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Reiter's Condition P_1 and Approximate Identities for Polynomial Hypergroups

By

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Abstract. Let K be a commutative hypergroup with the Haar measure μ. In the present paper we investigate whether the maximal ideals in $L^1(K,\mu)$ have bounded approximate identities. We will show that the existence of a bounded approximate identity is equivalent to the existence of certain functionals on the space $L^\infty(K,\mu)$. Finally we apply the results to polynomial hypergroups and obtain a rather complete solution for this class.

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1. Introduction

Let G be a locally compact group. An important problem in spectral synthesis is to analyze whether points of the character space $\hat{G}$ are Wiener-Ditkin sets. It is well-known, see e.g. [21], [12] or [3], that for G the answer is yes. In the far more general case of commutative hypergroups K the problem is unsolved. There are results by Chilana and Ross [6] and Chilana and Kumar [5] on spectral synthesis on hypergroups with $\hat{K}$ bearing a dual hypergroup structure. This additional assumption is restrictive. Especially for polynomial hypergroups it means that their results are available only for hypergroups generated by certain Jacobi polynomials. This class of hypergroups has been studied by Wolfenstetter in some detail, see [30]. For general hypergroups one can find related results in [27] and some counterexamples in [14]. For hypergroups induced by the automorphism group on a locally compact group one can find results on spectral synthesis in [15], [16] and a general discussion in [2]. All these contributions are focusing more or less on the Wiener-part of the problem, that means the question is studied whether there is only one closed ideal with the cospectrum consisting of one point.

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In the present paper we deal with the Ditkin-part of the problem, i.e., we investigate whether the maximal ideals in $L^1(K,\mu)$ have bounded approximate identities. This problem has been attacked by one of the authors with help of a modified Reiter condition of type $\mathscr{P}_1$, see [9]. In [9] we have proved that a bounded approximate identity in a maximal ideal exists if and only if the modified Reiter condition holds for the character α which generates the maximal ideal. It is well known that in the group case the Reiter condition is equivalent to the existence of an invariant mean. In this paper we present an extension of this results to commutative hypergroups. We are going to prove that the modified Reiter condition is equivalent to the existence of a generalized mean.

We want to point out that the existence of generalized means for characters of K could be the basic tool to carry on harmonic analysis for example on the bidual of $L^1(K,\mu)$ or to study the function spaces of almost periodic, weakly almost periodic or uniformly continuous functions on K in more detail. On the other hand the existence of approximate identities in maximal ideals allows to apply Cohen's factorization theorem, which is very useful in harmonic analysis.

In [9] we started the study the modified Reiter condition for polynomial hypergroups. The second part of this paper is devoted to the continuation of this work. With help of our new characterization we are able to study the problem in case of polynomial hypergroups in much more detail as in [9]. So we arrive at a rather complete solution for this class.

The contents of the paper are as follows. After recalling some basic facts on commutative hypergroups in Section 2 we will present one of our main results in Section 3. The remaining part of the paper is devoted to an exhaustive study of polynomial hypergoups. A detailed study especially for polynomial hypergroups generated by Jacobi polynomials will be presented in this part. The appendix contains an alternative and simpler proof of a result of Máté and Nevai on the convergence of Turán determinats.

2. Preliminaries

Throughout this paper, K will denote a commutative hypergroup. For the convolution of two elements $x, y \in K$ we write $\delta_x * \delta_y$, where δ_x is the point measure at the point x. The involution of an element $x \in K$ will be denoted by $\tilde{x}$. For a given $y \in K$ and a function $f \in C_c(K)$ the translation $T_y f$ of f is given by

$$T_y f(x) = \delta_y * \delta_x(f).$$

The commutativity of K ensures the existence of a Haar measure μ on K. We denote by $\mathscr{X}_b(K)$ the set of all characters of K, i.e., the set

$$\mathscr{X}_b(K) = \{\alpha \in C_b(K) : \alpha \neq 0 \text{ and } T_y\alpha(x) = \alpha(x)\alpha(y) \quad \text{for all } x, y \in K\}.$$

By $\hat{K}$ we denote the set of all hermitian characters, i.e.,

$$\hat{K} = \{\alpha \in \mathscr{X}_b(K) : \alpha(x) = \overline{\alpha(\tilde{x})}\}.$$

The convolution of two functions $f, g \in L^1(K,\mu)$ is defined by

$$f * g\,(x) = \int_K f(y)\, T_{\tilde{y}} g(x)\, d\mu(y).$$

With this product and the $*$-operation $f^*(x) = \overline{f(\tilde{x})}$ the Banach space $L^1(K,\mu)$ becomes a Banach $*$-algebra. The multiplicative functionals of this Banach $*$-algebra are given by

$$\phi_\alpha(f) = \int_K f(x)\,\overline{\alpha(x)}\,d\mu(x)\,, \quad f \in L^1(K,\mu)\,, \quad \alpha \in \mathscr{X}_b(K).$$

The structure space, which is the space of all multiplicative functionals, will be denoted by $\Delta(L^1(K,\mu))$. The map $\mathscr{X}_b(K) \to \Delta(L^1(K,\mu)),\ \alpha \to \phi_\alpha$ is a homeomorphism. The structure space $\Delta(L^1(K,\mu))$ can also be identified with $\mathfrak{M}(L^1(K,\mu))$, the set of all maximal ideals of the Banach algebra $L^1(K,\mu)$ via the injective map $\Delta(L^1(K,\mu)) \to \mathfrak{M}(L^1(K,\mu)),\ \phi_\alpha \mapsto \ker(\phi_\alpha)$. We shall write $I(\alpha) = \ker(\phi_\alpha)$.

The Fourier transform of a function $f \in L^1(K,\mu)$ is given by $\mathscr{F}f(\alpha) = \phi_\alpha(f)$, $\alpha \in \mathscr{X}_b(K)$. We denote the Fourier transform by $\hat{f}(\alpha)$ in case $\alpha \in \hat{K}$.

A net $(u_\lambda)_{\lambda \in \Lambda}$ in $I(\alpha)$ is a bounded approximate identity in $I(\alpha)$ if there is a constant $M > 0$ such that $\|u_\lambda\|_1 \leqslant M$ for all $\lambda \in \Lambda$ and $\lim_\lambda \|u_\lambda * f - f\|_1 = 0$ for all $f \in I(\alpha)$.

For more details on hypergroups we refer to [13] and the monograph [4].

3. Existence of Bounded Approximate Identities

In [9] we gave a necessary and sufficient condition for the existence of a bounded approximate identity in the maximal ideal $I(\alpha)$. For getting this result we modified the so-called Reiter condition $\mathscr{P}_1$ which reads as follows.

The hypergroup K satisfies the condition $\mathscr{P}_1$ if for each $\varepsilon > 0$ and every compact subset $C \subseteq K$ there exists a function $g \in L^1(K,\mu)$ with the properties $g \geqslant 0,\ \|g\|_1 = 1$, and $\|T_y g - g\|_1 < \varepsilon$ for every $y \in C$.

This condition was introduced by Reiter [21], [22] in the group case and later studied by Skantharajah [23] in the context of hypergroups.

The modified condition $\mathscr{P}_1$ reads as follows:

Definition 3.1. Let $\alpha \in \mathscr{X}_b(K)$ be fixed. We say that the $\mathscr{P}_1$-condition with bound $M > 0$ is satisfied in $\alpha \in \mathscr{X}_b(K)$ ($\mathscr{P}_1(\alpha, M)$ for the sake of brevity) if for each $\varepsilon > 0$ and every compact subset $C \subseteq K$ there exists $g \in L^1(K,\mu)$ with the following properties:

(i) $\mathscr{F}g(\alpha) = 1$
(ii) $\|g\|_1 \leqslant M$
(iii) $\|T_{\tilde{y}} g - \overline{\alpha(y)} g\|_1 < \varepsilon$ for all $y \in C$.

Remark 3.2. (a) It is easy to see that for a locally compact abelian group G the condition $\mathscr{P}_1(\alpha, M)$ is fulfilled exactly when $\mathscr{P}_1(1, M)$ is satisfied. Indeed, assume $g \in L^1(G)$ satisfies (i)–(iii) of $\mathscr{P}_1(1, M)$ for a given $\varepsilon > 0$ and a compact set $C \subseteq G$ then $f = \alpha\ g$ fulfills all requirements for $\mathscr{P}_1(\alpha, M)$. So in the case of a locally compact abelian group we do not get new insights.

(b) In [10] we considered a Reiter condition of type $\mathscr{P}_2$. We showed how this condition can be used for a characterisation of the support of the Plancherel measure of a commutative hypergroup. In contrast to the group case the condition

$\mathscr{P}_2$ can not be obtained from the condition $\mathscr{P}_1$. The reason for this lies in the different behaviour of the translation operator in the hypergroup case.

In [9] we were able to prove the following result which gives a sufficient and necessary condition for the existence of a bounded approximate identity in a maximal ideal $I(\alpha)$.

Theorem 3.3. *Let $\alpha \in \mathscr{X}_b(K)$. Then $I(\alpha)$ has a bounded approximate identity $(u_\lambda)_{\lambda \in \Lambda}$ with bound M if and only if $\mathscr{P}_1(\alpha, M')$ is satisfied, for some constant $M' > 0$.*

It is known that Reiter's condition $\mathscr{P}_1$ is equivalent to the existence of an invariant mean, i.e., a positive functional m on $L^\infty(K,\mu)$ with $m(1) = 1$ and $m(T_y f) = m(f)$, $f \in L^\infty(K,\mu)$, see [23] and [29].

We now will give the relation between the modified Reiter condition $\mathscr{P}_1(\alpha, M)$ and the existence of certain linear functional m_α on $L^\infty(K,\mu)$. This functional is not an invariant mean in the sense above but it has similar properties. Therefore we will call it a generalized mean.

Theorem 3.4. *Let $\alpha \in \hat{K}$. There is a bounded approximate identity with bound $M > 0$ in the maximal ideal $I(\alpha)$ if and only if there exists $m_\alpha \in (L^\infty(K,\mu))^*$ with*

(i) $m_\alpha(\alpha) = 1$,
(ii) $\| m_\alpha \| \leqslant M$,
(iii) $m_\alpha(T_y f) = \alpha(y)\, m_\alpha(f)$ *for all* $f \in L^\infty(K,\mu)$, $y \in K$.

Proof. Assume there is a bounded approximate identity with bound $M > 0$ in $I(\alpha)$. Then by Theorem 3.3 the condition $\mathscr{P}_1(\alpha, M)$ is fulfilled. For $\varepsilon > 0$ and a compact set $C \subset K$ let $g \in L^1(K,\mu)$ according to $\mathscr{P}_1(\alpha, M)$. We define the functional $m_{\varepsilon,C}$ on $L^\infty(K,\mu)$ by the rule

$$m_{\varepsilon,C}(f) = \int_K f(\tilde{x}) g(x)\, d\mu(x).$$

We have $m_{\varepsilon,C}(\alpha) = 1$ and

$$\|m_{\varepsilon,C}\| \leqslant \|g\|_1 \leqslant M.$$

Hence the functionals $m_{\varepsilon,C}$ are uniformly bounded. Moreover, for $y \in C$ we have

$$m_{\varepsilon,C}(T_y f) = \int_K T_y f(\tilde{x}) g(x)\, d\mu(x) = \int_K f(\tilde{x}) T_y g(x)\, d\mu(x).$$

Thus

$$\begin{aligned} |m_{\varepsilon,C}(T_y f) - \alpha(y) m_{\varepsilon,C}(f)| &\leqslant \int_K |f(x)| \cdot |T_y g(x) - \alpha(y) g(x)|\, d\mu(x) \\ &\leqslant \|f\|_\infty \|T_y g - \alpha(y) g\|_1 \leqslant \varepsilon \|f\|_\infty \end{aligned} \tag{1}$$

The family of functionals $m_{\varepsilon,C}$ form a net, where the indices (ε, C) are partially ordered by

$$(\varepsilon, C) \prec (\varepsilon', C') \quad \text{if } \varepsilon' \leqslant \varepsilon,\ C \subset C'.$$

Let m_α be an accumulation point of this net. Then $\|m_\alpha\| \leqslant M$ and $m_\alpha(\alpha) = 1$. Moreover, from (1) we obtain

$$m_\alpha(T_y f) = \alpha(y) m_\alpha(f).$$

Conversely assume that a generalized mean m_α exists. Since m_α belongs to the second dual of $L^1(K, \mu)$ by the Goldstine theorem [8, p. 424] there is a net $(f_i)_{i \in I}$ $*$-weakly convergent to m_α such that $\|f_i\|_1 \leqslant M$. In particular, we have $\hat{f}_i(\alpha) \to m_\alpha(\alpha)$. Since $m_\alpha(\alpha) = 1$ we can assume $\hat{f}_i(\alpha) = 1$. For any $y \in K$ and $f \in L^\infty(K, \mu)$ we have

$$\int_K T_{\tilde{y}} f_i(x) \overline{f(x)}\, d\mu(x) = \int_K f_i(x)\, \overline{T_y f(x))}\, d\mu(x) \to m_\alpha(T_y f) = \alpha(y) m_\alpha(f).$$

Therefore

$$\int_K \Big(T_{\tilde{y}} f_i(x) - \overline{\alpha(y)} f_i(x) \Big) \overline{f(x)}\, d\mu(x) \to 0.$$

Fix $y_1, y_2, \ldots, y_m \in K$ and write $F_{k,i} = T_{\widetilde{y_k}} f_i - \overline{\alpha(y_k)} f_i$. The m-tuple

$$\mathbf{F}_i = (F_{1,i}, F_{2,i}, \ldots, F_{m,i})$$

forms a net weakly convergent to 0 in the product space $L^1(K, \mu) \times \cdots \times L^1(K, \mu)$. By [8, Corollary 14, p. 422] there is a sequence of convex combinations of $\mathbf{F}_i$ convergent to 0 in norm. Hence for every $\varepsilon > 0$ there is a function $g \in L^1(K, \mu)$, a convex linear combination of f_i's, such that $\hat{g}(\alpha) = 1$, $\|g\|_1 \leqslant M$ and

$$\|T_{\tilde{y}_i} g - \overline{\alpha(y_i)} g\|_1 < \varepsilon \quad \text{for } i = 1, \ldots, m.$$

Now our assertion follows from [9, Proposition 3.2]. □

From the characterization above we immediately get the next statement.

Corollary 3.5. *Let K be compact. Then there is a bounded approximate identity in $I(\alpha)$ for every $\alpha \in \hat{K}$.*

Proof. We observe that

$$m_\alpha(f) = \frac{1}{\|\alpha\|_2^2} \int_K f(x) \overline{\alpha(x)}\, d\mu(x)$$

is a functional on $L^\infty(K, \mu)$ which fulfills all conditions of Theorem 3.4. □

4. Application to Polynomial Hypergroups

Polynomial hypergroups are a very interesting class since one can find hypergroups in this class which are quite different from groups, see for example [18]. In [9] we already studied polynomial hypergroups in view of the $\mathscr{P}_1(\alpha, M)$ condition. This work will be continued in this section. We will present here a much more detailed study of the problem. For this investigations we use Theorem 3.4. To have a good reference and for the sake of completeness we recall the basic facts for polynomial hypergroups. For more details and the proofs we refer to [17] and [18].

Let $(R_n)_{n \in \mathbb{N}_0}$ be a polynomial sequence defined by a recurrence relation of the type

$$R_1(x)\, R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x) \tag{2}$$

for $n \in \mathbb{N}$ with starting polynomials $R_0(x) = 1$, $R_1(x) = \frac{1}{a_0}(x - b_0)$ and $a_n > 0$, $b_n \geqslant 0$ for all $n \in \mathbb{N}_0$ and $c_n \geqslant 0$ for all $n \in \mathbb{N}$. We assume that $a_n + b_n + c_n = 1$ for $n \in \mathbb{N}$ and $a_0 + b_0 = 1$. It follows from this assumptions that

$$R_n(1) = 1$$

for all $n \in \mathbb{N}_0$. In particular the R_n's are not orthonormal. By the Theorem of Favard there is a probability measure π with bounded support with respect to which $(R_n)_{n \in \mathbb{N}_0}$ is orthogonal, i.e.,

$$\int_{\mathbb{R}} R_n(x)\, R_m(x)\; d\pi(x) = \frac{1}{h(n)}\, \delta_{n,m}. \tag{3}$$

The relationship between the R_n's and the orthonormal polynomials which we will always denote by p_n is given by $p_n(x) = \sqrt{h(n)}\, R_n(x)$.

The recurrence relation (2) is a special case of the linearization formula

$$R_n(x) R_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m;k)\, R_k(x),$$

where we assume that the coefficients $g(n,m;k)$ are non-negative for all $n,m,k \in \mathbb{N}_0$. There are many orthogonal polynomial systems which have this property (cf. [1], [11], [17], [18], [25], [26]).

An easy calculation shows that for every orthogonal polynomial system we have $h(n) = g(n,n,0)^{-1}$. In particular we obtain $h(0) = 1$. It is also possible to express the h's in terms of the coefficients of the recurrence relation (2), namely

$$h(1) = \frac{1}{c_1}, \qquad h(n) = \frac{a_1 a_2 \cdot \dots \cdot a_{n-1}}{c_1 c_2 \cdot \dots \cdot c_n}. \tag{4}$$

Since the non-negative coefficients $g(n,m;k)$ satisfy $\sum_{k=|n-m|}^{n+m} g(n,m;k) = 1$ we can define a convolution on $\mathbb{N}_0$ by:

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g(n,m;k)\, \delta_k.$$

With this convolution, the involution $\tilde{n} = n$ and the discrete topology, the set of natural numbers $\mathbb{N}_0$ is a commutative hypergroup generated by the orthogonal polynomial system $(R_n)_{n \in \mathbb{N}_0}$. Such a hypergroup is called polynomial hypergroup, see [17]. The Haar measure is the counting measure with weights $h(n)$ at the points $n \in \mathbb{N}_0$.

In this case the translation reads as

$$T_n \beta(m) = \sum_{k=|n-m|}^{n+m} g(n,m;k)\, \beta(k).$$

The dual $\mathcal{X}_b(\mathbb{N}_0)$ (resp. $\widehat{\mathbb{N}_0}$) can be identified with the set

$$D = \{x \in \mathbb{C} : |R_n(x)| \leqslant 1\} \quad (\text{resp. } D_s = D \cap \mathbb{R})$$

via the map $x \mapsto \alpha_x$, where $\alpha_x(n) := R_n(x)$.

The remaining part of the paper is concerned with a detailed investigation of conditions under which the condition $\mathcal{P}_1(x, M)$ holds. We start with a result which has been proved in [9].

Theorem 4.1. *Let $(R_n)_{n \in \mathbb{N}_0}$ define a polynomial hypergroup on $\mathbb{N}_0$ and let $x \in D_s$. Then the $\mathcal{P}_1(x, M)$-condition is satisfied if and only if for every $\varepsilon > 0$ there exists $\beta \in \ell^1(\mathbb{N}_0, h)$ with*

(i) $\sum_{n=0}^{\infty} \beta(n)\, R_n(x)\, h(n) = 1$,
(ii) $\|\beta\|_1 \leqslant M$,
(iii) $\|T_1\beta - R_1(x)\beta\|_1 < \varepsilon$.

The crucial point here is that we can restricted ourselves to the translation T_1 instead of the translation T_n in the definition of the $\mathcal{P}_1(\alpha, M)$ condition, see Definition 3.1.

Let

$$\beta_n(k) := \begin{cases} \Lambda_n(x)\, R_k(x) & 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Lambda_n(x) = \left(\sum_{j=0}^{n} R_j^2(x) h(j) \right)^{-1}$. Using the recurrence relation we obtain

$$\|T_1\beta_n - R_1(x)\beta_n\|_1 = \Lambda_n(x) \left(|R_{n+1}(x)|\, c_{n+1} h(n+1) + |R_n(x)| a_n h(n) \right). \tag{5}$$

For a proof of this statement, see [9]. In view of Theorem 4.1 and Eq. (5) we have the following Proposition.

Proposition 4.2. *Let $(R_n)_{n \in \mathbb{N}_0}$ define a polynomial hypergroup and let $x \in D_s$. Then the $\mathcal{P}_1(x, M)$-condition is fulfilled if the following two conditions hold*

(i) *there is a constant $M > 0$ such that*

$$\frac{\sum_{k=0}^{n} |R_k(x)|\, h(k)}{\sum_{k=0}^{n} R_k^2(x)\, h(k)} \leqslant M$$

for all $n \in \mathbb{N}_0$.

(ii)

$$\liminf_{n \to \infty} \frac{|R_n(x)|\, a_n h(n) + |R_{n+1}(x)|\, c_{n+1} h(n+1)}{\sum_{k=0}^{n} R_k^2(x)\, h(k)} = 0$$

The next Proposition provides a positive result for the condition $\mathcal{P}_1(x, M)$ to hold.

Proposition 4.3. *Let $x \in D_s$. If $\sum_{n=0}^{\infty} |R_n(x)|\, h(n)$ is convergent then the condition $\mathscr{P}_1(x,M)$ is fulfilled.*

Proof. Since $|R_n(x)| \leqslant 1$ the series $\sum_{n=0}^{\infty} |R_n(x)|^2\, h(n)$ is convergent. Thus the orthogonalization measure π has positive mass at x

$$\pi(\{x\}) = \frac{1}{\sum_{n=0}^{\infty} R_n^2(x)\; h(n)}.$$

Let $\beta(n) := \pi(\{x\})\; R_n(x)$. It can be easily checked that $\beta := (\beta(n))_{n\,\in\,\mathbb{N}_0}$ satisfies all conditions in Theorem 4.1. □

In contrast to the result of the preceding Proposition we can establish a negative result by using the Theorem 3.4.

Theorem 4.4. *Let $x \in D_s$ with $\pi(\{x\}) = 0$ and $R_n(x) \to 0$. Then the generalized mean m_x at the point x does not exist.*

Proof. Assume a generalized mean m_x associated with $x \in D_s$ exists. Since $T_n(\delta_0) = \frac{\delta_n}{h(n)}$ we have from Theorem 3.4 (iii)

$$m_x(\delta_n) = h(n)\, R_n(x)\, m_x(\delta_0). \tag{6}$$

Let $\varepsilon_n := \operatorname{sign}(R_n(x))$. Then, since $|R_n(x)| \leqslant 1$,

$$M \geqslant \left| m_x\left(\sum_{n=0}^{N} \varepsilon_n \delta_n \right) \right| = |m_x(\delta_0)| \sum_{n=0}^{N} h(n)\, |R_n(x)| \geqslant |m_x(\delta_0)| \sum_{n=0}^{N} h(n)\, |R_n(x)|^2.$$

If $m_x(\delta_0) \neq 0$ we obtain the estimate

$$\sum_{n=0}^{\infty} h(n)\, |R_n(x)|^2 \leqslant \frac{M}{|m_x(\delta_0)|}. \tag{7}$$

By assumption we have

$$0 = \pi(\{x\}) = \left(\sum_{n=0}^{\infty} h(n)\, |R_n(x)|^2 \right)^{-1}$$

which contradicts (7). Therefore $m_x(\delta_0) = 0$ and by (6) we get $m_x(\delta_n) = 0$.

For $\alpha_x = (R_n(x))_{n\,\in\,\mathbb{N}_0}$ let α_x^N be the truncated sequence defined by

$$\alpha_x^N(n) = \begin{cases} 0 & n \leqslant N, \\ R_n(x) & n > N. \end{cases}$$

Then

$$m_x(\alpha_x) = \sum_{n=0}^{N} R_n(x)\, m_x(\delta_n) + m_x(\alpha_x^N) = m_x(\alpha_x^N).$$

Hence

$$|m_x(\alpha_x)| = |m_x(\alpha_x^N)| \leqslant M \sup_{n > N} |R_n(x)|.$$

Since $R_n(x)$ tends to zero we get $m_x(\alpha_x) = 0$ in contradiction to $m_x(\alpha_x) = 1$. $\square$

In the proof of the above Proposition we have shown that $\pi(\{x\}) = 0$ implies $m_x(\delta_n) = 0$ for each $n \in \mathbb{N}_0$. This fact implies the following statement.

Proposition 4.5. *Let $x \in D_s$ with $\pi(\{x\}) = 0$. If there exists a generalized mean m_x at the point x, then $m_x|_{c_0} \equiv 0$, where c_0 is the subspace of $\ell^\infty(\mathbb{N}_0)$ consisting of sequences tending to zero.*

Note that in view of Theorem 3.4 the Theorem 4.4 shows that the condition $\mathscr{P}_1(x, M)$ does not hold for any point $x \in D_s$ with $R_n(x) \to 0$ and $\pi(\{x\}) = 0$.

Example 4.6. We consider the Jacobi polynomials $R_n^{(\alpha,\beta)}$ with parameters $\alpha \geqslant \beta > -1$ and $g(2,2,2) \geqslant 2$. By [11] these polynomials form a discrete hypergroup. By [24, (4.1.1) and (8.21.18)] we have

$$|R_n^{(\alpha,\beta)}(x)| = \mathcal{O}(n^{-\alpha-1/2})$$

for $x \in (-1,1)$. Thus $R_n^{(\alpha,\beta)}(x) \to 0$ for $x \in (-1,1)$ provided $\alpha > -1/2$. This and Theorem 4.4 imply that a generalized invariant mean does not exist for any $x \in (-1,1)$. Also when we additionally assume that $\alpha > \beta$ we have by [24, (4.1.1) and (4.1.4)]

$$R_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n} \Big/ \binom{n+\alpha}{n}.$$

Hence $R_n^{(\alpha,\beta)}(-1)$ tends to zero which implies that the generalized mean at $x = -1$ does not exist for $\alpha > \beta$. In case of $\alpha = \beta > -1/2$ we see that conditions (i) and (ii) are fulfilled for $x = -1$, since $h(n)$ is of polynomial growth. Finally for the case $\alpha = \beta = -1/2$ (the Chebyshev polynomials of the first kind) we apply Theorem 4.11 to obtain that $\mathscr{P}_1(x, M)$ holds for all $x \in (-1,1)$. Since the Reiter condition at $x = 1$ is always fulfilled (see Corollary 3.5 (i)) we have for the Jacobi polynomials $R_n^{(\alpha,\beta)}$ a complete description for those $x \in [-1,1]$ enjoying or not enjoying Reiter's condition $\mathscr{P}_1(x, M)$.

Proposition 4.7. *Assume π is continuous, i.e. $\pi(\{x\}) = 0$ for every $x \in D_s$. If $\sum_{n=0}^{\infty} \frac{1}{h(n)}$ is convergent then a generalized mean does not exist for π-almost every $x \in D_s$.*

Proof. We have from (3)

$$\int_{D_s} \sum_{n=0}^{\infty} R_n^2(x)\, d\pi(x) = \sum_{n=0}^{\infty} \frac{1}{h(n)} < +\infty.$$

Hence, $\sum_{n=0}^{\infty} R_n^2(x)$ is convergent π-almost everywhere. In particular $R_n(x) \to 0$ π-almost everywhere and we get the conclusion by Theorem 4.4. $\square$

Let us make some remarks concerning the last result.

Remark 4.8. (a) We conjecture that Proposition 4.7 holds if $h(n) \to +\infty$.

(b) The statement of Proposition 4.7 is in general not valid for all $x \in D_s$. For example, considering the Jacobi polynomials $R_n^{(\alpha,\beta)}$ of example Example 4.6 we have $h(n) = \mathcal{O}(n^{2\alpha+1})$. Therefore the series $\sum_{n=0}^{\infty} \frac{1}{h(n)}$ is convergent for $\alpha > 0$. But we know that $\mathscr{P}_1(1,M)$ holds in any case. To have an example for which the assumptions of Proposition 4.7 hold and $\mathscr{P}_1(x,M)$ is fulfilled for an interior point of D_s we put

$$x\,R_{2n}(x) = (1-c_n)\,R_{2n+1}(x) + c_n\,R_{2n-1}(x),$$
$$x\,R_{2n-1}(x) = \frac{1}{2}\,R_{2n}(x) + \frac{1}{2}\,R_{2n-2}(x),$$

where $c_0 = 0$. If $c_n \nearrow \frac{1}{2}$ then by [26] the polynomials give rise to a hypergroup. By [28, Theorem 8.2] (see also [18, Theorem 2]) we have $\operatorname{supp}\pi = D_s = [-1,1]$. The recurrence relation gives

$$R_{2n}(0) = (-1)^n \quad \text{and} \quad R_{2n+1}(0) = 0.$$

In this case the condition (i) of Proposition 4.2 is fulfilled with no assumptions on $h(n)$. The condition (ii) of Proposition 4.2 reduces to

$$\frac{h(2n)}{\sum_{k=0}^{n} h(2k)} \to 0. \tag{8}$$

By (4) we have

$$h(2n) = \frac{(1-c_1)\cdot \cdots \cdot (1-c_{n-1})}{c_1 \cdot \cdots \cdot c_n}$$

and hence $\frac{h(2n-2)}{h(2n)} \to 1$. Now it is straightforward to prove condition (8). In order to have an example with $\sum_{n=0}^{\infty} \frac{1}{h(n)}$ convergent choose $c_n = \frac{n}{2n+2}$.

Our next investigations are concerned with a special class of polynomial hypergoups which contain Jacobi polynomial hypergroups and their q-analogues. We will see that for this broad class the Haar weights $h(n)$ being bounded or not decides whether $\mathscr{P}_1(x,M)$ holds for every x of the interior of D_s. For the following considerations it is more convenient to use the orthonormal polynomials $p_n(x) = \sqrt{h(n)}\,R_n(x)$, which satisfy the recurrence relation

$$x\,p_n(x) = \lambda_{n+1}\,p_{n+1}(x) + \beta_n\,p_n(x) + \lambda_n\,p_{n-1}(x) \quad \text{for } n \in \mathbb{N}_0 \tag{9}$$

with $p_0(x) = 1$ and $\lambda_n = a_0\sqrt{c_n a_{n-1}}$ for $n \geqslant 2$, $\lambda_1 = a_0\sqrt{c_1}$, $\lambda_0 = 0$ and $\beta_n = a_0 b_n + b_0$ for $n \geqslant 1$, $\beta_0 = b_0$.

Definition 4.9. The polynomial system $(p_n)_{n\in\mathbb{N}_0}$ is of bounded variation type ((BV) for the sake of brevity) if the sequences (λ_n) and (β_n) have bounded variation, i.e.,

$$\sum_{n=1}^{\infty} (|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|) < +\infty. \tag{10}$$

The polynomial system $(p_n)_{n\in\mathbb{N}_0}$ is an element of the Nevai class $M(0,1)$ if $\lim_{n\to\infty} \lambda_n = \frac{1}{2}$ and $\lim_{n\to\infty} \beta_n = 0$.

The condition (10) implies in particular that λ_n and β_n are convergent.

Theorem 4.10. *Let the polynomial hypergroup $(R_n)_{n\in\mathbb{N}_0}$ be of type (BV) and belong to the class $M(0,1)$. If the Haar weights $h(n)$ tend to infinity the generalized mean does not exist for any $x\in(-1,1)$.*

Proof. By [19] the Turán determinants

$$\Delta_n = p_n^2 - \frac{\lambda_{n+1}}{\lambda_n} p_{n-1}p_{n+1} \tag{11}$$

are convergent for $x\in(-1,1)$. By Theorem 5.1 (see appendix below) the sequence $p_n(x)$ is bounded for any $x\in(-1,1)$. Hence

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{p_n(x)}{\sqrt{h(n)}} = 0$$

for $x\in(-1,1)$. Also by [19] the orthogonalization measure is absolutely continuous in $(-1,1)$. Now the conclusion follows from Theorem 4.4. □

In contrast to the above result we have the following.

Theorem 4.11. *Let the polynomial hypergroup $(R_n)_{n\in\mathbb{N}_0}$ be of type (BV) and belong to the class $M(0,1)$. If the Haar weights $h(n)$ are bounded the condition $\mathscr{P}_1(x,M)$ holds for every $x\in(-1,1)$.*

Proof. By [19] we have

$$\lim_{n\to\infty}\left(p_n^2(x) - \frac{\lambda_{n+1}}{\lambda_n} p_{n+1}(x)p_{n-1}(x)\right) = f(x) > 0$$

for $x\in(-1,1)$. Since $\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n} = 1$ there exists N such that for $n>N$

$$\max\{|p_{n-1}(x)|, |p_n(x)|, |p_{n+1}(x)|\} > \frac{f(x)}{2} =: \eta. \tag{12}$$

We will show that $\mathscr{P}_1(x,M)$ holds by using Proposition 4.2. We have

$$\frac{\sum_{k=0}^n |R_k(x)|\, h(k)}{\sum_{k=0}^n R_k^2(x)\, h(k)} = \frac{\sum_{k=0}^n |p_k(x)|\, \sqrt{h(k)}}{\sum_{k=0}^n p_k^2(x)} \leqslant C\, \frac{\sum_{k=0}^n |p_k(x)|}{\sum_{k=0}^n p_k^2(x)},$$

where $C = \sup_k \sqrt{h(k)}$.

For $n>N$ we obtain from (12)

$$\begin{aligned}
|p_{n-1}(x)| + |p_n(x)| + |p_{n+1}(x)| &\leqslant 3\eta \max\left\{\frac{|p_{n-1}(x)|}{\eta}, \frac{|p_n(x)|}{\eta}, \frac{|p_{n+1}(x)|}{\eta}\right\} \\
&\leqslant 3\eta \max\left\{\frac{p_{n-1}^2(x)}{\eta^2}, \frac{p_n^2(x)}{\eta^2}, \frac{p_{n+1}^2(x)}{\eta^2}\right\} \\
&\leqslant \frac{3}{\eta}\left(p_{n-1}^2(x) + p_n^2(x) + p_{n+1}^2(x)\right).
\end{aligned}$$

This implies that there exists some $M > 0$ such that

$$\frac{\sum_{k=0}^{n} |p_k(x)|}{\sum_{k=0}^{n} p_k^2(x)} \leqslant M$$

for all $n \in \mathbb{N}_0$. Hence assumption (i) of Proposition 4.2 follows.

We have for all n belong to $\mathbb{N}_0$

$$\frac{|R_n(x)|h(n)}{\sum_{k=0}^{n} R_k^2(x)\, h(k)} = \frac{\sqrt{h(n)}}{|p_n(x)|} \frac{p_n^2(x)}{\sum_{k=0}^{n} p_k^2(x)} \leqslant \frac{C}{|p_n(x)|} \frac{p_n^2(x)}{\sum_{k=0}^{n} p_k^2(x)}.$$

Since there is an infinite subsequence n_i such that

$$|p_{n_i}(x)| > \eta,$$

we obtain the estimate

$$\frac{|R_{n_i}(x)|\, h(n_i)}{\sum_{k=0}^{n_i} R_k^2(x)\, h(k)} \leqslant \frac{C}{\eta} \frac{p_{n_i}^2(x)}{\sum_{k=0}^{n_i} p_k^2(x)}.$$

By [20, Theorem 11 (ii), p. 32]

$$\lim_{n\to\infty} \frac{p_n^2(x)}{\sum_{k=0}^{n} p_k^2(x)} = 0$$

for $x \in (-1, 1)$. This gives (ii) of Proposition 4.2. □

Example 4.12. We consider a subclass of the so-called Bernstein-Szegö polynomials, see [24]. The polynomials $Q_n^{(\nu,\kappa)}$ under consideration are orthogonal with respect to the measure

$$d\pi(x) = \frac{\chi_{[-1,1]}(x)}{\rho(x)\sqrt{1-x^2}}\, dx, \quad x = \cos t,$$

where $\rho(x) = |\nu e^{2it} + \kappa e^{it} + 1|^2$ is a polynomial with $\rho(\cos t) > 0$ for all $t \in [0, \pi]$. By [24] it is well known that these polynomials can be represented explicitly in the form

$$\begin{aligned} Q_n^{(\nu,\kappa)}(x) &= \frac{1}{\nu+\kappa+1}\Big(T_n(x) + \kappa\, T_{n-1}(x) + \nu T_{n-2}(x)\Big), \quad n \geqslant 2, \\ Q_1^{(\nu,\kappa)}(x) &= \frac{1}{\nu+\kappa+1}\Big((\nu+1)T_1(x) + \kappa T_0(x)\Big), \quad Q_0^{(\nu,\kappa)}(x) = 1, \end{aligned}$$

where $T_n(x)$ are the Chebychev polynomials of the first kind. An easy calculation shows

$$\begin{aligned} Q_1^{(\nu,\kappa)}(x) Q_n^{(\nu,\kappa)}(x) = {} & \frac{\nu+1}{2(\nu+\kappa+1)} Q_{n+1}^{(\nu,\kappa)}(x) + \frac{\kappa}{\nu+\kappa+1} Q_n^{(\nu,\kappa)}(x) \\ & + \frac{\nu+1}{2(\nu+\kappa+1)} Q_{n-1}^{(\nu,\kappa)}(x) \end{aligned}$$

for $n>2$. It can be easily checked that this polynomials induce a polynomial hypergroup on $\mathbb{N}_0$ provided that $(\kappa,\nu)\in\{(x,y)\in\mathbb{R}^2 : x,y\geqslant 0,\ y<1,\ x-1<y\}$. The polynomial system $Q_n^{(\nu,\kappa)}$ is of type (BV) and using (4) we obtain the boundedness of $h(n)$.

5. Appendix

We will give an alternative and simpler proof of a result of Maté and Nevai (see [19]) on convergence of Turán determinants. Our proof is based on ideas of Dombrowski and Nevai, see [7].

Theorem 5.1. (Maté and Nevai, [19]) *Let the polynomials p_n satisfy*

$$xp_n(x)=\lambda_{n+1}p_{n+1}(x)+b_np_n(x)+\lambda_np_{n-1}(x).$$

Assume that $\lambda_n\to\frac{1}{2},\ b_n\to 0$ *and*

$$\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|+|b_{n+1}-b_n|<+\infty.$$

Then the sequence of Turán determinants

$$\Delta_n(x)=p_n^2(x)-\frac{\lambda_{n+1}}{\lambda_n}p_{n-1}(x)p_{n+1}(x)$$

is convergent uniformily on closed subintervals of $(-1,1)$. *Moreover the polynomials p_n are uniformly bounded on each closed subinterval of* $(-1,1)$.

Proof. By using the recurrence relation we get

$$\Delta_n(x)=p_n^2(x)+p_{n-1}^2(x)-\frac{x-b_n}{\lambda_n}p_{n-1}(x)p_n(x), \tag{13}$$

$$\Delta_n(x)=p_n^2(x)+\frac{\lambda_{n+1}^2}{\lambda_n^2}p_{n+1}^2(x)-\frac{x-b_n}{\lambda_n^2}\lambda_{n+1}\,p_n(x)p_{n+1}(x). \tag{14}$$

Applying (13) to Δ_n and (14) to Δ_{n-1} gives

$$\begin{aligned}\Delta_n-\Delta_{n-1}=&\frac{1}{\lambda_{n-1}^2}(\lambda_{n-1}^2-\lambda_n^2)\,p_n^2\\&+\frac{1}{\lambda_n\lambda_{n-1}^2}\Big[(x-b_{n-1})(\lambda_n^2-\lambda_{n-1}^2)\ +\ (b_n-b_{n-1})\lambda_{n-1}^2\Big]p_{n-1}p_n\end{aligned}$$

Hence

$$|\Delta_n-\Delta_{n-1}|\leqslant C_1\big(|\lambda_n-\lambda_{n-1}|+|b_n-b_{n-1}|\big)\,(p_{n-1}^2+p_n^2), \tag{15}$$

where C_1 is a constant independent of n and x.

By (13) we have

$$\Delta_n=\left(p_n-\frac{x-b_n}{2\lambda_n}p_{n-1}\right)^2+\left(1-\frac{(x-b_n)^2}{4\lambda_n^2}\right)p_{n-1}^2,$$

$$\Delta_n=\left(p_{n-1}-\frac{x-b_n}{2\lambda_n}p_n\right)^2+\left(1-\frac{(x-b_n)^2}{4\lambda_n^2}\right)p_n^2.$$

Hence we obtain

$$\Delta_n \geqslant \left(1 - \frac{(x-b_n)^2}{4\lambda_n^2}\right) p_n^2,$$

$$\Delta_n \geqslant \left(1 - \frac{(x-b_n)^2}{4\lambda_n^2}\right) p_{n-1}^2.$$

Fix $\delta > 0$ and let $|x| \leqslant 1-\delta$. Since $\lambda_n \to \frac{1}{2}$ and $b_n \to 0$ there is a constant $C_2 > 0$ and a number N such that

$$\Delta_n \geqslant C_2(p_{n-1}^2 + p_n^2) \tag{16}$$

for all $n > N$. By (15) and (16) we obtain

$$|\Delta_n - \Delta_{n-1}| \leqslant \varepsilon_n \ \Delta_n$$

for all $n > N$, where $\varepsilon_n := \frac{C_1}{C_2}\left(|\lambda_n - \lambda_{n-1}| + |b_n - b_{n-1}|\right)$. Thus

$$\frac{1}{1+\varepsilon_n}\ \Delta_{n-1} \leqslant \Delta_n \leqslant \frac{1}{1-\varepsilon_n}\ \Delta_{n-1}$$

for all $n > N$. We can assume $\varepsilon_n < 1$. Since the series $\sum \varepsilon_n$ is convergent. Δ_n is convergent and the limit is positive. Moreover the convergence is uniform for $|x| \leqslant 1-\delta$. By (16) we get that $p_n(x)$ is bounded uniformly with respect to n and $|x| \leqslant 1-\delta$. □

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References

[1] Askey R (1970) Linearization of the product of orthogonal polynomials. In: Gunning R (ed) Problems in Analysis, pp 223–228. Princton: Univ Press

[2] Barnes BA (1983) Ditkin's theorem and [*SIN*]-groups. Monatsh Math **96**: 1–15

[3] Benedetto JJ (1975) Spectral Synthesis. Stuttgart: Teubner

[4] Bloom WR, Heyer H (1994) Harmonic Analysis of Probability Measures on Hypergroups. Berlin: De Gruyter

[5] Chilana AK, Kumar A (1979) Ultra-strong Ditkin sets in hypergroups. Proc Amer Math Soc **77**: 353–358

[6] Chilana AK, Ross KA (1978) Spectral synthesis in hypergroups. Pacific J Math **76**: 313–328

[7] Dombrowski J, Nevai PG (1986) Orthogonal polynomials, measures and recurrence relations. SIAM J Math Anal **17**: 752–759

[8] Dunford N, Schwartz JT (1988) Linear Operators I. New York: Wiley

[9] Filbir F (2000) Existence of approximate identities in maximal ideals of hypergroup-algebras. In: Dunkl C et al (eds) Special Functions. Proc of the International Workshop Hong Kong 1999, pp 88–97. Singapore: World Scientific

[10] Filbir F, Lasser R (2000) Reiter's conditions P_2 and the Plancherel measure for hypergroups. Illinois J Math **44**: 20–31

[11] Gasper G (1970) Linearization of the product of Jacobi polynomials II. Canad J Math **32**: 582–593

[12] Hewitt E, Ross KA (1970) Abstract Harmonic Analysis II. Berlin Heidelberg New York: Springer

[13] Jewett RI (1975) Spaces with an abstract convolution of measures. Adv Math **18**: 1–101

[14] Kumar A, Singh AI (1988) Counterexamples in spectral analysis on hypergroups. Rend Math **8**: 329–338

[15] Lasser R (1977) Primary ideals in center of group algebras. Math Ann **229**: 53–58

[16] Lasser R (1978) Zur Idealtheorie in Gruppenalgebren von $[FIA]_B^-$-Gruppen. Monatsh Math **96**: 59–79
[17] Lasser R (1983) Orthogonal polynomials and hypergroups. Rend Mat Appl, VII Ser **3**: 185–209
[18] Lasser R (1994) Orthogonal polynomials and hypergroups II-the symmetric case. Trans Amer Math Soc **341**: 749–769
[19] Máté A, Nevai PG (1983) Orthogonal polynomials and absolutely continuous measures. In: Chui CK et al (eds) Approximation Theory IV, pp 611–617. New York: Academic Press
[20] Nevai PG (1979) Orthogonal Polynomials. Mem Amer Math Soc **213**
[21] Reiter H (1968) Classical Harmonic Analysis and Locally Compact Groups. Oxford: Univ Press
[22] Reiter H (1970) L^1-Algebras and Segal-Algebras. Lect Notes Math **231**. Berlin Heidelberg New York: Springer
[23] Skantharajah M (1992) Amenable hypergroups. Illinois J Math **36**: 15–46
[24] Szegö G (1975) Orthogonal polynomials. Amer Math Colloq Publ **23**. Providence, RI: Amer Math Soc
[25] Szwarc R (1992) Orthogonal polynomials and discrete boundary value problem I. SIAM J Math Anal **23**: 959–964
[26] Szwarc R (1992) Orthogonal polynomials and discrete boundary value problem II. SIAM J Math Anal **23**: 965–969
[27] Vogel M (1987) Spectral synthesis on algebras of orthogonal polynomial series. Math Z **194**: 99–116
[28] Voit M (1991) Factorization of probability measures on symmetric hypergroups. J Austral Math Soc (Series A) **50**: 417–467
[29] Wolfenstetter S (1984) Jacobi-Polynome und Bessel-Funktionen unter dem Gesichtspunkt der harmonischen Analyse. PhD Thesis, Technische Universität München
[30] Wolfenstetter S (1984) Spectral synthesis on algebras of Jacobi polynomial series. Arch Math **43**: 364–369

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