CONNECTION COEFFICIENTS OF ORTHOGONAL POLYNOMIALS WITH APPLICATIONS TO CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. New criteria for nonnegativity of connection coefficients between to systems of orthogonal polynomials are given. The results apply to classical orthogonal polynomials.

1. Introduction

Let μ be a positive measure on the real line $\mathbb{R}$ with all moments finite. Let $\{p_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials obtained from the sequence of consecutive monomials 1, x, $x^2, \ldots$ by the Gram-Schmidt procedure. We normalize p_n so its leading coefficients is 1. We call p_n the monic orthogonal polynomials.

If $\mathcal{P} = \{p_n\}_{n=0}^{\infty}$ and $\mathcal{Q} = \{q_n\}_{n=0}^{\infty}$ are two systems of monic orthogonal polynomials we can express p_n as linear combinations of q_n as

$$p_n = \sum_{m=0}^{n} c(n,m) q_m.$$

The numbers $c(n,m)$ are called the connection coeffcients from $\mathcal{P}$ to $\mathcal{Q}$. Many problems in harmonic analysis related to nontrigonometric orthogonal expansions depend on nonnegativity of connection coefficients (see [2, Lecture 7], [5]). Also nonnegativity of connection coefficients from a given system of orthogonal polynomials $\mathcal{P}$ to Tchebyshev polynomials (so-called property $\mathbf{T}$) was used in [9] to derive nonnegativity of linearization of the system $\mathcal{P}$. Property $\mathbf{T}$ was used in [10] in proving central limit theorems related to random walks associated with $\mathcal{P}$.

There are a few criterion for nonnegativity of connection coefficents. Some of them are given in terms of corresponding spectral measures

1991 *Mathematics Subject Classification.* Primary 42C05, 33C45.

Supported by a grant from KBN..

This paper is in final form and no version of it will be submittefed for publication elsewhere.

The paper was completed while the author was visiting the Department of Mathematics and Computer Science, University of Missouri–St. Louis during the Fall 1993.

([6, 11]) the other impose conditions on coefficients in the recurrence formula satisfied by the polynomials ([1, 8, 9])

In this paper we derive several criteria for nonnegativity of connection coeffcients. All of them are stated in terms of the supports of spectral measures associated with $\mathcal{P}$ and $\mathcal{Q}$.

As an application we prove the positivity of connection coeffcients from dilated Legendre polynomials into standard Legendre polynomials.

Acknowledgement. I'd like to thank George Gasper for calling the formula (3.2) to my attention.

2. General results

In the sequel μ and ν will be positive measures on $\mathbb{R}$ and $\mathcal{P} = \{p_n\}_{n=0}^{\infty}$, $\mathcal{Q} = \{q_n\}_{n=0}^{\infty}$ will denote the corresponding systems of monic orthogonal polynomials.

Definition 2.1. ?? *We say that $\mathcal{P}$ is subordinate to $\mathcal{Q}$, if the connection coefficients from $\mathcal{P}$ to $\mathcal{Q}$ are nonnegative, i.e. for every $n \in \mathbb{N}$ the coefficients $c(n,m)$ in the expansion*

$$p_n = \sum_{m=0}^{n} c(n,m) q_m \tag{2.1}$$

are nonnegative. In this case we will write

$$\mathcal{P} \prec \mathcal{Q}.$$

The sum in (2.1) is orthogonal hence multiplying both sides of (2.1) by q_m and integrating with respect to ν gives

$$\left(\int_{-\infty}^{\infty} q_m^2 d\nu \right) c(n,m) = \int_{-\infty}^{\infty} p_n q_m d\nu$$

This implies that positivity of $c(n,m)$ is equivalent to positivity of

$$d(n,m) = \int_{-\infty}^{\infty} p_n q_m d\nu \tag{2.2}$$

Lemma 2.1.

(i) *Let μ be a positive measure on $\mathbb{R}$ such that* $\operatorname{supp} \mu \subset (-\infty, a]$ *and $b \geq a$. Let $\nu = \mu + \varepsilon\delta_b$, where $\varepsilon > 0$. Then $\mathcal{P} \prec \mathcal{Q}$.*

(ii) *Let μ be a symmetric positive measure on $\mathbb{R}$ such that* $\operatorname{supp} \mu \subset$ *$[-a, a]$ and $b \geq a$. Let $\nu = \mu + \varepsilon\delta_{-b} + \varepsilon\delta_b$, where $\varepsilon > 0$. Then $\mathcal{P} \prec \mathcal{Q}$.*

Proof. We have to show that $d(n,m)$ are nonnegative for $m \le n$. For $n = m$ we have

$$d(n,n) = \int_{-\infty}^{\infty} p_n q_n d\nu = \int_{-\infty}^{\infty} x^n q_n d\nu = \int_{-\infty}^{\infty} q_n^2 d\nu > 0$$

as the polynomials have unit leading coefficients, and q_n is orthogonal to polynomials of degree less than n.

(i) If $m < n$, then

$$d(n,m) = \int_{-\infty}^{\infty} p_n q_m d\nu = \int_{-\infty}^{\infty} p_n q_m d\mu + \varepsilon p_n(b) q_m(b) = \varepsilon p_n(b) q_m(b) > 0$$

Here we used the fact that polynomials take positive values at the point b, since the supports of corresponding measures lie to the left of b.

(ii) The coefficient $d(n,m)$ is zero unless $n - m$ is an even number. Also the values of the polynomials at b are positive while their values at $-b$ have alternating signs. If $m < n$, and $n - m$ is even then similarly as in (i) we get

$$d(n,m) = \varepsilon p_n(-b) q_m(-b) + \varepsilon p_n(b) q_m(b) > 0$$

as both summands are positive due to the same parity of n and m. ■

Lemma 2.2.

(i) *μ is as in Lemma 2.1(i). Let $a \le b_1 \le b_2 \le \ldots \le b_N$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ be positive numbers. Let*

$$\nu = \mu + \sum_{i=1}^{N} \varepsilon_i \delta_{b_i}$$

Then $\mathcal{P} \prec \mathcal{Q}$.

(ii) *μ is as in Lemma 2.1(ii). Let $a \le b_1 \le b_2 \le \ldots \le b_N$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ be positive numbers. Let*

$$\nu = \mu + \sum_{i=1}^{N} \{\varepsilon_i \delta_{-b_i} + \varepsilon_i \delta_{b_i}\}$$

Then $\mathcal{P} \prec \mathcal{Q}$.

Proof. We will show part (i) only. Define the measures ν_j as

$$\nu_j = \sum_{i=1}^{j} \varepsilon_i \delta_{b_i}$$

and let $\mathcal{Q}_j$ denote the corresponding system of monic orthogonal polynomials. By Lemma 2.1 we have

$$\mathcal{P} \prec \mathcal{Q}_1 \prec \mathcal{Q}_2 \prec \ldots \prec \mathcal{Q}_N = \mathcal{Q}.$$

Since the relation $\prec$ is obviously transitive the conclusion follows. ∎

We say that a sequence of measures ν_k is *weakly convergent* to the measure ν_0, which we denote by $\nu_k \Rightarrow \nu$, if for every $n \in \mathbb{N}$

$$\lim_{k\to\infty} \int_{-\infty}^{+\infty} x^n d\nu_k(x) = \lim_{k\to\infty} \int_{-\infty}^{+\infty} x^n d\nu_0(x).$$

Lemma 2.3. *Let $\mu,\ \nu, \nu_1,\ \nu_2\ \ldots$ be positive measures on $\mathbb{R}$, and $\mathcal{P},\ \mathcal{Q},\ \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ denote the corresponding systems of orthogonal monic polynomials. If $\mathcal{P} \prec\ Q_k$ for every $k \in \mathbb{N}$ and $\nu_k \Rightarrow \nu$ then $\mathcal{P} \prec \mathcal{Q}$.*

Proof. The coefficients of the polynomials $\mathcal{Q}_k = \{q_{k,n}\}_{n=0}^{\infty}$ depend only on the moments of the measure ν_k. (see [3, Theorem 3.1]) and the moments of ν_k are convergent to the moments of the measure ν. Hence

$$\lim_{k\to\infty} \int_{-\infty}^{+\infty} p_n q_{k,m} d\nu_k = \int_{-\infty}^{+\infty} p_n q_m d\nu.$$

This implies the conclusion of the lemma. ∎

Theorem 2.1.

(i) *Let μ and μ_0 be positive measures such that* $\operatorname{supp}\mu \subset (-\infty, a]$ *and* $\operatorname{supp}\mu_0 \subset [a, +\infty)$. *Let $\nu = \mu + \mu_0$. Then $\mathcal{P} \prec \mathcal{Q}$.*

(ii) *Let μ and μ_0 be symmetric positive measures such that* $\operatorname{supp}\mu \subset [-a, a]$ *and* $\operatorname{supp}\mu_0 \cap (-a, a) = \emptyset$. *Let $\nu = \mu + \mu_0$. Then $\mathcal{P} \prec \mathcal{Q}$.*

Proof. (i) There is a sequence of measures μ_k such that $\operatorname{supp}\mu_k$ is a finite set contained in $[a, +\infty)$ and

$$\lim_{k\to\infty} \int_{-\infty}^{+\infty} x^n d\mu_k = \lim_{k\to\infty} \int_{-\infty}^{+\infty} x^n d\mu_0.$$

This can be achieved in the following way. First approximate μ_0 by the restrictions of μ_0 to the intervals $[a, k]$, and then approximate the latter by discrete measures supported in $[a, k]$.

Let $\nu_k = \mu + \mu_k$. Denote the system of corresponding orthogonal polynomials by $\mathcal{Q}_k$. By Lemma 2.2 we have $\mathcal{P} \prec \mathcal{Q}_k,$ for $k = 1, 2, \ldots$. Now Lemma 2.3 implies $\mathcal{P} \prec \mathcal{Q}$. ∎

Theorem 2.2.

(i) *Let μ and ν be positive measures such that* $\operatorname{supp}\mu \subset (-\infty, a]$ *and* $\operatorname{supp}\nu \subset [a, +\infty)$. *Then $\mathcal{P} \prec \mathcal{Q}$.*

(ii) *Let μ and ν be symmetric positive measures such that* $\operatorname{supp}\mu \subset [-a, a]$ *and* $\operatorname{supp}\nu \cap (-a, a) = \emptyset$. *Then $\mathcal{P} \prec \mathcal{Q}$.*

Proof. Let $\nu_k = \frac{1}{k}\mu + \nu$, and denote the corresponding system of orthogonal polynomials by $\mathcal{Q}_k$. Then $\nu_k \Rightarrow \nu$. By Theorem 2.1 we have $\mathcal{P} \prec \mathcal{Q}_k$. Now we get the conclusion by applying Lemma 2.3. ∎

For $\mathcal{P} = \{p_n(x)\}_{n=0}^{\infty}$ a system of monic orthogonal polynomials let $\mathcal{P}^{\vee} = \{(-1)^n p_n(-x)\}_{n=0}^{\infty}$. Obviously $\mathcal{P}^{\vee}$ is again a system of monic orthogonal polynomials. Moreover if $\mathcal{P}$ is orthogonal with respect to μ the system $\mathcal{P}^{\vee}$ is orthogonal with respect to $\mu^{\vee}$, where $d\mu^{\vee}(x) = d\mu(-x)$.

Corollary 2.1. *Let μ and ν be positive measures such that* $\operatorname{supp}\nu \subset (-\infty, a]$ *and* $\operatorname{supp}\mu \subset [a, +\infty)$. *Then $\mathcal{P}^{\vee} \prec \mathcal{Q}^{\vee}$.*

Proof. It suffices to observe that the measures $\mu^{\vee}$ and $\nu^{\vee}$ satisfy the assumptions of Theorem 2.1. ∎

3. Applications to classical orthogonal polynomials

For $\alpha,\ \beta > -1$ let $\mu_{\alpha,\beta}$ denote the measure

$$d\mu_{\alpha,\beta}(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}\,dx \qquad -1 \le x \le 1.$$

The corresponding monic orthogonal polynomials $T_n^{(\alpha,\beta)}$ are called the generalized Tchebyshev polynomials. They are related to the Jacobi polynomials $P_n^{(\alpha,\beta)}$ by the quadratic formula

$$T_{2n}^{(\alpha,\beta)}(x) = 2^{-n} P_n^{(\alpha,\beta)}(2x^2-1) \tag{3.1}$$

(see [7]).

Theorem 3.1. *Let $\beta > -1$, and $\lambda > 1$. The coefficients $c(n,m)$ and $d(n,m)$ in the expansions*

$$\begin{aligned} T_n^{(0,\beta)}(\lambda x) &= \sum_{m=1}^{n} c(n,m) T_m^{(0,\beta)}(x) \\ P_n^{(0,\beta)}(\lambda x - 1) &= \sum_{m=1}^{n} d(n,m) P_m^{(0,\beta)}(x-1) \end{aligned}$$

are nonnegative.

Proof. Let $d\mu(x) = |x|^{2\beta+1}\chi_{[-(1/\lambda),(1/\lambda)]}dx$ and $d\nu(x) = |x|^{2\beta+1}\chi_{[-1,1]}dx$. Then μ and ν satisfy the assumptions of Theorem 2.1 for $\lambda > 1$. Furthermore $T_n^{(0,\beta)}(\lambda x)$ are orthogonal with respect to μ. This shows nonnegativity of $c(n,m)$. The nonnegativity of $d(n,m)$ follows from the quadratic transformation (3.1). One can observe also that $d(n,m) = c(2n,2m)$. ∎

Applying Theorem 2.2 with $\beta = -\frac{1}{2}$ gives the following.

Corollary 3.1. *Let P_n be the Legendre polynomials. Then the coefficients in the expansion*

$$P_n(\lambda x) = \sum_{m=0}^{n} c(n,m) P_m(x)$$

are nonnegative for any $\lambda \geq 1$.

Corollary 3.2. *Let $\alpha > -1$, and $\lambda \geq 1$. The coefficients $c(n,m)$ and in the expansion*

$$P_n^{(\alpha,0)}(\lambda x + 1) = \sum_{m=0}^{n} c(n,m) P_m^{(\alpha,0)}(x+1)$$

have alternating sign, i.e. $(-1)^{n+m} c(n,m) > 0$.

Proof. It suffices to observe that

$$P_n^{(\alpha,0)}(x) = (-1)^n P_n^{(0,\alpha)}(-x)$$

and apply Corollary 2.1. ∎

Remarks. It is surprising that Theorem 3.1 is new even for the Legendre polynomials. A similar result is known for the Laguerre polynomials L_n^α, and coefficients are given explicitly. Namely by [4, p. 192, (40)]

$$L_n^\alpha(\lambda x) = \sum_{m=0}^{n} \binom{n+\alpha}{m} \lambda^{n-m} (1-\lambda)^m L_{n-m}^\alpha(x) \tag{3.2}$$

This shows that the coefficients are positive for $0 < \lambda < 1$ and alternating for $\lambda > 1$.

In case of the generalized Tchebyshev or Legendre polynomials we were unable to determine the behaviour of connection coeffcients in Corollary 3.1 when $0 < \lambda < 1$.

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