$. *Moreover there is a constant* c *such that*

$$|a_n(x)| \le c$$

for $n \ge 0$ *and* $x \in [a, b]$.

Proof. Let

$$\Delta_n(x) = a_n^2(x) - a_{n-1}(x)a_{n+1}(x),$$

for $n \ge N$. By using the recurrence relation one can show that

$$\Delta_{n+1} - \Delta_n = \left(1 - \frac{\Lambda_n}{\Lambda_{n-1}}\right) a_{n+1}^2 + \left(1 - \frac{\Lambda_n}{\Lambda_{n+1}}\right) a_n^2 + B\left(\frac{1}{\Lambda_{n+1}} - \frac{1}{\Lambda_{n-1}}\right) a_n a_{n+1}.$$

Hence

$$|\Delta_{n+1} - \Delta_n| \le c(|\Lambda_{n-1} - \Lambda_n| + |\Lambda_n - \Lambda_{n+1}|)(a_n^2 + a_{n+1}^2). \tag{2}$$

On the other hand

$$\begin{aligned} \Delta_n &= a_n^2 + \frac{\Lambda_n}{\Lambda_{n-1}} a_{n+1}^2 + \frac{B}{\Lambda_{n-1}} a_n a_{n+1} \\ &= \left(a_n + \frac{B}{2\Lambda_{n-1}}\right)^2 + \left(\frac{\Lambda_n}{\Lambda_{n-1}} - \frac{B^2}{4\Lambda_{n-1}^2}\right) a_{n+1}^2 \\ &= \frac{\Lambda_n}{\Lambda_{n-1}}\left(a_{n+1} + \frac{B}{2\Lambda_n}\right)^2 + \left(1 - \frac{B^2}{4\Lambda_{n-1}\Lambda_n}\right) a_n^2. \end{aligned}$$

Since $\Lambda_n \stackrel{n}{\to} \frac{1}{2}$, uniformly for $x \in [a, b]$, and $|B| < 1$ we have

$$a_n^2 + a_{n+1}^2 \le 2c'\Delta_n, \qquad \text{where} \quad (c')^{-1} = \frac{1}{2} - \frac{B^2}{2}, \tag{3}$$

for n sufficiently large. Combining this with (2) gives

$$|\Delta_{n+1} - \Delta_n| \le 2cc'(|\Lambda_{n-1} - \Lambda_n| + |\Lambda_n - \Lambda_{n+1}|)\Delta_n.$$

Let

$$\varepsilon_n = 2cc'(|\Lambda_{n-1} - \Lambda_n| + |\Lambda_n - \Lambda_{n+1}|).$$

Then

$$(1 - \varepsilon_n)\Delta_n \le \Delta_{n+1} \le (1 + \varepsilon_n)\Delta_n,$$

for n sufficiently large. Thus the sequence Δ_n is convergent uniformly to a positive function $f(x)$ for $x \in [a, b]$. Moreover by (3) we obtain the second part of the statement. ∎

The main result of this note is following.

Theorem 2.2 *Assume the sequences* λ_n *and* β_n *satisfy* $\lambda_n \to +\infty$, $|\beta_n| \xrightarrow{n} +\infty$, $\beta_n/\beta_{n-1} \xrightarrow{n} 1$ *and*

$$\frac{\lambda_n^2}{\beta_{n-1}\beta_n} \xrightarrow{n} \frac{1}{4B^2} > \frac{1}{4}.$$

Let the sequences

$$\frac{\lambda_n^2}{\beta_{n-1}\beta_n}, \quad \frac{\beta_{n-1}+\beta_n}{\lambda_n^2}, \quad \frac{1}{\lambda_n^2}$$

have bounded variation. Then the corresponding Jacobi matrix J *is essentially selfadjoint if and only if* $\sum \lambda_n^{-1} = \infty$. *In that case the spectrum of* J *coincides with the whole real line and the spectral measure is absolutely continuous.*

Proof. We may assume that $\beta_n \xrightarrow{n} +\infty$. Assume that J is essentially selfadjoint. Let μ denote the spectral measure of J. Fix a real number x. Consider the difference equation

$$xy_n = \lambda_{n+1}y_{n+1} + \beta_n y_n + \lambda_{n-1}y_{n-1}, \tag{4}$$

for $n \geq 1$. By [7] the measure μ is absolutely continuous on the set of those x for which the ratio

$$\frac{\sum_{k=1}^{n} |u_k|^2}{\sum_{k=1}^{n} |v_k|^2} \tag{5}$$

remains bounded above for any n, for any fixed solutions u_n and v_n of (4). We are going to show that this ratio is always bounded. Let a_n satisfy (4). Let N be large enough so that $\beta_n > x$ for $n \geq N$. Set

$$a_n(x) = y_n\sqrt{\beta_n - x}, \quad \text{for } n \geq N. \tag{6}$$

The equation (4) can be transformed into the following.

$$\Lambda_{n+1}(x)a_{n+1}(x) + Ba_n(x) + \Lambda_n a_{n-1}(x) = 0, \tag{7}$$

for $n \geq N$, where

$$\Lambda_n(x) = B\frac{\lambda_n}{(\beta_{n-1}-x)(\beta_n - x)}. \tag{8}$$

By assumptions we have $\Lambda_n \xrightarrow{n} \frac{1}{2}$ and $|B| < 1$. Moreover $\Lambda_n(x)$ has bounded variation if and only if $\Lambda_n^{-2}(x)$ has bounded variation. But

$$\Lambda_n^{-2}(x) = \frac{\beta_{n-1}\beta_n}{\lambda_n^2} - \frac{\beta_{n-1}+\beta_n}{\lambda_n^2}\,x + \frac{1}{\lambda_n^2}\,x^2.$$

Theorem 2.1 implies

$$a_n^2(x) - a_{n-1}(x)a_{n+1}(x) \xrightarrow{n} C > 0$$

and $a_n(x)$ is a bounded sequence. Using (6), the boundedness of $a_n(x)$ and the assumptions on β_n we obtain that

$$\beta_n(y_n^2 - y_{n-1}y_{n+1}) \xrightarrow{n} C$$

and $\beta_n y_n^2$ is bounded. Therefore there exist positive constants c and M such that

$$\begin{array}{rcl} \beta_n y_n^2 & \leq & c \\ \beta_n(y_n^2 - y_{n-1}y_{n+1}) & \geq & c^{-1} \end{array}$$

for $n \geq M$. If J is essentially selfadjoint there exists a solution y_n of (4) which is not square summable. Thus $\sum \beta_n^{-1} = +\infty$. Hence $\sum \lambda_n^{-1} = +\infty$.

We have

$$c \leq \beta_n(y_n^2 - y_{n-1}y_{n+1}) \leq \beta_n(y_{n-1}^2 + y_n^2 + y_{n+1}^2) \leq c'.$$

for $n \geq M$. Now if u_n and v_n are arbitrary nonzero solutions of (4) we have

$$\frac{u_{n-1}^2 + u_n^2 + u_{n+1}^2}{v_{n-1}^2 + v_n^2 + v_{n+1}^2} \leq \frac{c'}{c}.$$

This implies the ratio in (5) is bounded. ■

Remark 2.3 Let p_n be the polynomials satisfying

$$xp_n = \lambda_{n+1}p_{n+1} + \beta_n p_n + \lambda_n p_{n-1}.$$

By the proof of Theorem 2.2 we get that

$$\beta_n[p_n^2(x) - p_{n-1}(x)p_{n+1}(x)] \xrightarrow{n} f(x) > 0,$$

uniformly on any bounded interval. and

$$\beta_n p_n^2(x) \leq c$$

on any bounded interval. In the case of bounded λ_n and β_n Maté and Nevai showed that the limit $f(x) = \lim_n[p_n^2(x) - p_{n-1}(x)p_{n+1}(x)]$ is closely related with the density of the spectral measure of J, which coincides with the orthogonality measure for the polynomials p_n. Namely they showed that if $\lambda_n \xrightarrow{n} 1/2$ and $\beta_n \xrightarrow{n} 0$ then the orthogonality measure μ is absolutely continuous in the interval $(-1, 1)$ and its density is given by

$$\frac{2\sqrt{1-x^2}}{\pi f(x)}, \quad -1 < x < 1.$$

Remark 2.4 Similar result has been obtained recently by Janas and Moszyski [6] under stronger assumptions that the sequences

$$\frac{\lambda_{n-1}}{\lambda_n}, \quad \frac{1}{\lambda_n}, \quad \frac{\beta_{n-1}}{\lambda_n}$$

have all bounded variation. It can be verified easily that these assumptions imply the assumptions of Theorem 2.2. Moreover there are examples showing that our assumptions are actually weaker. Indeed, let

$$\beta_n = n + 1 + (-1)^n, \qquad \lambda_n = \sqrt{\beta_{n-1}\beta_n}.$$

One can verify that β_n/λ_n does not have bounded variation while the assumptions of Theorem 2.2 are satisfied.

Example 2.5 Let $\lambda_n = n^\kappa$ and $\beta_n = \beta n^\kappa$, where $|\beta| < 1$ and $0 < \kappa \leq 1$ (see [5])). By the Carleman criterion the corresponding Jacobi matrix is essentially selfadjoint. Moreover all the assumptions of Theorem 2.2 are satisfied. Hence the spectrum of J cover the whole real line and the spectral measure is absolutely continuous. Also we have that the corresponding orthonormal polynomials satisfy

$$\begin{aligned} &n^\kappa \left[p_n^2(x) - p_{n-1}(x)p_{n+1}(x)\right] \to f(x) > 0, \\ &n^\kappa |p_n(x)| \leq c, \end{aligned}$$

uniformly with respect to x from any bounded interval $[a, b]$.

The next theorem is known (see [3]). We give an operator theoretic proof. Finding such a proof was one of the open problems stated in [5].

Theorem 2.6 (Chihara) *Let J be a Jacobi matrix given by (1) and satisfying*

$$\frac{\lambda_n^2}{\beta_{n-1}\beta_n} \to \frac{1}{4B^2} < \frac{1}{4}.$$

Let $\lambda_n \to +\infty$ and $\beta_n \to \infty$. Assume J is essentially selfadjoint. Then the spectrum of J is discrete and consists of a sequence of points convergent to $+\infty$.

Proof. It suffices to show that for every real number M there are only finitely many points in the spectrum $\sigma(J)$ which are less than M. Fix M. By assumptions there is N such that $\beta_{n+N-1} > M$ and

$$\frac{\lambda_{n+N}^2}{(\beta_{n+N-1} - M)(\beta_{n+N} - M)} \leq \frac{1}{4}, \tag{9}$$

for $n \geq 0$. Let J_N be the Jacobi matrix defined as

$$J_N = \begin{pmatrix} \beta_{N-1} & \lambda_N & 0 & 0 & \cdots \\ \lambda_N & \beta_N & \lambda_{N+1} & 0 & \cdots \\ 0 & \lambda_{N+1} & \beta_{N+1} & \lambda_{N+2} & \ddots \\ 0 & 0 & \lambda_{N+2} & \beta_{N+2} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

We will show that $\sigma(J_N) \subseteq [M, +\infty)$ by estimating the quadratic form $(J_N x, x)_{\ell^2(\mathbf{N})}$ from below by $M(x,x)_{\ell^2(\mathbf{N})}$. Let x be a real valued sequence. Set $\beta'_n = \beta_{n+N-1} - M$ and $\lambda'_n = \lambda_{n+N}$. Then by (9) we have

$$\begin{aligned}
(J_N x, x)_{\ell^2(\mathbf{N})} - M(x,x)_{\ell^2(\mathbf{N})} &= \sum_{n=0}^{\infty} \beta'_n x_n^2 + 2\sum_{n=0}^{\infty} \lambda'_n x_n x_{n+1} \\
&\geq \sum_{n=0}^{\infty} \beta'_n x_n^2 - 2\sum_{n=0}^{\infty} \lambda'_n |x_n||x_{n+1}| \\
&\geq \sum_{n=0}^{\infty} \beta'_n x_n^2 - \sum_{n=0}^{\infty} \sqrt{\beta'_n}\sqrt{\beta'_{n+1}}|x_n||x_{n+1}| \\
&\geq \sum_{n=0}^{\infty} \beta'_n x_n^2 - \frac{1}{2}\sum_{n=0}^{\infty} \beta'_n x_n^2 - \frac{1}{2}\sum_{n=1}^{\infty} \beta'_n x_n^2 \\
&= \frac{1}{2}\beta'_0 x_0^2 \geq 0.
\end{aligned}$$

Hence

$$(J_N x, x)_{\ell^2(\mathbf{N})} \geq M(x,x)_{\ell^2(\mathbf{N})},$$

and consequently $\sigma(J_N) \subseteq [M, +\infty)$. Let 0_N denote the $N \times N$ matrix with all entries equal to zero. Observe that the Jacobi matrix J can be written in the form

$$J = J_0 + (0_N \oplus J_N),$$

where J_0 is a finite dimensional Jacobi matrix. We have

$$\sigma(0_N \oplus J_N) = \{0\} \cup \sigma(J_N).$$

By the Weyl perturbation theorem the spectra of J and $0_N \oplus J_N$ may differ by at most N points. Hence $\sigma(J)$ can have at most $N+1$ points to the left of M.

■

References

[1] Akhiezer, N. I., *The Classical Moment Problem,* Hafner Publ. Co., New York, 1965.

[2] T. Chihara, *An Introduction to Orthogonal Polynomials,* Mathematics and Its Applications, Vol. **13**, Gordon and Breach, New York, London, Paris, 1978.

[3] T. Chihara, *Chain sequences and orthogonal polynomials,* Trans. Amer. Math. Soc., **104** (1962), 1–16.

[4] J. Dombrowski and P. Nevai, *Orthogonal polynomials, measures and recurrence relations,* SIAM J. Math. Anal., **17** (1986), 752–759.

[5] E. K. Ifantis, *On the spectral measure of a class of orthogonal polynomials,* J. Comp. Appl. Math., **133** (2001), 688–689.

[6] J. Janas and M. Moszyński, *Alternative approaches to the absolute continuity of Jacobi matrices with monotonic weights,* Int. Eq. Oper. Theory, (to appear).

[7] S. Khan and D. B. Pearson, *Subordinacy and spectral theory for inifinite matrices,* Helv. Phys. Acta **65** (1992), 505–527.

[8] A. Maté and P. Nevai, *Orthogonal polynomials and absolutely continuous measures,* In C. K. Chui et. al., editor, Approximation Theory IV, Vol. **103**, Academic Press, New York, (1983), 611–617.

[9] B. Simon, *The classical moment problem as a self-adjoint finite difference operator,* Advances Math. **137** (1998), 82–203.

Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50–384 Wrocław, Poland
Email address: szwarc@math.uni.wroc.pl