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Linearization and Connection Coefficients of Orthogonal Polynomials

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Abstract. Let $\{P_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials. LASSER [5] observed that if the linearization coefficients of $\{P_n\}_{n=0}^{\infty}$ are nonnegative then each of the $P_n(x)$ is a linear combination of the Tchebyshev polynomials with nonnegative coefficients. The aim of this paper is to give a partial converse to this statement. We also consider the problem of determining when the polynomials P_n can be expressed in terms of Q_n with nonnegative coefficients, where $\{Q_n\}_{n=0}^{\infty}$ is another system of orthogonal polynomials. New proofs of well known theorems are given as well as new results and examples are presented.

Introduction

The aim of this paper is to give a new criterion for the nonnegative linearization of orthogonal polynomials. This criterion is related to the question whether polynomials can be expressed in terms of the Tchebyshev polynomials with nonnegative coefficients. The relation between linearization coefficients and connection coefficients relative to the Tchebyshev polynomials was stated by LASSER [5] but can be traced to NEVAI's work [7].

In the second section we reprove well-known theorems concerning connection coefficients. We also provide some new results and examples.

1. Linearization Coefficients

Let $P_n(x)$ be the polynomials orthogonal with respect to probability measure $d\mu(x)$ on the real line, normalized so that the leading coefficients are positive. The $P_n(x)$ satisfy a recurrence relation

$$x P_n(x) = \gamma_n P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x), \tag{1}$$

where α_n and γ_n are positive. The product $P_n(x) P_m(x)$ is a polynomial of degree $n + m$, so it can be expressed in the form

$$P_n(x) P_m(x) = \sum_{k=|n-m|}^{n+m} a(n, m, k) P_k(x). \tag{2}$$

The coefficients in (2) are called the *linearization coefficients* of the polynomials $P_n(x)$.

Let $T_n(x)$ be the Tchebyshev polynomials of the first kind, i.e.

$$T_n(\cos\theta) = \cos n\theta.$$

By the well known cosine identity the Tchebyshev polynomials satisfy

$$x T_n(x) = \frac{1}{2} T_{n+1}(x) + \frac{1}{2} T_{n-1}(x), \quad n \geqslant 1.$$

Let us consider the connection coefficients from $T_n(x)$ to $P_n(x)$. Let

$$P_n(x) = \sum_{m=0}^{n} c(n, m) T_m(x). \tag{3}$$

Following NEVAI [7] we say that a measure $d\mu(x)$ belongs to the class $M(0, 1)$ if

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = \frac{1}{2}, \tag{4}$$

$$\lim_{n\to\infty} \beta_n = 0. \tag{5}$$

LASSER [5] observed that NEVAI's result ([7], Theorem 4.2.13) implies that if $d\mu(x) \in M(0, 1)$, then the nonnegativity of the linearization coefficients $a(n, m, k)$ from (2) implies that of the connection coefficients $c(n, m)$ from (3). We are going to show that the result has a partial converse.

Let $P_n(x)$ be the polynomials orthonormal with respect to the measure $d\mu(x)$. In this case they satisfy

$$x P_n(x) = \lambda_n P_{n+1}(x) + \beta_n P_n(x) + \lambda_{n-1} P_{n-1}(x). \tag{6}$$

Theorem 1. *Let the orthogonal polynomials $P_n(x)$ satisfy (6) and $d\mu(x) \in M(0, 1)$. Assume that*

(i) *the sequences λ_n and β_n are decreasing;*

(ii) *the connection coefficients* $c(n, m)$ *in* (3) *are nonnegative.*
Then the linearization coefficients $a(n, m, k)$ *in* (2) *are also nonnegative.*

Proof. Let us renormalize the polynomials $P_n(x)$ (i.e. multiply each $P_n(x)$ by a positive coefficients) in two different ways. Let $\hat{P}_n(x)$ be the monic version of $P_n(x)$, and $\check{P}_n(x)$ be the version satisfying

$$\int_{-\infty}^{\infty} x^n \check{P}_n(x)\, d\mu(x) = 1.$$

An easy verification gives that

$$x\hat{P}_n(x) = \hat{P}_{n+1}(x) + \beta_n \hat{P}_n(x) + \lambda_{n-1}^2 \hat{P}_{n-1}(x), \quad n \geqslant 0, \tag{7}$$

$$x\check{P}_n(x) = \lambda_n^2 \check{P}_{n+1}(x) + \beta_n \check{P}_n(x) + \check{P}_{n-1}(x), \quad n > 0. \tag{8}$$

Fix a natural number k, and define the matrix $u(n, m)$ by

$$u(n, m) = \int_{-\infty}^{\infty} \hat{P}_m(x) \check{P}_n(x) P_k(x)\, d\mu(x). \tag{9}$$

Since $\hat{P}_m(x)$ and $\check{P}_n(x)$ are positive multiples of $P_m(x)$ and $P_n(x)$, respectively, nonnegativity of $u(n, m)$ is equivalent to that of $c(n, m, k)$. Without loss of generality we can assume that $n \geqslant m$. We will prove that $u(n, m) \geqslant 0$, and $u(n-1, m) - u(n, m+1) \leqslant 0$, by induction on the difference $k - (n - m)$. If $k - (n - m) < 0$ then $u(n, m) = 0$ by (9). Thus we can assume that $k - (n - m) \geqslant 0$. Observe that (7), (8) and (9) imply

$$u(n, m+1) + \beta_m u(n, m) + \lambda_{m-1}^2 u(n, m-1) =$$
$$= \lambda_n^2 u(n+1, m) + \beta_n u(n, m) + u(n-1, m).$$

This gives

$$u(n-1, m) - u(n, m+1) = (\beta_m - \beta_n) u(n, m) + (\lambda_{m-1}^2 - \lambda_n^2) u(n, m-1)$$
$$+ \lambda_n^2 [u(n, m-1) - u(n+1, m)]. \tag{10}$$

Assume that $u(s, t+1) \geqslant 0$, and $u(s-1, t) - u(s, t+1) \geqslant 0$, for $s > t$ and $k - (s - t) \leqslant l$. Let $n > m$ and $k - (n - m) = l + 1$. Then by (10), the assumptions (i), (ii) and by induction hypothesis we get

$$u(n-1, m) - u(n, m+1) \geqslant 0. \tag{11}$$

Define the sequence

$$a_r = u(k+r, l+r).$$

By (11) the sequence a_r is decreasing. Let $a_\infty = \lim_{r\to\infty} a_r$. We will show that a_∞ is nonnegative. This will imply $a_r \geqslant 0$.

Observe that by (9) a_r is a positive multiple of $\int P_{k+r} P_{l+r} P_k \, d\mu$. Thus

$$a_r = \sigma_{k,l,r} \int_{-\infty}^{\infty} P_{k+r}(x) P_{l+r}(x) P_k(x) \, d\mu(x),$$

where $\sigma_{k,l,r} > 0$. Now using [7] ((3), p. 45) we get

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} P_{k+r}(x) P_{l+r}(x) P_k(x) \, d\mu(x) = \frac{1}{\pi} \int_{-1}^{1} P_k(x) T_{k-l}(x) (1-x^2)^{-1/2} \, dx.$$

The last integral is exactly the connection coefficient $c(k, k-l)$, which by assumption is nonnegative. Summarizing the sequence a_r is the product of a positive sequence $\sigma_{k,l,r}$ and a sequence having nonnegative limit. Thus $a_\infty \geqslant 0$. This completes the proof of the theorem.

Example 1. Consider the Gegenbauer polynomials $C_n^\lambda(x)$. Let $\tilde{C}_n^\lambda(x)$ denote the orthonormal polynomials. Then

$$x\tilde{C}_n^\lambda(x) = \sqrt{\frac{(n+1)(n+2\lambda)}{4(n+\lambda+1)(n+\lambda)}}\, \tilde{C}_{n+1}^\lambda(x) + \\ + \sqrt{\frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)}}\, \tilde{C}_{n-1}^\lambda(x).$$

When $0 \leqslant \lambda \leqslant 1$, then the sequence

$$\lambda_n = \sqrt{\frac{(n+1)(n+2\lambda)}{4(n+\lambda+1)(n+\lambda)}}$$

is decreasing. Moreover since $T_n(x) = C_n^0(x)$, we get by Example 1 that the connection coefficients from $T_n(x)$ to $C_n^\lambda(x)$ are nonnegative. Thus, by Theorem 1 also the linearization coefficients of $C_n^\lambda(x)$ are nonnegative for $0 \leqslant \lambda \leqslant 1$. The case $\lambda > 1$ is more handy and can be derived from ASKEY's result ([2], Theorem 5.2; see also [9] and [10]). Actually the linearization coefficients for the Gegenbauer polynomials are known explicitely (see [2], Lecture 5).

2. Connection Coefficients

Let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be polynomials orthogonal with respect to different measures $d\mu(x)$ and $d\nu(x)$ respectively, on the real line. Every polynomial can be represented as a linear combination of the polynomials $Q_n(x)$. In particular we have

$$P_n(x) = \sum_{m=0}^{n} c(n, m)\, Q_m(x), \quad n = 0, 1, 2, \ldots. \tag{12}$$

The coefficients $c(n, m)$ from (12) are called the *connection coefficients* from the Q_ns to the P_ns.

We are interested in finding conditions ensuring the nonnegativity of the connection coefficients. One way is to impose conditions on the coefficients in the recurrence formulas that the polynomials P_n and Q_n satisfy. This was done in [1] and [8], but applications were rather modest, however in some cases that was the only method available so far (see [8], Corollary 1).

Another direction is to explore the relation between the measures $d\mu(x)$ and $d\nu(x)$. To be more specific, it was being assumed that $d\mu(x)$ was absolutely continuous with respect to $d\nu(x)$, and conditions were imposed on the density function to secure $c(n, m)$ were nonnegative. MICCHELI [6] showed that if the derivative of this density is a completely monotonic function on the positive half-axis then the connection coefficients are nonnegative. He also proved that the condition is necessary provided that the conclusion holds for every measure $d\mu(x)$.

One of the tools which was used for instance by MICCHELI [6] was the following result of Karlin and McGregor which we would like to furnish with a new proof.

Theorem 2. (KARLIN and MCGREGOR, [4]). *Let $d\mu(x)$ be a probability measure on the halfline $[0, +\infty)$. Let $P_n(x)$ be the orthogonal polynomials with respect to $d\mu(x)$, normalized so that $P_n(0) > 0$. Then*

$$\int_0^{\infty} e^{-tx} P_n(x)\, P_m(x)\, d\mu(x) > 0, \tag{13}$$

for every $t > 0$.

Proof. We start by showing the weak inequality in (13) which is sufficient for further applications. First we consider a measure with bounded support. Let $d\mu(x)$ be supported on the interval $[0, a]$. Set $d\mu_a(x) = d\mu(a-x)$. Then we have

$$\int_0^\infty e^{-tx} P_n(x) P_m(x)\, d\mu(x) = \int_0^a e^{-tx} P_n(x) P_m(x)\, d\mu(x) =$$

$$= \int_0^a e^{-t(a-x)} P_n(a-x) P_m(a-x)\, d\mu_a(x) =$$

$$= e^{-ta} \int_0^a e^{tx} P_n(a-x) P_m(a-x)\, d\mu_a(x) =$$

$$= e^{-ta} \sum_{k=0}^\infty \frac{t^k}{k!} \int_0^a x^k P_n(a-x) P_m(a-x)\, d\mu_a(x)$$

We will show that every term of the series is nonnegative, and the terms $k = |n-m|$ and $k = n+m$ are positive. To this end observe that the polynomials

$$\tilde{P}_n(x) = P_n(a-x)$$

are orthogonal with respect to the measure $d\mu_a(x)$ and have positive leading coefficients as they are normalized by $\tilde{P}_n(a) > 0$ and the corresponding measure is supported on $[0, a]$. Hence by the Favard theorem they satisfy a recurrence formula

$$x \tilde{P}_n(x) = \gamma_n \tilde{P}_{n+1}(x) + \beta_n \tilde{P}_n(x) + \alpha_n \tilde{P}_{n-1}(x), \tag{14}$$

with γ_n and α_n positive. Multiplying (14) by $\tilde{P}_n(x)$ and integrating against $d\mu_a(x)$ we get

$$\beta_n = \left(\int_0^a \tilde{P}_n^2(x)\, d\mu_a(x) \right)^{-1} \int_0^a x \tilde{P}_n^2(x)\, d\mu_a(x). \tag{15}$$

Thus the coefficients β_n are nonnegative. Applying (14) succesively k times we obtain that the integral

$$\int_0^a x^k \tilde{P}_n(x) \tilde{P}_m(x)\, d\mu_a(x)$$

vanishes for $k < |n-m|$ and it is strictly positive otherwise. This proves (13) in case of compactly supported measures.

If $d\mu(x)$ is an arbitary measure supported on $[0, \infty)$ then the sequence of measures $d\mu_N(x) = \chi_{[0,N]}(x)\, d\mu(x)$ converges to $d\mu(x)$

weakly. Let $P_{n,N}(x)$ be the polynomials orthogonal with respect to the measure $d\mu_N(x)$, such that $P_{n,N}(0) = 1$. Since the moments of $d\mu_N(x)$ tend to the corresponding moments of $d\mu(x)$ and the coefficients of orthogonal polynomials depend only on the moments of the measure (see [3], Theorem 3.1), we have

$$\int_0^\infty e^{-tx} P_n(x) P_m(x) \, d\mu(x) = \lim_{N \to \infty} \int_0^\infty e^{-tx} P_{n,N}(x) P_{m,N}(x) \, d\mu_N(x). \quad (16)$$

This shows that the integral in (13) is nonnegative. Now we are going to prove that actually we have strict inequality in (13). To this end observe that the polynomials $P_n(x)$ satisfy the following.

$$x P_n(x) = -\gamma_n P_{n+1}(x) + \beta_n P_n(x) - \alpha_n P_{n-1}(x), \quad (17)$$

where γ_n, $\alpha_n > 0$, except for $\alpha_0 = 0$. Let $m < n$. Consider the function

$$f(t) = \int_0^{+\infty} e^{-xt} P_n(x) P_m(x) \, d\mu(x).$$

We have that $f(t) \geqslant 0$. Assume that $f(t_0) = 0$, at some point t_0. Then $f(t)$ has a minimum at t_0. Thus $f'(t_0) = 0$. Observe that by (17) we have

$$\begin{aligned} f'(t_0) &= -\int_0^{+\infty} e^{-xt_0} [x P_n(x)] P_m(x) \, d\mu(x) = \\ &= \gamma_n \int_0^{+\infty} e^{-xt_0} P_{n+1}(x) P_m(x) \, d\mu(x) - \\ &\quad - \beta_n \int_0^{+\infty} e^{-xt_0} P_n(x) P_m(x) \, d\mu(x) + \\ &\quad + \alpha_n \int_0^{+\infty} e^{-xt_0} P_{n-1}(x) P_m(x) \, d\mu(x) \end{aligned}$$

The first and the third integral are nonnegative by the first part of the proof, while the second integral vanishes by our assumption. As $f'(t_0) = 0$, we get that

$$\int_0^{+\infty} e^{-xt_0} P_{n-1}(x) P_m(x) \, d\mu(x) = 0.$$

We can now repeat the argument several times till we get

$$\int_0^{+\infty} e^{-xt_0} P_m(x) P_m(x)\, d\mu(x) = 0,$$

which gives the contradiction.

Next we are going to give an alternative proof to the following result of Wayne Wilson. Our proof doesn't make use of the Stieltjes theorem on matrices with negative entries off the main diagonal.

Theorem 3. (WILSON, [11]). *Let $P_n(x)$ and $Q_n(x)$ be the polynomials orthogonal with respect to $d\mu(x)$ and $d\nu(x)$ respectively, having positive leading coefficients. If*

$$\int_{-\infty}^{\infty} Q_n(x) Q_m(x)\, d\mu(x) \leqslant 0, \quad n \neq m, \tag{18}$$

then the connection coefficients in (12) *are nonnegative.*

Theorem 3 is a straightforward consequence of the following proposition.

Proposition 1. *Under assumptions of Theorem* 3 *we have*

$$Q_n(x) = b(n, n) P_n(x) + \sum_{m=0}^{n-1} b(n, m) P_m(x), \quad n = 0, 1, 2, \ldots, \tag{19}$$

where $b(n, n) > 0$ and $b(n, m) \leqslant 0$, for $m = 0, 1, 2, \ldots, n-1$.

Proof. Let $n > m$. Without loss of generality we can assume that $P_n(x)$ are orthonormal with positive coefficients of the highest power of x. Then

$$0 \geqslant \int_{-\infty}^{\infty} Q_n(x) Q_m(x)\, d\mu(x) = \int_{-\infty}^{\infty} Q_n(x) \sum_{k=0}^{m} b(m, k) P_k(x)\, d\mu(x) =$$

$$= \sum_{k=0}^{m} b(m, k) \int_{-\infty}^{\infty} Q_n(x) P_k(x)\, d\mu(x) = \sum_{k=0}^{m} b(m, k) b(n, k).$$

We prove that $b(n, m)$ is negative by induction on n and m. First observe that $b(n, 0) \leqslant 0$, because

$$b(n, 0) = \int_{-\infty}^{\infty} Q_n(x)\, d\mu(x) \leqslant 0.$$

Observe also that $b(n, n) > 0$ as the leading coefficients of both sides of (19) have the same sign.

Assume that $b(l, k)$ is nonpositive for all l less than n and $k < l$; and that $b(n, k)$ is nonpositive for all k less than m. Then

$$0 \geqslant \sum_{k=0}^{m} b(m, k) b(n, k) = b(m, m) b(n, m) + \sum_{k=0}^{m-1} b(m, k) b(n, k) \geqslant$$
$$\geqslant b(m, m) b(n, m).$$

As $b(m, m)$ is positive we get $b(n, m) \leqslant 0$.

The Wilson theorem yields the following (cf. [8], remarks preceding the Example).

Theorem 4. *Let* $d\mu(x) = h(x)\, dv(x)$, *and*

$$h(x) = h_0 - \sum_{n=1}^{\infty} h_n x^n,$$

where $h_0, h_1, h_2, \ldots$ *are nonnegative and let the series be uniformly convergent on the support of the measure* $dv(x)$. *Let* $Q_n(x)$ *be the polynomials orthogonal with respect to* $dv(x)$ *with positive leading coefficients. Assume that in the recurrence formula*

$$x Q_n(x) = \gamma_n Q_{n+1}(x) + \beta_n Q_n(x) + \alpha_n Q_{n-1}(x), \tag{20}$$

the coefficients β_n *are nonnegative* (*the coefficients* α_n *and* γ_n *are always nonnegative due to the fact that the leading coefficients are positive*). *Then*

$$\int_{-\infty}^{\infty} Q_n(x) Q_m(x)\, d\mu(x) \leqslant 0, \quad n \neq m.$$

In particular, the conclusion holds if the measure $dv(x)$ *is symmetric about* 0, *i.e.* $dv(-x) = dv(x)$.

Proof. Let $n > m$. Then

$$\int_{-\infty}^{\infty} Q_n(x) Q_m(x)\, dv(x) = \int_{-\infty}^{\infty} Q_n(x) Q_m(x) h(x)\, d\mu(x) =$$
$$= h_0 \int_{-\infty}^{\infty} Q_n(x) Q_m(x)\, dv(x) - \sum_{k=0}^{\infty} h_k \int_{-\infty}^{\infty} x^k Q_n(x) Q_m(x)\, dv(x) =$$
$$= -\sum_{k=0}^{\infty} h_k \int_{-\infty}^{\infty} x^k Q_n(x) Q_m(x)\, dv(x).$$

Applying (20) k times and using the orthogonality relations we get that the integrals

$$\int_{-\infty}^{\infty} x^k Q_n(x) Q_m(x)\, d\nu(x)$$

are nonnegative. This completes the proof.

Combining Wilson's theorem and Theorem 4 gives the following.

Corollary 1. *Let $P_n(x)$ and $Q_n(x)$ be the polynomials orthogonal with respect to the measures $d\mu(x)$ and $d\nu(x)$ respectively. Under the assumptions of Theorem* 4, *the connection coefficients in* (12) *are nonnegative.*

Example 2. Let $C_n^\lambda(x)$ and C_n^α be the Gegenbauer polynomials corresponding to the measures $d\mu(x) = (1 - x^2)_+^{\lambda - (1/2)}\,dx$ and $d\nu(x) = (1 - x^2)_+^{\alpha - (1/2)}\,dx$. Assume that $0 < \sigma = \alpha - \lambda < 1$. Put $h(x) = (1 - x^2)^{\alpha - \lambda}$. Then

$$h(x) = 1 - \sum_{n=1}^{\infty} \binom{\sigma}{n} x^n.$$

As σ is between 0 and 1, the binomial coefficients are positive. Thus the assumptions of Corollary 1 are satisfied and we have

$$C_n^\alpha(x) = \sum_{m=0}^{n} c(n, m)\, C_m^\lambda(x),$$

where $c(n, m) \geqslant 0$. Iterating this we can get that the same is true if only $\alpha > \lambda$.

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