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ORTHOGONAL POLYNOMIALS AND A DISCRETE BOUNDARY VALUE PROBLEM I*

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Abstract. Let $\{P_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials with respect to a measure μ on the real line. Sufficient conditions are given under which any product P_nP_m is a linear combination of P_k's with positive coefficients.

Key words. orthogonal polynomials, recurrence formula

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Let us consider the following problem: we are given a probability measure μ on the real line $\mathbf{R}$ all of whose moments $\int x^{2n}d\mu(x)$ are finite. Let $\{P_n(x)\}$ be an orthonormal system in $L^2(\mathbf{R}, d\mu)$ obtained from the sequence $1, x, x^2, \cdots$ by the Gram-Schmidt procedure. We assume that the support of μ is an infinite set so that $1, x, x^2, \ldots$ are linearly independent. Clearly P_n is a polynomial of degree n which is orthogonal to all polynomials of degree less than n. It can be taken to have positive leading coefficients. The product P_nP_m is a polynomial of degree $n+m$ and it can be expressed uniquely as a linear combination of polynomials $P_0, P_1, \cdots, P_{n+m}$,

$$P_nP_m = \sum_{k=0}^{n+m} c(n, m, k)P_k$$

with real coefficients $c(n, m, k)$. Actually, if $k<|n-m|$ then $c(n, m, k)=0$. This is because

$$c(n, m, k) = \langle P_nP_m, P_k\rangle_{L^2(d\mu)} = \langle P_n, P_mP_k\rangle_{L^2(d\mu)} = \langle P_m, P_nP_k\rangle_{L^2(d\mu)}.$$

Hence if $k<|n-m|$ then either $k+m<n$ or $k+n<m$ and one of the above scalar products vanishes. Finally we get

$$P_nP_m = \sum_{k=|n-m|}^{n+m} c(n, m, k)P_k. \tag{1}$$

We ask when the coefficients $c(n, m, k)$ are nonnegative for all $n, m, k=0, 1, 2, \cdots$. The positivity of coefficients $c(n, m, k)$ (called also the linearization coefficients) gives rise to a convolution structure on $l^1(N)$ and if some additional boundedness condition is satisfied then l^1 with this new operation resembles l^1 of the circle (see [2]).

Analogously to (1), we have

$$xP_n = \gamma_nP_{n+1} + \beta_nP_n + \alpha_nP_{n-1} \quad \text{for } n=0, 1, 2, \cdots \tag{2}$$

(we apply the convention $\alpha_0=\gamma_{-1}=0$). The coefficients α_n and γ_n are strictly positive. If the measure μ is symmetric, i.e., $d\mu(x)=d\mu(-x)$, then $\beta_n=0$. When P_n are normalized so that $\|P_n\|_{L^2(\mu)}=1$ then we can check easily that $\alpha_{n+1}=\gamma_n$. Hence, if we put $\lambda_n=\gamma_n$ we get

$$xP_n = \lambda_nP_{n+1} + \beta_nP_n + \lambda_{n-1}P_{n-1} \quad \text{for } n=0, 1, 2, \cdots. \tag{3}$$

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Favard [4] proved that the converse is also true, i.e., any system of polynomials satisfying (3) is orthonormal with respect to a probability measure μ (not necessarily unique). In case of bounded sequences λ_n and β_n we can recover the measure μ in the following way. Consider a linear operator L on $l^2(N)$ given by

$$La_n = \lambda_n a_{n+1} + \beta_n a_n + \lambda_{n-1} a_{n-1}, \qquad n = 0, 1, 2, \cdots. \tag{4}$$

Then L is a self-adjoint operator on $l^2(N)$. Let $dE(x)$ be the spectral resolution associated with L. Then the system $\{P_n\}$ is orthonormal with respect to the measure $d\mu(x) = d\langle E(x)\delta_0, \delta_0\rangle$.

The statement of the positivity of $c(n, m, k)$ does not require orthonormalization of the polynomials P_n. We can as well consider another normalization, i.e., let $\tilde{P}_n = \sigma_n P_n$ where σ_n is a sequence of positive numbers. The problem of positive coefficients in the product of $\tilde{P}'_n s$ is equivalent to that of $P'_n s$. Moreover, it is easy to check that the polynomials $\tilde{P}_n$ satisfy the recurrence relation of the form

$$x\tilde{P}_n = \gamma_n \tilde{P}_{n+1} + \beta_n \tilde{P}_n + \alpha_n \tilde{P}_{n-1} \quad \text{for } n = 0, 1, 2, \cdots \tag{5}$$

and the unique relation connecting α_n, γ_n and the coefficients λ_n from (3) is $\alpha_{n+1}\gamma_n = \lambda_n^2$; the sequence of diagonal coefficients β_n remains unchanged. From this observation it follows that if polynomials $\tilde{P}_n$ satisfy (5) then after appropriate renormalization the polynomials $\bar{P}_n = c_n \tilde{P}_n$ satisfy

$$x\bar{P}_n = \alpha_{n+1} \bar{P}_{n+1} + \beta_n \bar{P}_n + \gamma_{n-1} \bar{P}_{n-1}. \tag{6}$$

Consider the particular case of monic normalization, i.e., assume that the leading coefficient of any P_n is 1. Then the recurrence formula is

$$xP_n = P_{n+1} + \beta_n P_n + \lambda_{n-1}^2 P_{n-1}. \tag{7}$$

In 1970 Askey proved the following theorem concerning the monic case.

THEOREM (Askey [1]). *Let P_n satisfy (6) and let the sequences λ_n and β_n be increasing ($\lambda_n \geqq 0$); then the linearization coefficients in the formula*

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k$$

are nonnegative.

This theorem applies to the Hermite, Laguerre, and Jacobi polynomials with $\alpha + \beta \geqq 1$ (see [7]). However, it does not cover the symmetric Jacobi polynomials with $\alpha = \beta$ when $-\frac{1}{2} \leqq \alpha \leqq \frac{1}{2}$ (and, in particular, the Legendre polynomials when $\alpha = \beta = 0$). Recall that the problem of positive linearization for Jacobi polynomials was completely solved by Gaspar in [5] and [6]. In particular, $c(n, m, k)$ are positive for $\alpha \geqq \beta$ and $\alpha + \beta + 1 \geqq 0$.

The aim of this paper is to give a generalization of Askey's result so it would cover the symmetric Jacobi polynomials for $a \geqq -\frac{1}{2}$. One of the results is as follows.

THEOREM 1. *If polynomials P_n satisfy*

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}$$

and

(i) *α_n, β_n, and $\alpha_n + \gamma_n$ are increasing sequences ($\gamma_n, \alpha_n \geqq 0$),*

(ii) *$\alpha_n \leqq \gamma_n$ for $n = 0, 1, 2, \cdots$,*

then $c(n, m, k) \geqq 0$ (see (1)).

It is remarkable that the assumptions on α_n's and γ_n's are separated from that on β_n.

Before giving a proof let us explain how Askey's theorem can be derived from Theorem 1. If polynomials $\tilde{P}_n$ satisfy the assumptions of Askey's theorem then after orthonormalization of $\tilde{P}_n$'s we get the system of polynomials P_n satisfying (3), i.e.,

$$xP_n = \lambda_n P_{n+1} + \beta_n P_n + \lambda_{n-1} P_{n-1} \quad \text{for } n = 0, 1, 2, \cdots,$$

and if λ_n and β_n are increasing then putting $\alpha_n = \lambda_{n-1}$ and $\gamma_n = \lambda_n$ we can see easily that the assumptions of Theorem 1 are also satisfied.

Example. Consider the symmetric Jacobi polynomials $R_n^{(\alpha,\alpha)}$ normalized by $R_n^{\alpha,\alpha}(1) = 1$. They satisfy the following recurrence formula:

$$xR_n^{(\alpha,\alpha)} = \frac{n+2\alpha+1}{2n+2\alpha+1} R_{n+1}^{(\alpha,\alpha)} + \frac{n}{2n+2\alpha+1} R_{n-1}^{(\alpha,\alpha)}.$$

In this case

$$\alpha_n = \frac{n}{2n+2\alpha+1}, \quad \gamma_n = \frac{n+2\alpha+1}{2n+2\alpha+1}, \quad \beta_n = 0.$$

Observe that $\alpha_n + \gamma_n = 1$ and α_n is increasing when $\alpha \geqq -\frac{1}{2}$. We have also $\alpha_n \leqq \gamma_n$ when $\alpha \geqq -\frac{1}{2}$.

Instead of showing Theorem 1 we will prove a more general result.

THEOREM 2. *Let polynomials P_n satisfy*

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}$$

and let for some sequence of positive numbers σ_n polynomials $\bar{P}_n = \sigma_n P_n$ satisfy

$$x\bar{P}_n = \gamma'_n \bar{P}_{n+1} + \beta_n \bar{P}_n + \alpha'_n \bar{P}_{n-1}.$$

Assume also that

(i) $\beta_m \leqq \beta_n$ *for any* $m \leqq n$,

(ii) $\alpha_m \leqq \alpha'_n$ *for any* $m < n$,

(iii) $\alpha_m + \gamma_m \leqq \alpha'_n + \gamma'_n$ *for any* $m < n-1$,

(iv) $\alpha_m \leqq \gamma'_n$ *for any* $m \leqq n$.

Then the linearization coefficients $c(n, m, k)$ in the formula

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k$$

are nonnegative.

Setting $\alpha'_n = \alpha_n$ and $\gamma'_n = \gamma_n$, we can easily see that Theorem 2 implies Theorem 1.

Proof. First observe that we have $\alpha_{n+1}\gamma_n = \alpha'_{n+1}\gamma'_n$. Moreover, by the remarks preceding (6) we may assume that P_n and $\bar{P}_n$ satisfy

$$xP_n = \alpha_{n+1} P_{n+1} + \beta_n P_n + \gamma_{n-1} P_{n-1},$$

$$x\bar{P}_n = \alpha'_{n+1} \bar{P}_{n+1} + \beta_n \bar{P}_n + \gamma'_{n-1} \bar{P}_{n-1}.$$

The rest of the proof will follow from the maximum principle for a discrete boundary value problem.

Let L and L' be linear operators acting on sequences $\{a_n\}_{n \in N}$ by the rule

$$\begin{aligned} La_n &= \alpha_{n+1} a_{n+1} + \beta_n a_n + \gamma_{n-1} a_{n-1}, \\ L'a_n &= \alpha'_{n+1} a_{n+1} + \beta'_n a_n + \gamma'_{n-1} a_{n-1}. \end{aligned} \tag{8}$$

Let L_m and L'_n denote the operators acting on complex functions $u(n, m)$, n, $m \in N$, as L and L' but with respect to the m- or n-variable treating the other variable as a parameter.

Let us consider the following problem: $N \times N \ni (n, m) \mapsto u(n, m) \in C$ and

$$(L'_n - L_m)u = 0, \qquad u(n, 0) \geqq 0. \tag{9}$$

THEOREM 3. *Assume that* $\alpha_n > 0$ *for* $n \geqq 1$ (*we follow the convention* $\alpha_0 = \alpha'_0 = 0$) *and*

(i) $\beta_m \leqq \beta'_n$ *for any* $m \leqq n$,

(ii) $\alpha_m \leqq \alpha'_n$ *for any* $m < n$,

(iii) $\alpha_m + \gamma_m \leqq \alpha'_n + \gamma'_n$ *for any* $m < n - 1$,

(iv) $\alpha_m \leqq \gamma'_n$ *for any* $m \leqq n$.

Then $u(n, m) \geqq 0$ *for* $m \leqq n$.

Proof. On the contrary, assume that u is negative at some points. Let $(n, m+1)$ be the lowest point in the domain $\{(s, t)\colon s \geqq t\}$ for which $u(n, m+1) < 0$. It means that $u(s, t)$ is nonnegative if $t \leqq m$. Consider the rectangular triangle with vertices $A(n, m)$, $B(n-m, 0)$ and $C(n+m, 0)$, as illustrated in Fig. 1.

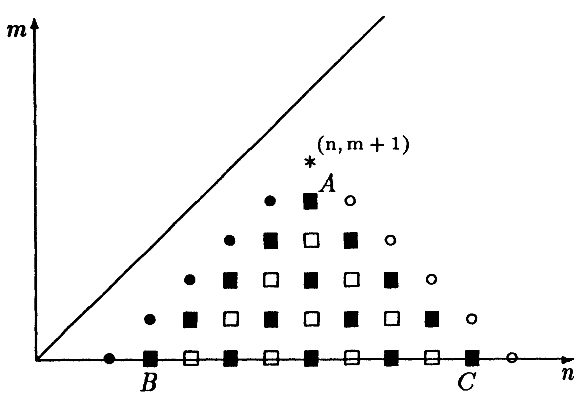

FIG. 1

All lattice points in ΔABC we divide into two subsets: Ω_1, consisting of the points (k, l) such that $k - l = n - m \pmod 2$, and the rest Ω_2. In the figure the points of Ω_1 are marked by ■ while the points of Ω_2 are marked by □. Let Ω_3 denote the lattice points connecting $(n-m-1, 0)$ and $(n, m+1)$ (except $(n, m+1)$) and Ω_4 denote those which connect $(n+m+1, 0)$ with $(n, m+1)$ (except $(n, m+1)$). The points of Ω_3 and Ω_4 are marked by ● and ○, respectively.

Assume that $(L'_n - L_m)u = 0$. Thus $\sum_{(x,y)\in\Omega_1} (L'_n - L_m)u(x, y) = 0$. If we calculate the terms $(L'_n - L_m)u(x, y) = 0$ and we sum them up we will obtain a sum of the values of the function $u(s, t)$ with some coefficients $c_{s,t}$ where (s, t) runs throughout the sets $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \{(n, m+1)\}$. Namely,

$$\begin{aligned} 0 &= \sum_{(x,y)\in\Omega_1} (L'_n - L_m)u(x, y) \\ &= \sum_{i=1}^{4} \sum_{(s,t)\in\Omega_1} c_{s,t}u(s, t) + c_{n,m+1}u(n, m+1). \end{aligned}$$

It is not hard to compute the coefficients $c_{s,t}$, so we just list them below.

(i) $(s, t) \in \Omega_1$; $c_{s,t} = \beta'_s - \beta_t$.
(ii) $(s, t) \in \Omega_2$; $c_{s,t} = \alpha'_s + \gamma'_s - (\alpha_t + \gamma_t)$.
(iii) $(s, t) \in \Omega_3$; $c_{s,t} = \gamma'_s - \alpha_t$.
(iv) $(s, t) \in \Omega_4$; $c_{s,t} = \alpha'_s - \alpha_t$.
(v) $c_{n,m+1} = -\alpha_{m+1}$.

By the assumptions of the theorem all coefficients $c_{s,t}$ are nonnegative while $c_{n,m+1}$ is strictly negative. Since $u(s, t) \geqq 0$ for $(s, t) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ and $u(n, m+1) < 0$ then the sum we were dealing with cannot be zero. It gives a contradiction.

Let us return to the proof of Theorem 2. Let P_n and $\bar{P}_n$ satisfy (8) and $\bar{P}_n = \sigma_n P_n$ for a strictly positive sequence σ_n. If

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k,$$

then

$$\bar{P}_n P_m = \sum_{k=|n-m|}^{n+m} \bar{c}(n, m, k) P_k,$$

where $\bar{c}(n, m, k) = c(n, m, k)\sigma_n$. Therefore in order to prove $c(n, m, k) \geqq 0$ it suffices to show that $\bar{c}(n, m, k) \geqq 0$ for $n > m$. Since

$$L'_n(\bar{P}_n P_m) = x\bar{P}_n P_m = L_m(\bar{P}_n P_m)$$

and the polynomials P_n are linearly independent then for any k the function $u(n, m) = \bar{c}(n, m, k)$ is a solution of (9). Obviously,

$$u(n, 0) = c(n, 0, k)\sigma_n = \begin{cases} \sigma_n & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $u(n, 0) \geqq 0$. Hence by Theorem 3 we get $u(n, m) = \bar{c}(n, m, k) \geqq 0$. This completes the proof of Theorem 2.

COROLLARY. *Let polynomials P_n satisfy $xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}$ and let*
(i) *β_n and α_n be increasing ($\alpha_n > 0$ for $n \geqq 1$, $\alpha_0 = 0$);*
(ii) *$\alpha_m + \gamma_m \leqq \alpha_{n+1} + \gamma_{n-1}$ for $m < n-1$;*
(iii) *$\alpha_m \leqq \gamma_n$ for $m < n$.*

Then the linearization coefficients $c(n, m, k)$ in (1) are nonnegative.

Proof. By remarks preceding (6) after appropriate renormalization of P_n we obtain polynomials P'_n satisfying (6). Then we get the required result by applying Theorem 2.

Example. Consider Jacobi polynomials $P_n^{(\alpha,\beta)}$. They satisfy the recurrence formula

$$xP_n^{(\alpha,\beta)} = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{(\alpha,\beta)} + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} P_n^{(\alpha,\beta)} + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}.$$

Applying the corollary yields that for $\alpha \geqq \beta$ and $\alpha + \beta \geqq 0$ we get positive linearization coefficients. However, for $\alpha \geqq \beta$ and $\alpha + \beta < 0$ the sequence

$$\beta_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$$

is decreasing and we cannot apply any of the preceding results, although we know from [5] and [6] that the condition $\alpha+\beta+1\geqq 0$ is sufficient.

In part II of this paper we will discuss the problem of positive linearization under assumption β_n is decreasing when starting from $n=1$. This is more delicate because assumptions on α_n's and γ_n's cannot be separated from those on β_n's.

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