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Simultaneous Preservation of Orthogonality of Polynomials by Linear Operators Arising from Dilation of Orthogonal Polynomial Systems[1]

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For an orthogonal polynomial system $p = (p_n)_{n \in \mathbb{N}_0}$ and a sequence $d = (d_n)_{n \in \mathbb{N}}0$ of nonzero numbers, let $S_{p,d}$ be the linear operator defined on the linear space of all polynomials via $S_{p,d}\, p_n = d_n\, p_n$ for all $n \in \mathbb{N}_0$.We investigate conditions on p and d under which $S_{p,d}$ can simultaneously preserve the orthogonality of different polynomial systems. As an application, we get that for $p = (\,L_n^{\alpha}\,)$, a generalized Laguerre polynomial system, no d can simultaneously preserve the orthogonality of two additional Laguerre systems, $(\,L_n^{\alpha+t_1}\,)$ and $(\,L_n^{\alpha+t_2}\,)$, where $t_1, t_2 \neq 0$ and $t_1 \neq t_2$. On the other hand, for $p = (\,T_n\,)$, the Chebyshev polynomial system and $d = (\,(-1)^n)$, $S_{p,d}$ simultaneously preserves the orthogonality of uncountably many kernel polynomial systems associated with p. We study many other examples of this type.

KEY WORDS: Orthogonal polynomials; dilation map; kernel polynomials; Jacobi polynomials; Laguerre polynomials.

1. INTRODUCTION

We start with some notation. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set X, $\sharp\, X$ will denote the cardinality of X. Let $\mathcal{P}_1$ be the linear space of all polynomials and let $\mathcal{P}$ be the set of sequences $p = (p_n)_{n \in \mathbb{N}_0}$ in $\mathcal{P}_1$ with $p_0 = 1$ and $\deg p_n = n$ for all $n \in \mathbb{N}_0$. We shall

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often write $(p_n)_{n\in\mathbb{N}_0}$ simply as (p_n) or even p. S will usually stand for a degree preserving linear operator on $\mathcal{P}_1$ to itself with $S(1)=1$. Orthogonality will always be with respect to some *quasi-definite* moment functional, whereas positive orthogonality will be with respect to some *positive* definite moment functional and in the context of real polynomials only (see [5], Theorem I.3.3). The word orthonormality will be used only in the context of positive orthogonality, where, in addition, the positive moment functional $\mathcal{L}$ is normalized so that $\mathcal{L}(1)=1$. For any such pair $(p,\mathcal{L})$ and any polynomial q, $(\mathcal{L}(|q|^2))^{1/2}$ will be denoted by $\|q\|_{\mathcal{L}}$ or $\|q\|_p$; the suffix $\mathcal{L}$ or p will be suppressed if no confusion arises. Orthogonality with respect to a symmetric moment functional (see [5], Definition I.4.1) will be termed symmetric orthogonal. We refer to [2], [5], and [13] for the basics on orthogonal polynomials.

Operators preserving orthogonality of polynomials have been studied by various authors, among them [1], [8], and [10]. We start by quoting two simple recent results from [8] which, to a certain extent, provided the motivation for the present paper. For $\alpha\in\mathbb{R}\setminus(-\mathbb{N})$, let (L_n^α) be the generalized Laguerre polynomial system, given by

$$L_n^\alpha(x)=\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!},\qquad x\in\mathbb{R},\qquad n\in\mathbb{N}_0,$$

where

$$\binom{t}{0}=1,\qquad \binom{t}{k}=\frac{t(t-1)\cdots(t-k+1)}{k!}$$

$$=\frac{(t-k+1)_k}{(1)_k},\qquad t\in\mathbb{R},\qquad k\in\mathbb{N}.$$

From Ref. [8]: Corollary 1. If S preserves the orthogonality of (L_n^α) for $\alpha=\alpha_0-1,\ \alpha_0,\ \alpha_0+1,\ \alpha_0+2$ (with some α_0) and, moreover, $\|\,S\,L_1^\alpha\,\|^2=1+\alpha$ for at least two of $\{\alpha_0,\alpha_0+1,\alpha_0+2\}$, then for some a, $b\in\mathbb{R}$, $(Sp)(x)=p(ax+b)$ for $x\in\mathbb{R}$ and $p\in\mathcal{P}_1$.

From Ref. [8]: Theorem 3. For any $\beta\in\mathbb{R}$, the linear operator S_β defined on $\mathcal{P}_1$ via $(S_\beta L_n^\alpha)(x)=L_n^{\alpha+\beta}(x+\beta),\ x\in\mathbb{R},\ n\in\mathbb{N}$, is independent of α. Further, S_β simultaneously preserves the orthogonality of $(\,L_n^\alpha\,)$ for all α.

Let $\mathcal{P}_m$, $\mathcal{P}_o$, $\mathcal{P}_{mo}$, and $\mathcal{P}_{+o}$ be the subsets of $\mathcal{P}$ consisting of monic, orthogonal, monic orthogonal, and positive orthogonal systems, respectively. Let $\mathcal{D}$ be the set of nonconstant sequences $d=(d_n)_{n\in\mathbb{N}_0}$ of non-zero numbers with $d_0=1$.

In Section 2 we shall define a dilation-type linear operator $S_{p,d}$ on $\mathcal{P}_1$ to itself associated to a p in $\mathcal{P}_o$ and d in $\mathcal{D}$. In fact, $S_{p,d}$ is the linear operator on $\mathcal{P}_1$ to itself, given by $S_{p,d}\, p_n = d_n\, p_n$ for all $n \in \mathbb{N}_0$. We first observe that if $d \in \mathcal{D}$ is the sequence of eigenvalues of a linear differential operator $\mathcal{F}$ such that the corresponding sequence of eigenfunctions consists of an orthogonal polynomial system p, then $S_{p,d}$ coincides with $\mathcal{F}$. (For these topics, one can consult [7] and the related expository articles [4] and [11], or the books [3], Section 3.5, and [5], Section V.2 and Section V.3.) This is illustrated by the following simple examples.

(a) Let $\alpha, \beta > -1$ and $p = (P_n^{(\alpha,\beta)})$, the Jacobi polynomial system. Let $d_n = -n\,(n + \alpha + \beta + 1) + 1$ for $n \in \mathbb{N}_0$. Then $S_{p,d}$ is the differential operator $\mathcal{F}$ determined by

$$(\mathcal{F}f)(x) = (1 - x^2)\, f''(x) + (\beta - \alpha - (\alpha + \beta + 2)\, x) f'(x) + f(x).$$

(b) For $p = (H_n)$, the Hermite polynomial system, and d given by $d_n = -2n + 1$ for $n \in \mathbb{N}_0$, $S_{p,d}$ coincides with the differential operator $\mathcal{F}$ defined by

$$(\mathcal{F}f)(x) = f''(x) - 2xf'(x) + f(x).$$

(c) Let $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$ and $p = (L_n^{\alpha})$, the generalized Laguerre system. If $d_n = -2n + 1$ for $n \in \mathbb{N}_0$, then $S_{p,d}$ coincides with the differential operator $\mathcal{F}$ determined by

$$(\mathcal{F}f)(x) = 2xf''(x) + 2(\alpha + 1 - x)f'(x) + f(x).$$

For a fixed $(p, d) \in \mathcal{P}_o \times \mathcal{D}$ we shall look for those subsets $\mathcal{S}$ of $\mathcal{P}$, which are mapped into $\mathcal{P}_o$ by $S_{p,d}$, i.e., where orthogonality is simultaneously induced by $S_{p,d}$. Also, for a class $\mathcal{S}$ contained in $\mathcal{P}_o$ and a p in $\mathcal{P}_o$, we shall study the question of existence of a d in $\mathcal{D}$ such that $S_{p,d}$ preserves the orthogonality of $\mathcal{S}$.

To motivate, we may note the following result, which follows readily from the connection coefficients of the Jacobi system (cf. [2], 7.32–7.34).

Theorem 1.1. Let $\alpha, \beta > -1$ and p be the Jacobi polynomial system $(P_n^{(\alpha,\beta)})$. Let

$$d_n = (-1)^n \frac{(\alpha + 1)_n}{(\beta + 1)_n} = (-1)^n \frac{\prod_{j=1}^n (\alpha + j)}{\prod_{j=1}^n (\beta + j)}$$

and $d'_n = 1/d_n$ for $n \in \mathbb{N}_0$.

(a) Then, for $t > 0$ and $n \in \mathbb{N}_0$,

$$S_{p,d}\, P_n^{(\alpha,\beta+t)} = d_n\, P_n^{(\alpha+t,\beta)}$$

and

$$S_{p,d'}\, P_n^{(\alpha+t,\beta)} = d'_n\, P_n^{(\alpha,\beta+t)}\ .$$

Therefore $S_{p,d}$ simultaneously preserves the orthogonality of $\{P_n^{(\alpha,\beta+t)}\ ,\ t > 0\}$ and $S_{p,d'}$ that of $\{P_n^{(\alpha+t,\beta)}\ ,\ t > 0\,\}$.

(b) For $\alpha = \beta$, we have $d_n = (-1)^n$, $n \in \mathbb{N}_0$, and $d = d'$. In this case, $S_{p,d}\, P_n^{(\gamma,\gamma)} = (-1)^n P_n^{(\gamma,\gamma)}$ for all $\gamma > -1$, and thus $S_{p,d} = S_{(P_n^{(\gamma,\gamma)}),d}$ for $\gamma > -1$.

This is true for any sequence d satisfying $d_{2n} = 1$ and $d_{2n+1} = d_1$ for all $n \in \mathbb{N}_0$.

Further, for such d's, it is enough to consider the case $p = (P_n^{(-1/2,-1/2)}) = (T_n)$ instead of the whole class $\{P_n^{(\gamma,\gamma)}\ :\ \gamma > -1\}$ of symmetric Jacobi polynomials.

In Section 3, we shall define and study the notion of a $q = (q_n)$ in $\mathcal{P}$ to be analytic with respect to a $p = (p_n)$ in $\mathcal{P}$. Let $\mathcal{P}_{ap}$ be the class of those q's in $\mathcal{P}$, which are analytic with respect to p. It turns out that for a p in $\mathcal{P}_{mo}$ and q in $\mathcal{P}_{ap}$, $q \in \mathcal{P}_o$ if, and only if, q is a kernel polynomial $K(p;y)$ associated with p where $y \notin Z_p$, Z_p being the set of zeros of p_n's. An equivalent form for a p in $\mathcal{P}_{mo}$ is obtained using the general method developed in Section 2. This enables us to have nice transparent conditions on d for $S_{p,d}$ to simultaneously preserve the orthogonality of various subsets of $\mathcal{P}_{ap} \cap \mathcal{P}_o$. For some special d's we can obtain many different subsets of $\mathcal{P}_{ap}$, whose orthogonality is simultaneously preserved by $S_{p,d}$.

In the last section, we apply the results of the second and third sections to the class of generalized Laguerre polynomial systems and obtain a striking contrast to the situation for the Chebyshev polynomials!

2. DILATION-TYPE OPERATORS AND ORTHOGONALITY

2.1. Discussion and Definition

Let $p = (p_n)$ be an orthogonal polynomial system. Then the set $\mathcal{P}$ of all polynomial systems is in one–one correspondence with the set $\mathcal{A}$ of (infinite) matrices $A = (a_{j,k})_{j,k \in \mathbb{N}_0}$ with $a_{j,k} = 0$ for $k > j$, $a_{0,0} = 1$ and $a_{j,j} \neq 0$ for all j via $q = Ap$, i.e., for all $n \in \mathbb{N}_0, q_n = \sum_{k=0}^{n} a_{n,k} p_k$.

(a) Clearly, in the case p has real coefficients, q has real coefficients if, and only if, A is real.
(b) In the case $p \in \mathcal{P}_m$, i.e., p is monic and $A \in \mathcal{A}$, $q = Ap \in \mathcal{P}_m$ if, and only if, $a_{j,j} = 1$ for all j. Such matrices will be called *monic*. The set of monic matrices will be denoted by $\mathcal{A}_m$.
(c) For a $d = (d_n) \in \mathcal{D}$ let $S_{p,d}$ be the linear operator on $\mathcal{P}_1$ to itself given by $S_{p,d}\, p_n = d_n p_n$ for all $n \in \mathbb{N}_0$. The map induced by $S_{p,d}$ on $\mathcal{P}$ will be called the (p, d)-*dilation* and denoted by $S_{p,d}$ only.
(d) Clearly for an $A \in \mathcal{A}$ and $q = Ap$, $S_{p,d}\, q$ corresponds to the matrix $A_d = (a_{j,k}\, d_k) = A\, D$, D being the diagonal matrix given by the diagonal d. In the case p is monic, the associated monic polynomial system Q has the matrix $\big((a_{j,k}\ d_k)/(a_{j,j}\ d_j)\big)$.
(e) For a fixed p, the sequence d and the operator $S_{p,d}$ determine each other. However, for a fixed d, $S_{p,d} = S_{q,d}$ can hold without $p = q$, as is seen from Theorem 1.1. In fact, for $q = Ap$ and $d, d' \in \mathcal{D}$, $S_{p,d} = S_{q,d'}$ if, and only if, $a_{j,k}\,(d'_j - d_k) = 0$ for all $j, k \in \mathbb{N}_0$. As a consequence, if d has all terms distinct and p, q are monic, then $p = q \Longleftrightarrow S_{p,d} = S_{q,d}$. On the other hand, if d has at least two terms coincident then $\{q \text{ monic} : S_{p,d} = S_{q,d}\}$ is infinite.
(f) Monic orthogonal polynomial systems (p_n) can be characterized via their recurrence relation

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0$$

with recurrence sequences $b = (b_n)$, $c = (c_n)$ and $c_n \neq 0$ for all $n \in \mathbb{N}_0$ (see again [5]). For notational convenience, we set $p_{-1} = q_{-1} \equiv 0$, $a_{j,k} = 0$ for $j < 0$ or $k < 0$, $d_{-1} = 0$, and we take empty sums to be zero and empty products to be 1.
(g) I. M. Sheffer [12] defined polynomial systems of type zero as follows. The system $q = (q_n)$ is of *type zero* if there are numbers α_n with $\alpha_1 \neq 0$ such that $\alpha_1 q'_n + \alpha_2 q''_n + \cdots + \alpha_n q_n^{(n)} = q_{n-1}$ for all $n \in \mathbb{N}$. Among other things, he connected systems of type zero to orthogonal systems by dilations. He proved ([12], Theorem 4.1): Assume that a polynomial system $q = (d_n\, p_n)$ is given with $d \in \mathcal{D}$. Then q is of type zero if, and only if, there exist numbers $a_1, \ldots, a_4$ with

$$b_n = a_1 + a_2\, n \quad \text{and} \quad c_n = n\,(a_3 + a_4\, n) \qquad \text{for all } n \in \mathbb{N}_0.$$

Theorem 2.2. Let $p = (p_n)$ be a monic orthogonal polynomial system with recurrence sequences $b = (b_n)$ and $c = (c_n)$. Let $A = (a_{j,k})$ be a monic matrix and $q = (q_n) = Ap$.

(a) The system q is orthogonal if, and only if,

(i) $a_{2,0} - a_{2,1}a_{1,0} \neq c_1 + (b_0 - b_1)a_{1,0} - (a_{1,0})^2$,

(ii) $a_{n+1,n-1} - a_{n+1,n}a_{n,n-1}$
$$\neq c_n + (b_{n-1} - b_n)a_{n,n-1} - (a_{n,n-1})^2 + a_{n,n-2}, \quad n \geq 2,$$

(iii) $(a_{n+1,0} - a_{n+1,n}a_{n,0}) - (a_{n+1,n-1} - a_{n+1,n}a_{n,n-1})a_{n-1,0}$
$$= [(b_0 - b_n) - a_{n,n-1}]a_{n,0} + c_1 a_{n,1} - [c_n + (b_{n-1} - b_n)a_{n,n-1} - (a_{n,n-1})^2 + a_{n,n-2}]a_{n-1,0}, \quad n \geq 2,$$

and

(iv) $(a_{n+1,j} - a_{n+1,n}a_{n,j}) - (a_{n+1,n-1} - a_{n+1,n}a_{n,n-1})a_{n-1,j}$
$$= a_{n,j-1} + [b_j - b_n - a_{n,n-1}]a_{n,j} + c_{j+1}a_{n,j+1} - [c_n + (b_{n-1} - b_n)a_{n,n-1} - (a_{n,n-1})^2 + a_{n,n-2}]a_{n-1,j},$$
$$n \geq 3, \qquad 1 \leq j \leq n-2.$$

(b) Suppose p is positive orthogonal and A is real. Then q is positive orthogonal if, and only if, all the conditions in (a) above are satisfied with $\neq$ in (i) and (ii) strengthened to $<$.

Proof. (a) By Favard's Theorem ([5], Theorem I.4.4), combined with the fundamental recurrence relation ([5], Theorem I.4.1), q is orthogonal if, and only if, there exist sequences $\beta = (\beta_n)_{n \in \mathbb{N}_0}$, $\gamma = (\gamma_n)_{n \in \mathbb{N}_0}$, γ_0 arbitrary (and ineffective) such that for each $n \in \mathbb{N}$, $\gamma_n \neq 0$

$$x q_n(x) = q_{n+1}(x) + \beta_n q_n(x) + \gamma_n q_{n-1}(x), \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N}_0.$$

Using the recurrence relation for the p_n's, this happens if, and only if, for all $n \in \mathbb{N}_0$,

$$\sum_{j=0}^{n} a_{n,j}(p_{j+1} + b_j p_j + c_j p_{j-1}) = \sum_{j=0}^{n+1} a_{n+1,j}p_j + \beta_n \sum_{j=0}^{n} a_{n,j}p_j + \gamma_n \sum_{j=0}^{n-1} a_{n-1,j}p_j.$$

Since the p_j's are linearly independent, we can equate their coefficients on both sides, and simple computations then give that the conditions stated in the theorem are necessary and sufficient. Along the way we also have that

$$\beta_0 = b_0 - a_{1,0},$$

$$\beta_1 = b_1 + a_{1,0} - a_{2,1},$$

$$\gamma_1 = c_1 + \big[b_0 - b_1 - a_{1,0}\big]a_{1,0} - (a_{2,0} - a_{2,1}a_{1,0}),$$

and for $n \geq 2$,

$$\beta_n = b_n + a_{n,n-1} - a_{n+1,n}\,,$$

$$\gamma_n = c_n + \big[(b_{n-1} - b_n) - a_{n,n-1}\big]a_{n,n-1}$$

$$+ a_{n,n-2} - (a_{n+1,n-1} - a_{n+1,n}a_{n,n-1}).$$

(b) We shall again use the fundamental recurrence relation and Favard's Theorem. We have that for each n, b_n is real and $c_{n+1} > 0$. Further, q is positive orthogonal if, and only if, for each n, β_n is real and $\gamma_{n+1} > 0$. This gives us that $\neq$ in (i) and (ii) of (a) have to be replaced by $<$. □

Remark 2.3.

(a) The above theorem tells us how to construct all infinite matrices which preserve orthogonality or positive orthogonality.

Take any sequence $s = (s_n)_{n\in\mathbb{N}_0}$ of numbers and another sequence $t = (t_n)_{n\in\mathbb{N}_0}$ of numbers satisfying

$$t_0 \neq c_1 + s_0[(b_0 - b_1) - (s_0 - s_1)],$$

$$t_n - t_{n-1} \neq c_{n+1} + s_n[(b_n - b_{n+1}) - (s_n - s_{n+1})], \qquad n \geq 1.$$

Set $a_{n+1,n} = s_n$ and $a_{n+2,n} = t_n$ for all n. Define, for $n \geq 2,\ 0 \leq j \leq n-2$, the numbers $a_{n+1,j}$ recursively as in (a) (iii) and (iv). Finally, take $a_{j,j} = 1$ and $a_{j,k} = 0$ for $k > j$, and obtain $A_{s,t} = (a_{j,k})$. Then $q = A_{s,t}\,p$ is orthogonal. Also, every monic matrix A having this preservation property has this form for a unique pair (s,t).

(b) In case p is positive orthogonal, we have to restrict s and t to be real sequences and take

$$t_0 < c_1 + s_0[(b_0 - b_1) - (s_0 - s_1)],$$

$$t_n - t_{n-1} < c_{n+1} + s_n[(b_n - b_{n+1}) - (s_n - s_{n+1})], \qquad n \geq 1.$$

(c) On the other hand, if p is symmetric orthogonal, then $b_n = 0$ for all n. Thus q is symmetric orthogonal if, and only if, $\beta_n = 0$ is satisfied together with (a) of Theorem 2.2, i.e., the sequence $s = (s_n)$ is constant. This simplifies the condition on t to $t_n - t_{n-1} \neq c_{n+1}$ for all $n \in \mathbb{N}$, and it simplifies (a) as follows:

(i) $a_{2,0} \neq c_1$,
(ii) $a_{n+1,n-1} \neq a_{n,n-2} + c_n, \quad n \geq 2$,
(iii) $a_{n+1,0} = c_1\, a_{n,1} + (a_{n+1,n-1} - a_{n,n-2} - c_n)\, a_{n-1,0}, \quad n \geq 2$,

and

(iv) $a_{n+1,j} = a_{n,j-1} + c_{j+1}\, a_{n,j+1} + (a_{n+1,n-1} - a_{n,n-2} - c_n)\, a_{n-1,j}, \quad n \geq 3, \quad 1 \leq j \leq n-2.$

As before, for positive orthogonality, in (i) and (ii), $\neq$ has to be replaced by $<$.

(d) Given a monic matrix A, the above theorem also gives us conditions on the sequences b and c for the polynomial system $q = Ap$ to be orthogonal.

Theorem 2.4. Let A be a monic infinite matrix, d a sequence of nonzero numbers with $d_0 = 1$ and $p = (p_n)$ a monic orthogonal system with recurrence sequences b and c. Let $q = A\,p$ and $r = S_{p,d}\, q$.

(a_d) The system r is orthogonal if, and only if,

(i) $$\frac{1}{d_2}(a_{2,0} - a_{2,1}a_{1,0}) \neq c_1 + \frac{1}{d_1}(b_0 - b_1)a_{1,0} - \left(\frac{1}{d_1}a_{1,0}\right)^2,$$

(ii) $$\frac{d_{n-1}}{d_{n+1}}(a_{n+1,n-1} - a_{n+1,n}a_{n,n-1}) \neq c_n + \frac{d_{n-1}}{d_n}(b_{n-1} - b_n)a_{n,n-1} - \left(\frac{d_{n-1}}{d_n}a_{n,n-1}\right)^2 + \frac{d_{n-2}}{d_n}a_{n,n-2}, \; n \geq 2,$$

(iii) $$\frac{1}{d_{n+1}}\left[(a_{n+1,0} - a_{n+1,n}a_{n,0}) - (a_{n+1,n-1} - a_{n+1,n}a_{n,n-1})a_{n-1,0}\right] = \frac{1}{d_n}\left[\left(b_0 - b_n - \frac{d_{n-1}}{d_n}a_{n,n-1}\right)a_{n,0} + d_1c_1a_{n,1} - \frac{d_n}{d_{n-1}}\left(c_n + \frac{d_{n-1}}{d_n}(b_{n-1} - b_n)a_{n,n-1} - \left(\frac{d_{n-1}}{d_n}a_{n,n-1}\right)^2 + \frac{d_{n-2}}{d_n}a_{n,n-2}\right)a_{n-1,0}\right], \qquad n \geq 2,$$

$$\text{(iv)}\ \frac{1}{d_{n+1}}\left[(a_{n+1,j}-a_{n+1,n}a_{n,j})-(a_{n+1,n-1}-a_{n+1,n}a_{n,n-1})a_{n-1,j}\right]$$
$$=\frac{1}{d_n}\left[\frac{d_{j-1}}{d_j}a_{n,j-1}+\left(b_j-b_n-\frac{d_{n-1}}{d_n}a_{n,n-1}\right)a_{n,j}\right.$$
$$+\frac{d_{j+1}}{d_j}c_{j+1}a_{n,j+1}$$
$$-\left(\frac{d_n}{d_{n-1}}c_n+(b_{n-1}-b_n)a_{n,n-1}-\frac{d_{n-1}}{d_n}(a_{n,n-1})^2\right.$$
$$\left.\left.+\frac{d_{n-2}}{d_{n-1}}a_{n,n-2}\right)a_{n-1,j}\right],\ n\geq 3\,,\quad 1\leq j\leq n-2.$$

(b_d) Suppose p is positive orthogonal. Then q is positive orthogonal if, and only if, $\big((d_k/d_j)a_{j,k}\big)$ is a real matrix and all the conditions in (a_d) above are satisfied with $\neq$ in (i) and (ii) strengthened to $<$.

Proof. (a_d) We first note that r is orthogonal if, and only if, the corresponding monic polynomial system $R=(R_n)=((1/d_n)r_n)$ is orthogonal. The matrix associated with R is $\widetilde{A}=(\widetilde{a}_{j,k})$, given by $\widetilde{a}_{j,k}=(d_k/d_j)a_{j,k}$ for $j,k\in\mathbb{N}_0$. Thus, by Theorem 2.2 above, R is orthogonal if, and only if, $\widetilde{A}$ satisfies the Condition 2.2(a) where a is replaced by $\widetilde{a}$ everywhere. We already bracketed the expressions in 2.2(a) in such a way that (a) transforms to (a_d) immediately.

(b_d) By 2.1(a), $\widetilde{A}$ has to be real in the first place. The rest follows from (a_d) in the same manner as (b) does from (a) in Theorem 2.2. □

Remark 2.5.

(a) The above theorem is a two-edged sword, although not equally sharp! The sharper one says that if a d is given, then we can construct all those A's for which r is orthogonal (respectively, positive orthogonal in case p is so). All we have to do is to choose any sequence (s_n) of numbers and then take another sequence (t_n) satisfying

$$\frac{1}{d_2}t_0\neq c_1+\frac{1}{d_1}s_0\left[(b_0-b_1)-\left(\frac{1}{d_1}s_0-\frac{d_1}{d_2}s_1\right)\right],$$

and, for $n \geq 1$,

$$\frac{d_n}{d_{n+2}} t_n - \frac{d_{n-1}}{d_{n+1}} t_{n-1} \neq c_{n+1} + \frac{d_n}{d_{n+1}} s_n \left[(b_n - b_{n+1}) - \left(\frac{d_n}{d_{n+1}} s_n - \frac{d_{n+1}}{d_{n+2}} s_{n+1} \right) \right] .$$

Then we can set $a_{n+1,n} = s_n$ and $a_{n+2,n} = t_n$ for all $n \in \mathbb{N}$. Next we can define, for $n \geq 2$ and $0 \leq j \leq n-1$, the numbers $a_{n+1,j}$ recursively using (iii) and (iv) of (a_d). Finally, take $a_{j,j} = 1$ for all j and $a_{j,k} = 0$ for all $k > j$ and obtain $A_{s,t} = (a_{j,k})$. Then for $q = A_{s,t}\, p$, the system $S_{p,d}\, q$ is orthogonal. In addition, every monic matrix A having this preservation property has this form for a unique (s, t).

In case p is positive orthogonal, we have to restrict $s = (s_n)$ so that $(d_n/(d_{n+1}))s_n$ is real for all n and $t = (t_n)$ so that $(d_n/(d_{n+2}))t_n$ is real for all n, and strengthen the inequalities to $<$ in the above construction.

(b) We now come to the question of determining the sequence d when a monic matrix A is given. For $j, n \in \mathbb{N}$, we put

$$u_n = a_{n+1,n-1} - a_{n+1,n} a_{n,n-1} = - \begin{vmatrix} a_{n,n-1} & 1 \\ a_{n+1,n-1} & a_{n+1,n} \end{vmatrix},$$

$$v_{n,j} = a_{n+1,j} - a_{n+1,n} a_{n,j},$$

and for $n \geq 2$, $0 \leq j \leq n-2$, we put

$$w_{n,j} = v_{n,j} - u_n a_{n-1,j} = \begin{vmatrix} a_{n-1,j} & 1 & 0 \\ a_{n,j} & a_{n,n-1} & 1 \\ a_{n+1,j} & a_{n+1,n-1} & a_{n+1,n} \end{vmatrix} .$$

For each $n \geq 2$, one of the following two conditions is satisfied:

(n1) for some j_n, $0 \leq j_n \leq n-2$, $w_{n,j_n} \neq 0$, $w_{n,j} = 0$, $0 \leq j < j_n$.

(n2) $w_{n,j} = 0$, $0 \leq j_n \leq n-2$. This holds if, and only if, for $0 \leq j \leq n$, $a_{n+1,j} = a_{n+1,n} a_{n,j} + u_n a_{n-1,j}$ if, and only if, $q_{n+1} = p_{n+1} + a_{n+1,n} q_n + u_n q_{n-1}$.

Let

$$T_1 = \{ n : n \geq 2, \quad \text{(n1) holds} \},$$

and

$$T_2 = \{ n : n \geq 2, \quad \text{(n2) holds } \} \cup \{0, 1\}.$$

The problem reduces to finding a suitable function ξ on the set T_2 to the set of nonzero numbers such that, setting $d_{n+1} = \xi_n$ for $n \in T_2$ and then defining the remaining d_n's recursively using (a$_d$)(iii) if $j_{n-1} = 0$ and (a$_d$)(iv) if $j_{n-1} > 0$, the condition (a$_d$) [respectively, (b$_d$)] is satisfied.

Theorem 2.6. Let p, A, d, q and r be as in Theorem 2.4 above.

(a) We may combine the conditions (a) of Theorem 2.2 and (a$_d$) of Theorem 2.4 to obtain necessary and sufficient conditions on A and d for both q and r to be orthogonal.

(b) For positive orthogonality, we have to combine (b) of Theorem 2.2 with (b$_d$) of Theorem 2.4 which, in turn, put together are equivalent to:

(i) Both A and d are real,

(ii) $a_{2,0} - a_{2,1}a_{1,0} < c_1 + (b_0 - b_1)a_{1,0} - (a_{1,0})^2,$

and

$$\frac{1}{d_2}\left(a_{2,0} - a_{2,1}a_{1,0}\right) < c_1 + \frac{1}{d_1}(b_0 - b_1)a_{1,0} - \left(\frac{1}{d_1}a_{1,0}\right)^2;$$

(iii) $$a_{n+1,n-1} - a_{n+1,n}a_{n,n-1}$$

$$< c_n + (b_{n-1} - b_n)a_{n,n-1} - (a_{n,n-1})^2 + a_{n,n-2}$$

and

$$\frac{d_{n-1}}{d_{n+1}}\left(a_{n+1,n-1} - a_{n+1,n}a_{n,n-1}\right)$$
$$< c_n + \frac{d_{n-1}}{d_n}(b_{n-1} - b_n)a_{n,n-1} - \left(\frac{d_{n-1}}{d_n}a_{n,n-1}\right)^2$$
$$+ \frac{d_{n-2}}{d_n}a_{n,n-2}, \quad n \geq 2;$$

(iv) $$(a_{n+1,j} - a_{n+1,n}a_{n,j}) - (a_{n+1,n-1} - a_{n+1,n}a_{n,n-1})a_{n-1,j}$$

$$= a_{n,j-1} + (b_j - b_n - a_{n,n-1})a_{n,j} + c_{j+1}a_{n,j+1}$$
$$- \left[c_n + (b_{n-1} - b_n)a_{n,n-1} - (a_{n,n-1})^2 + a_{n,n-2}\right]a_{n-1,j},$$

and

$$\left(1 - \frac{d_{n+1}\,d_{j-1}}{d_n\,d_j}\right)a_{n,j-1}$$
$$+ \left[\left(1 - \frac{d_{n+1}}{d_n}\right)(b_j - b_n) - \left(1 - \frac{d_{n+1}\,d_{n-1}}{d_n^2}\right)a_{n,n-1}\right]a_{n,j}$$

$$+\left(1-\frac{d_{n+1}\,d_{j+1}}{d_n\,d_j}\right)c_{j+1}a_{n,j+1}$$
$$-\left[\left(1-\frac{d_{n+1}}{d_{n-1}}\right)c_n+\left(1-\frac{d_{n+1}}{d_n}\right)(b_{n-1}-b_n)a_{n,n-1}\right.$$
$$\left.+\left(1-\frac{d_{n+1}\,d_{n-2}}{d_n\,d_{n-1}}\right)a_{n,n-2}-\left(1-\frac{d_{n+1}\,d_{n-1}}{d_n^2}\right)(a_{n,n-1})^2\right]a_{n-1,j}=0,$$

$n\geq 2\,,\ 0\leq j\leq n-2.$

(c) For a set $\mathcal{A}$ of monic matrices, we can have a set of conditions on d in order that the members of $\{S_{p,d}\,Ap : A\in\mathcal{A}\}$ are all orthogonal or positive orthogonal systems.

In this general form these conditions look formidable. We shall discuss a few special cases here and in the next two sections.

Corollary 2.7. Let $d=((-1)^n)$, p a monic orthogonal system with recurrence sequences b and c, and A a monic infinite matrix. Let $q=A\,p$ and $r=S_{p,d}\,q$.

(a) (i) In case $(b_j-b_n)a_{n,j}=0$ for $0\leq j<n$, q is orthogonal if, and only if, r is orthogonal.
(ii) Suppose b is a constant sequence. Then $q\in\mathcal{P}_o$ if, and only if, $S_{p,d}\,q\in\mathcal{P}_o$. Thus $S_{p,d}\,\mathcal{P}_o=\mathcal{P}_o$. In particular, it is so if p is a symmetric orthogonal system. In this case, we also have that q is symmetric if, and only if, r is.
(iii) Suppose b is eventually constant, i.e. , for a least number $j_1\in\mathbb{N}_0\,,\quad b_n=b_j$ for all $n\geq j_1$. Then $(\,S_{p,d}\,\mathcal{P}_o\,)\cap\mathcal{P}_o$ is infinite.

(b) Suppose p is positive orthogonal. Then (i), (ii) and (iii) above hold with orthogonal replaced by positive orthogonal and $\mathcal{P}_o$ by $\mathcal{P}_{+o}$.

Proof. (a) (i) In this case Condition (a) of Theorem 2.2 and Condition (a_d) of Theorem 2.4 become equivalent.

(ii) Because b is a constant sequence, every monic matrix A satisfies the requirement in (i). For symmetric orthogonal polynomial systems, we have $b_n=0$ for all n.

(iii) Let $\mathcal{A}_{j_1}=\{\,A\in\mathcal{A}_m\;:\;a_{n,j}=0$ for $0\leq j\leq n\,,\ j<j_1\,\}$. Then every A in $\mathcal{A}_{j_1}$ satisfies the requirement in (i) and, therefore, $q=A\,p$ is orthogonal if, and only if, $S_{p,d}\,q$ is orthogonal if, and only if, A satisfies (a) of Theorem 2.2. There are infinitely many matrices in $\mathcal{A}_{j_1}$ satisfying

this condition, as can be easily seen, because a_{j_0+1,j_0} can be chosen arbitrarily.

(b) In the above proof of Part (a), we only have to use (b) of Theorem 2.2 and (b_d) of Theorem 2.4 instead of (a) and (a_d). □

Remark 2.8.

(a) Examples of p satisfying condition (iii) of Corollary 2.7 above are provided by Modified Lommel polynomials, Tricomi–Carlitz polynomials and polynomials related to Bernoulli numbers (cf. [5], VI.6, VI.7 and VI.8).

(b) For any d_1 with $0 \neq d_1^2 \neq 1$, define the sequence $d = (d_n) = (d_1^n)$. Let p be a symmetric orthogonal system and A, q, r, as in the above corollary. Then both q and r are symmetric orthogonal if, and only if, $a_{j,k} = 0$ for $0 \leq k \leq j-3\,,\ j \geq 3\,,\ a_{k+2,k} = a_{2,0} \neq c_1$ and $a_{k+1,k} = a_{1,0}\,,$ $k \in \mathbb{N}_0$. For the positive orthogonal case, the extra conditions $d_1\,,\ a_{1,0}\,,\ a_{2,0} \in \mathbb{R}$ and $a_{2,0} < \min\{\, c_1\,,\ d_1^2 c_1 \,\}$ are needed.

3. KERNEL POLYNOMIALS AND DILATIONS

3.1. Discussion and Definition

Let $p = (p_n)$ be a monic orthogonal polynomial system with the associated recurrence sequences $b = (b_n)$ and $c = (c_n)$. Let Z_p be the set of zeros of $p_n\,,\ n \in \mathbb{N}_0$, and I_p the open interval $(\inf Z_p\,,\ \sup Z_p\,)$. For a number y not in Z_p, let $p^y = (p_n^y)$ be the corresponding monic kernel polynomial system. We may refer to [5], Section 7, for this and related results to be used in this section.

(i)

$$p_n^y = \frac{c_1^n}{p_n(y)} \sum_{k=0}^{n} \frac{p_k(y)}{c_1^k}\, p_k\,, \qquad \text{where } c_j^k = \prod_{l=j}^{k} c_l\ .$$

Thus, the associated monic matrix $A^y = (a_{n,k}^y)$ satisfies

$$a_{n,k}^y = c_{k+1}^n \frac{p_k(y)}{p_n(y)} \qquad \text{for } 0 \leq k < n\ .$$

(ii) Also, $p^y = (p_n^y)$ is orthogonal.

(iii) Further, if p is positive orthogonal, then p^y is positive orthogonal if, and only if, $y \in \mathbb{R} \setminus I_p$. In this case, we denote by $\widetilde{p} = (\widetilde{p}_n)$ the corresponding orthonormal system. Then $\widetilde{p}_n = 1/\sqrt{c_1^n}\, p_n$ for $n \in \mathbb{N}$, and thus, for $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} \widetilde{p}_j(y)\,\widetilde{p}_j = (c_1^n)^{-1}\,p_n(y)\,p_n^y\,.$$

This motivates the definition and the name of the main object of study in this section.

(iv) A polynomial system $q = (q_n)$ will be called *analytic with respect to a polynomial system* $p = (p_n)$, in short, p-*analytic*, if there is a sequence $h = (h_n)$ of nonzero numbers with $h_0 = 1$ such that $q_n = \sum_{k=0}^{n} h_k\,p_k$, $n \in \mathbb{N}_0$. The sequence (h_n) will be called the *coefficient sequence* for q. If p is monic, then the monic system Q corresponding to q is given by the matrix $A = (a_{n,k})$ (with respect to p) that satisfies $a_{n,k} = h_k/h_n,\ 0 \le k \le n$. We shall say that q is *monic analytic with respect to* p, in short, p-*monic analytic*. Any monic matrix $A = (a_{n,k})$ having $a_{n,k}$ of the form $h_k/h_n,\ 0 \le k \le n$ for a sequence $(h_n) \in \mathcal{D}$ gives rise to a q that is p-monic analytic. We note that A and h determine each other uniquely.

(v) If q is p-analytic or p-monic analytic, then for $d, d' \in \mathcal{D}$, we have $S_{p,d} \neq S_{q,d'}$.

Theorem 3.2. Let p be a monic orthogonal system with recurrence sequences b and c, and let q be monic analytic with respect to p with monic analytic sequence h.

(a) The following conditions are equivalent:

(i) q is orthogonal,

(ii) $(b_0 + c_1 h_1)h_{n+1} = h_n + b_{n+1}h_{n+1} + c_{n+2}h_{n+2}\,,\ n \in \mathbb{N}_0$,

(iii) $h_n = 1/c_1^n\ p_n(\,b_0 + c_1 h_1\,)\,,\ n \in \mathbb{N}\,$,

(iv) $q = p^y$ with $y = b_0 + c_1 h_1$.

(b) Suppose p is positive orthogonal. Then q is positive ortho–gonal if, and only if, $q = p^y$ for some $y \in \mathbb{R} \setminus I_p$. Further, y and (h_n) are related via

$$y = b_0 + c_1 h_1 \qquad \text{and} \qquad h_n = \frac{1}{c_1^n}\,p_n(y)\,,\ n \in \mathbb{N}.$$

Proof. (a) We first note that the monic matrix A for q satisfies $a_{n,k}\,a_{k,l} = a_{n,l}\,,\ 0 \le l \le k \le n$. In this case (a)(iii) and (a)(iv) of Theorem 2.2 are equivalent.

Let q be orthogonal. Then (ii) follows from (a)(iii) of Theorem 2.2. Moreover, (ii) gives (a)(i) and (a)(ii) of Theorem 2.2 simply because $c_n \neq 0 \neq h_n$ for all n. This shows the equivalence of (i) and (ii).

Let, for $n \in \mathbb{N}$, $h'_n = c_1^n h_n$ and $h'_0 = 1$. We put $y = b_0 + c_1 h_1$. Then for $n \in \mathbb{N}_0$, $y\, h'_n = h'_{n+1} + b_n\, h'_n + c_n\, h'_{n-1}$. This gives that $h'_n = p_n(y)$ for $n \in \mathbb{N}_0$, which readily implies (iii).

(iii) ⇒ (iv) follows from 3.1(i), and (iv) ⇒ (i) is immediate from 3.1(ii).

(b) We only have to combine part (a) with 3.1(iii). □

Theorem 3.3. Let p be an orthogonal polynomial system with recurrence relation

$$xp_n(x) = \eta_n\, p_{n+1}(x) + \xi_n\, p_n(x) + \zeta_n\, p_{n-1}(x)\,, \qquad x \in \boldsymbol{R}, \qquad n \in \mathbb{N}_0\,,$$

$$p_0(x) = 1, \qquad p_{-1}(x) = 0\ .$$

Let $w_n = \prod_{j=0}^{n-1} \eta_j/\zeta_{j+1}$ for $n \in \mathbb{N}_0$ and $w_0 = 1$. Let q be p-analytic with coefficient sequence $h = (h_n)$.

(a) The system q is orthogonal if, and only if, for some number $y \notin Z_p$, we have $h_n = w_n\, p_n(y)$, $n \in \mathbb{N}_0$.

(b) Suppose p is a positive orthogonal polynomial system.

(i) The system q is orthogonal if, and only if, for some $y \in \mathbb{R} \setminus Z_p$,

$$h_n = \frac{p_n(y)}{\|p_n\|_p^2}\,, \qquad n \in \mathbb{N}_0.$$

In particular, if p is an orthonormal polynomial system, then the condition reduces to $h_n = p_n(y)$, $n \in \mathbb{N}_0$.

(ii) The system q is positive orthogonal if, and only if, y in (i) above satisfies $y \le \inf Z_p$ or $y \ge \sup Z_p$. Moreover, if p is monic then, in the former case, $(-1)^n\, h_n > 0$ for all n, while in the latter case, $h_n > 0$ for all n.

Proof. (a) Let b and c be the recurrence sequences for the monic polynomial system P corresponding to p. Then q is P-analytic as well with the coefficient sequence $(h_n\, (\prod_{j=0}^{n-1} \eta_j)^{-1})$. We note that $\eta_n = \sigma_n/\sigma_{n+1}$ and $\zeta_n = c_n(\sigma_{n+1}/\sigma_{n-1})\eta_n$, where σ_n is the leading coefficient of the polynomial p_n. From this we can derive $w_n = 1/c_1^n\, \sigma_n^2$. All that we have to do now is to apply the above theorem to P.

(b) In this case, w_n above has the value $\|p_n\|_p^{-2}$ for each n. Finally, $p_n(x)$ retains the same sign in $(-\infty, \inf Z_p\,]$ and the same sign in $[\,\sup Z_p, \infty)$. □

Theorem 3.4. Let $p = (p_n)$ be an orthogonal polynomial system with η, ξ, ζ, w as in Theorem 3.3 above and $d \in \mathcal{D}$.

(a) $S_{p,d}\,\mathcal{P}_{ap} \subset \mathcal{P}_{ap}$.

(b) $\big(S_{p,d}\,(\mathcal{P}_{ap} \cap \mathcal{P}_o)\big) \cap \mathcal{P}_o \neq \emptyset$ if, and only if, there exist distinct numbers y_1 and y_2, both not in Z_p, satisfying $d_n\,p_n(y_1) = p_n(y_2)$ for all n.

(c) $\sharp\,[(\,S_{p,d}\big(\mathcal{P}_{ap} \cap \mathcal{P}_o\big)\,) \,\cap\, \mathcal{P}_o] \geq 2$ if, and only if, there exist distinct pairs (y_1, y_2) and (y_3, y_4) with $y_i \notin Z_p$, $y_1 \neq y_2$, $y_1 \neq y_3$, $y_2 \neq y_4$, $y_3 \neq y_4$, such that

$$d_n = \frac{p_n(y_2)}{p_n(y_1)} = \frac{p_n(y_4)}{p_n(y_3)} \qquad \text{for all } n.$$

(d) If p is positive orthogonal, then we can replace $\mathcal{P}_o$ by $\mathcal{P}_{+o}$, provided we make the y_j's satisfy $y_j \in \mathbb{R} \setminus I_p$ instead of just $y_j \notin Z_p$.

Proof. For an $h \in \mathcal{D}$ and $q_n = \sum_{j=0}^{n} h_j\,p_j$ for $n \in \mathbb{N}_0$, we have $S_{p,d}\,q_n = \sum_{j=0}^{n} d_j h_j p_j$, $n \in \mathbb{N}_0$. This gives (a).

Further, by Theorem 3.3, both q and $S_{p,d}\,q$ can be in $\mathcal{P}_o$ if, and only if, there exist distinct numbers y_1 and y_2 not in Z_p satisfying

$$h_n = w_n\,p_n(y_1), \qquad h_n d_n = w_n\,p_n(y_2) \text{ for all } n, \text{ i.e.,}$$

$$h_n = w_n\,p_n(y_1), \qquad d_n p_n(y_1) = p_n(y_2) \text{ for all } n \;.$$

Thus we have (b).

(c) and (d) follow immediately from (b) in view of 3.1. □

Corollary 3.5. Let $p = (p_n)$ be a symmetric orthogonal polynomial system. Then for any two distinct nonzero numbers y_1 and y_2, both not in Z_p, and $d_n = p_n(y_2)/p_n(y_1)$, $n \in \mathbb{N}_0$, we have $\sharp[S_{p,d}(\mathcal{P}_{ap} \cap \mathcal{P}_o) \cap \mathcal{P}_o] \geq 2$.

Proof. We may take $y_3 = -y_1$ and $y_4 = -y_2$. □

Remark 3.6.

(a) If we choose $y_2 = -y_1$ in the above corollary, then we always get the sequence $d = ((-1)^n)$ and, thus, $S_{p,d}\big(\mathcal{P}_{ap} \cap \mathcal{P}_o\big) = \mathcal{P}_{ap} \cap \mathcal{P}_o$. This follows from Corollary 2.7, as well!

(b) We may use Mehler's formula ([13], Ex. 23, p. 377 or [2], 2.44 on p. 16) to deduce that in the case of Hermite polynomials there are no other solutions for the condition in Theorem 2.4(iii) except those given by the proof in Corollary 3.5 or (a) above. The same is true for Chebyshev polynomials of the first, as well as the second, kind, as can be easily checked by just considering the case $n = 1$ and 2. Thus, for these polynomials the following statements hold.

(i) For $d_n = (-1)^n$, we have $S_{p,d}(\mathcal{P}_{ap} \cap \mathcal{P}_o) = \mathcal{P}_{ap} \cap \mathcal{P}_o$.
(ii) For $d_n = p_n(y_2)/p_n(y_1)$ with y_1, y_2 two distinct numbers, both not in Z_p, and $y_2 \neq -y_1$, we have

$$S_{p,d}(\mathcal{P}_{ap} \cap \mathcal{P}_o) \cap \mathcal{P}_o = \{ p^{y_2}, p^{(-y_2)} \}.$$

(iii) For all other d's, we have $S_{p,d}(\mathcal{P}_{ap} \cap \mathcal{P}_o) \cap \mathcal{P}_o = \emptyset$.

(c) We can use [13], Theorem 5.1, or [2], 2.43, to show that there are no solutions to condition (iii) of Theorem 3.4 when we take p to be any Laguerre polynomial system and thus we have that for distinct y_1, y_2 not in Z_p and $d_n = p_n(y_2)/p_n(y_1)$ for all n, $S_{p,d}(\mathcal{P}_{ap} \cap \mathcal{P}_o) \cap \mathcal{P}_o = \{p^{y_2}\}$.

(d) As already noted in Theorem 3.3(b)(i), the coefficient sequence takes a simple form for an orthonormal polynomial system p. In fact, a direct proof of this part of Theorem 3.3 can be given using the orthogonality measure μ.

Theorem 3.7.

(a) Let p be an orthogonal polynomial system with recurrence sequences a, b, c, i.e.,

$$xp_n(x) = a_n\, p_{n+1}(x) + b_n\, p_n(x) + c_n\, p_{n-1}(x), \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N}_0.$$

Let g be a polynomial system such that p is g-analytic with constant one as the coefficient sequence. Then g is orthogonal if, and only if,

$$a_n + b_n + c_n = a_1 + b_1 + c_1 \neq a_0 + b_0, \qquad n \geq 2.$$

(b) Let (g, p, q) be a triple of polynomial systems such that p is g-analytic and q is p-analytic with constant one as the coefficient sequence in each case. Then g, p, q are all orthogonal if, and only if, p is orthogonal and the recurrence sequences a, b, c of p satisfy

(i) $a_n + b_n + c_n = a_1 + b_1 + c_1 \neq a_0 + b_0, \quad n \geq 2,$ and
(ii) $a_n + b_{n+1} + c_{n+2} = b_0 + c_1, \quad n \in \mathbb{N}_0.$

(c) The conditions in (b) are equivalent to

(i)′ $a_1 + b_1 + c_1 \neq a_0 + b_0,$
(ii)′ $a_{n-1} + (b_n - b_0) + (c_{n+1} - c_1) = 0, \quad n \geq 1,$ and
(iii)′ $(a_{n+1} - a_{n-1} - a_1) + [(b_{n+1} - b_n) - (b_1 - b_0)] = 0, \quad n \geq 1.$

Proof. (a) Suppose that g is orthogonal. Let α, β, γ be its recurrence sequences. Then $\alpha_n, \gamma_{n+1} \neq 0$, $n \in \mathbb{N}_0$. Also,

$$\sum_{j=0}^{n} (\alpha_j\, g_{j+1}(x) + \beta_j\, g_j(x) + \gamma_j\, g_{j-1}(x))$$

$$= \sum_{j=0}^{n} x g_j(x) = x\, p_n(x)$$

$$= \; a_n \sum_{j=0}^{n+1} g_j(x) + b_n \sum_{j=0}^{n} g_j(x)$$

$$+ c_n \sum_{j=0}^{n-1} g_j(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

This implies

$$\sum_{j=0}^{n} (\alpha_j\, g_{j+1} + \beta_j\, g_j + \gamma_j\, g_{j-1})$$

$$= \; a_n \sum_{j=0}^{n+1} g_j + b_n \sum_{j=0}^{n} g_j + c_n \sum_{j=0}^{n+1} g_j, \quad n \in \mathbb{N}_0.$$

Equating coefficients of g_j on both sides gives

$$\alpha_0 = a_0, \quad \beta_0 = a_0 + b_0,$$

$$\alpha_1 = a_1, \quad \alpha_0 + \beta_1 = a_1 + b_1, \quad \beta_0 + \gamma_1 = a_1 + b_1 + c_1,$$

$$\alpha_n = a_n, \quad \alpha_{n-1} + \beta_n = a_n + b_n,$$

$$\alpha_{j-1} + \beta_j + \gamma_{j+1} = a_n + b_n + c_n$$

and

$$\beta_0 + \gamma_1 = a_n + b_n + c_n$$

for $n \geq 2$ and $1 \leq j \leq n-1$. This implies $a_n + b_n + c_n = a_1 + b_1 + c_1$ for $n \geq 2$ and, because $\gamma_1 \neq 0$, it also implies $a_1 + b_1 + c_1 \neq a_0 + b_0$.

We also note that

$$\alpha_n = a_n, \quad n \in \mathbb{N}_0,$$

$$\beta_0 = a_0 + b_0, \qquad \beta_n = (a_n - a_{n-1}) + b_n, \qquad n \in \mathbb{N},$$

$$\gamma_1 = (a_1 + b_1 + c_1) - (a_0 + b_0), \quad \gamma_n = c_{n-1}, \quad n \geq 2.$$

Now, on the other hand, if a, b, c satisfy

$$a_n + b_n + c_n = a_1 + b_1 + c_1 \neq a_0 + b_0, \quad n \geq 2,$$

then α, β, γ, defined by the expressions given above, work as recurrence sequences for g and this shows orthogonality of g.

(b) As a byproduct of the computations in the proof of (a), we note that if α, β, γ are sequences satisfying

$$\alpha_n, \gamma_{n+1} \neq 0, \qquad \alpha_n + \beta_{n+1} + \gamma_{n+2} = \beta_0 + \gamma_1, \qquad n \in \mathbb{N}_0,$$

then the sequences a, b, c, defined by

$$a_n = \alpha_n, \quad n \in \mathbb{N}_0,$$

$$b_0 = \beta_0 - \alpha_0, \quad b_n = \beta_n - (\alpha_n - \alpha_{n-1}), \quad n \in \mathbb{N},$$

$$c_n = \gamma_{n+1}, \quad n \in \mathbb{N},$$

satisfy $a_n + b_n + c_n = a_1 + b_1 + c_1 \neq a_0 + b_0$, $n \geq 2$. Thus, (b) follows from (a) and its proof both applied to the pairs (g, p) and (p, q).

(c) is trivial. □

Definition 3.8. Let $p = (p_n)$ be an orthonormal polynomial system with the orthogonality measure μ. Let $y \in \mathbb{R} \setminus I_p$. Let $q = (q_n) = (\sum_{k=0}^{n} p_k(y) p_k)$ and Q be the corresponding orthonormal system, i.e., $Q = (Q_n) = (q_n / \|q_n\|_q)$, and let $r = (r_n) = (\sum_{k=0}^{n} Q_k(y) Q_k)$. Then (p, q, r) will be called an *analytic triple through* y.

Theorem 3.9. Let (p, q, r) be an analytic triple through y.

(a) The following are equivalent:

(i) There exists a $d \in \mathcal{D}$ such that $S_{p,d}$ simultaneously preserves the positive orthogonality of $\{q, r\}$.

(ii) There exist $y', y'' \in \mathbb{R} \setminus I_p$, $y \neq y'$, satisfying

$$\frac{p_1(y')}{p_n(y')\, p_{n+1}(y')} \left[\sum_{j=0}^{n} p_j(y'')\, p_j(y') \right]$$

$$= \frac{p_1(y)}{p_n(y)\, p_{n+1}(y)} \left[\sum_{j=0}^{n} (p_j(y))^2 \right] \quad \text{for all } n \in \mathbb{N}_0.$$

(iii) There exist $y', y'' \in \mathbb{R} \setminus I_p$, $y \neq y'$, such that

$$\text{either } (\alpha) \quad y' = y'' \quad \text{and} \quad p_1(y') \frac{p'_{n+1}(y')}{p_{n+1}(y')}$$

$$= p_1(y) \frac{p'_{n+1}(y)}{p_{n+1}(y)}, \ n \in \mathbb{N},$$

$$\text{or } (\beta) \quad y' \neq y'' \quad \text{and} \quad \frac{p_1(y')}{y'' - y'} \left[\frac{p_{n+1}(y'')}{p_{n+1}(y')} - 1 \right]$$

$$= \frac{p_1(y)\, p'_{n+1}(y)}{p_{n+1}(y)}, \ n \in \mathbb{N}.$$

The point y' (respectively y'') chosen in (ii) can be taken to be the same as in (iii), and vice versa, and in each case $d = (p_n(y')/p_n(y))$.

(b) For given $y', y'' \in \mathbb{R} \setminus I_p$, $y \neq y'$, Part (iii)(β) above is equivalent to each of the following, where σ_n is the leading coefficient of p_n, as before.

(i) $$y' \neq y'' = y' + \frac{1}{\sigma_2} \left[\frac{p_1(y)\, p_2(y')}{p_1(y')\, p_2(y)}\, p'_2(y) - p'_2(y') \right] \in \mathbb{R} \setminus I_p$$

and

$$\frac{p_1(y')}{y'' - y'} \left[\frac{p_{n+1}(y'')}{p_{n+1}(y')} - 1 \right] = \frac{p_1(y) p'_{n+1}(y')}{p_{n+1}(y)}, \quad n \geq 2.$$

(ii) $$y' \neq y' + \frac{1}{\sigma_2} \left[\frac{p_1(y)\, p_2(y')}{p_1(y')\, p_2(y)}\, p'_2(y) - p'_2(y') \right] \in \mathbb{R} \setminus I_p$$

and

$$\frac{p_1(y')}{p_{n+1}(y')} \left[\sum_{j=1}^{n+1} \frac{p_{n+1}^{(j)}}{j!} \left(\frac{1}{\sigma_2} \left[\frac{p_1(y)\, p_2(y')}{p_1(y')\, p_2(y)}\, p'_2(y) - p'_2(y') \right] \right)^{j-1} \right]$$

$$= \frac{p_1(y) p'_{n+1}(y)}{p_{n+1}(y)}, \quad n \geq 2.$$

(iii) $\dfrac{\sigma_n}{\sigma_{n+1}}\left[\left(\dfrac{\omega_{n+1}}{\omega_n}\,\dfrac{p_{n+2}(y')}{p_{n+1}(y')}+\dfrac{\omega_{n-1}}{\omega_n}\,\dfrac{p_{n-1}(y')}{p_n(y')}\right)-\left(\dfrac{p_n(y')}{p_{n+1}(y')}+\dfrac{p_{n+1}(y')}{p_n(y')}\right)\right]$

is a nonzero constant sequence, say $m \neq 0$, and $y'' := y' + m \in \mathbb{R}\setminus I_p$ satisfies

$$p_n(y'') = \omega_n\, p_{n+1}(y') - \omega_{n-1} p_{n-1}(y')\,,\ n \in \mathbb{N},$$

where

$$\omega_n = \frac{p_1(y)}{p_n(y)p_{n+1}(y)p_1(y')}\,\sum_{j=0}^{n}\,(p_j(y))^2\,,\ n \in \mathbb{N}_0\ .$$

Proof. (a) It is enough to consider the case $\inf Z_p > -\infty$ and $y \leq \inf Z_p$. Let μ_y be the measure given by $d\mu_y(x) = (x-y)\,d\mu(x)$. Let, for $n \in \mathbb{N}$, σ_n be the leading coefficient of p_n and $P_n = (1/\sigma_n)\,p_n$. Then, for $n \in \mathbb{N}_0$ (cf. [5], Section 7),

$$\begin{aligned}\int |\,q_n(x)|^2\,d\mu_y(x) &= \frac{\sigma_n}{\sigma_{n+1}}\int\left(\sum_{k=0}^{n} p_k(y)\,p_k(x)\right)\\ &\qquad\times\left(p_n(y)p_{n+1}(x) - p_{n+1}(y)p_n(x)\right)\,d\mu(x)\\ &= -\frac{\sigma_n}{\sigma_{n+1}}\,p_n(y)\,p_{n+1}(y).\end{aligned}$$

In particular, taking $n = 0$, we get $\mu_y(\mathbb{R}) = (-1)/(\sigma_1)\,p_1(y)$. So

$$\|q_n\|_q^2 = \frac{\sigma_1\sigma_n\ p_n(y)p_{n+1}(y)}{\sigma_{n+1}\ p_1(y)}, \qquad n \in \mathbb{N}_0. \tag{1}$$

Let $d \in \mathcal{D}$. By Theorem 3.3, $S_{p,d}\,q$ is positive orthogonal if, and only if, there is a $y' \in \mathbb{R}\setminus I_p$ such that $d_n\,p_n(y) = p_n(y')\,,\ n \in \mathbb{N}_0$. Since $d \in \mathcal{D}$, $y' \neq y$.

As in Definition 3.8, let $Q_n = q_n/(\|q_n\|_q)$. We next note that

$$S_{p,d}\,r_n = \sum_{k=0}^{n}\,Q_k(y)\,S_{p,d}\,Q_k\,, \qquad n \in \mathbb{N}_0\,.$$

Thus, $(S_{p,d}\,r_n)$ is analytic with respect to $(S_{p,d}\,Q_n)$. Therefore, by Theorem 3.3 again, for a d satisfying $d_n\,p_n(y) = p_n(y')\,,\ n \in \mathbb{N}_0\,$, with a $y' \in \mathbb{R}\setminus I_p$, $(\,S_{p,d}\,r_n\,)$ is positive orthogonal if, and only if, there exists a

$y'' \in \mathbb{R} \setminus I_p$ such that for $n \in \mathbb{N}_0$,

$$Q_n(y) = \frac{S_{p,d}\, Q_n(y'')}{\|S_{p,d}\, Q_n\|^2_{S_{p,d}Q}}, \qquad \text{i.e., } q_n(y) = \|q_n\|_q^2 \, \frac{S_{p,d}\, q_n(y'')}{\|S_{p,d}\, q_n\|^2_{S_{p,d}Q}}.$$

Replacing y by y' in (1), we have,

$$\| S_{p,d}\, q_n \|^2_{S_{p,d}Q} = \frac{\sigma_1 \sigma_n \; p_n(y')\, p_{n+1}(y')}{\sigma_{n+1}\, p_1(y')}, \quad n \in \mathbb{N} .$$

Hence, there exists a d with $S_{p,d}$ simultaneously preserving the positive orthogonality of $\{ q, r \}$ if, and only if, there exist y' , $y'' \in \mathbb{R} \setminus I_p$, $y \neq y'$ such that

$$\frac{p_1(y')}{p_n(y')\, p_{n+1}(y')} \sum_{j=0}^{n} p_j(y')\, p_j(y'') = \frac{p_1(y)}{p_n(y)\, p_{n+1}(y)} \sum_{j=0}^{n} (p_j(y))^2 , \quad n \in \mathbb{N}_0 .$$

In this case, $d_n = p_n(y')/p_n(y)$, $n \in \mathbb{N}_0$. This shows (i)$\Longleftrightarrow$(ii).

In (ii), either $y'' = y'$ or $y'' \neq y'$. Thus, using the confluent form of the Christoffel–Darboux identity, (ii) holds if, and only if, there exists a $y' \in \mathbb{R} \setminus I_p$ such that either

$$\frac{p_1(y')}{p_n(y')\, p_{n+1}(y')} \left[p'_{n+1}(y')\, p_n(y') - p'_n(y')\, p_{n+1}(y') \right]$$
$$= \frac{p_1(y)}{p_n(y)\, p_{n+1}(y)} \left[p'_{n+1}(y)\, p_n(y) - p'_n(y)\, p_{n+1}(y) \right], \; n \in \mathbb{N}_0 ,$$

or there exists a $y'' \in \mathbb{R} \setminus I_p$, $y'' \neq y'$, satisfying

$$\frac{p_1(y')}{p_n(y')\, p_{n+1}(y')} \, \frac{p_{n+1}(y'')\, p_n(y') - p_n(y'')\, p_{n+1}(y')}{y'' - y'}$$
$$= \frac{p_1(y)}{p_n(y)\, p_{n+1}(y)} \left[p'_{n+1}(y)\, p_n(y) - p'_n(y)\, p_{n+1}(y) \right], \; n \in \mathbb{N}_0 ,$$

i.e., either

$$p_1(y') \left[\frac{p'_{n+1}(y')}{p_{n+1}(y')} - \frac{p'_n(y')}{p_n(y')} \right] = p_1(y) \left[\frac{p'_{n+1}(y)}{p_{n+1}(y)} - \frac{p'_n(y)}{p_n(y)} \right], \quad n \in \mathbb{N}_0 ,$$

or there exists a $y'' \in \mathbb{R} \setminus I_p$, $y'' \neq y'$, satisfying

$$\frac{p_1(y')}{y'' - y'} \left[\frac{p_{n+1}(y'')}{p_{n+1}(y')} - \frac{p_n(y'')}{p_n(y')} \right] = \; p_1(y) \left[\frac{p'_{n+1}(y)}{p_{n+1}(y)} - \frac{p'_n(y)}{p_n(y)} \right], \quad n \in \mathbb{N}_0.$$

For $n=0$, both the conditions are trivially satisfied, so (i) is equivalent to the existence of $y' \in \mathbb{R} \setminus I_p$ such that either

$$p_1(y')\ \frac{p'_{n+1}(y')}{p_{n+1}(y')} = p_1(y)\ \frac{p'_{n+1}(y)}{p_{n+1}(y)}\,, \quad n \in \mathbb{N},$$

i.e., (iii)(α), or there exists $y'' \in \mathbb{R} \setminus I_p$ satisfying

$$\frac{p_1(y')}{y''-y'} \left[\frac{p_{n+1}(y'')}{p_{n+1}(y')} - 1 \right] = p_1(y)\ \frac{p'_{n+1}(y)}{p_{n+1}(y)}\,, \quad n \in \mathbb{N},$$

i.e., (iii)(β). This shows (ii)$\Longleftrightarrow$(iii).

(b) Condition (a)(iii)(β) is equivalent to

$$y' \neq y'' = y' + \frac{1}{\sigma_2} \left[\frac{p_1(y)\, p_2(y')}{p_1(y')\, p_2(y)}\ p'_2(y) - p'_2(y') \right] \ \in \mathbb{R} \setminus I_p$$

and

$$\frac{p_1(y')}{p_{n+1}(y')} \left[\frac{p_{n+1}(y'') - p_{n+1}(y')}{y''-y'} \right] = \frac{p_1(y)\, p'_{n+1}(y)}{p_{n+1}(y)}\,,\ n \geq 2\,,$$

which is (b)(i). Now,

$$\frac{p_{n+1}(y'') - p_{n+1}(y')}{y''-y'} = \sum_{j=1}^{n+1} \frac{p^{(j)}_{n+1}(y')}{j!}\ (y''-y')^{(j-1)}\,,\ n \in \mathbb{N},$$

so (a)(iii)(β) is further equivalent to

$$y' \neq y' + \frac{1}{\sigma_2} \left[\frac{p_1(y)\, p_2(y')}{p_1(y')\, p_2(y)}\ p'_2(y) - p'_2(y') \right] \ \in \mathbb{R} \setminus I_p$$

and

$$\frac{p_1(y')}{p_{n+1}(y')} \sum_{j=1}^{n+1} \left[\frac{p^{(j)}_{n+1}(y')}{j!\ \sigma_2^{j-1}} \left(\frac{p_1(y)\, p_2(y')}{p_1(y')\, p_2(y)}\ p'_2(y) - p'_2(y') \right)^{(j-1)} \right]$$
$$= \frac{p_1(y)\, p'_{n+1}(y)}{p_{n+1}(y)}\,,\ n \geq 2\,,$$

which is (b)(ii).

To prove the equivalence to (b)(iii), suppose first that (a)(iii)(β) holds. This implies (a)(ii), i.e., there exist $y'\,, y'' \in \mathbb{R} \setminus I_p\,,\ y' \neq y$ such that $\omega_n\, p_n(y')\, p_{n+1}(y') = \sum_{j=0}^{n} p_j(y') p_j(y''),\ n \in \mathbb{N}_0$.

So $\omega_0 = 1/p_1(y')$ and $\omega_n\, p_n(y')\, p_{n+1}(y') - \omega_{n-1}\, p_{n-1}(y')\, p_n(y') = p_n(y')\, p_n(y'')\,, n \in \mathbb{N}$. Thus, $\omega_n\, p_{n+1}(y') - \omega_{n-1}\, p_{n-1}(y') = p_n(y'')$. Also, by the Christoffel–Darboux identity, for $n \in \mathbb{N}$,

$$\omega_n\, p_n(y')\, p_{n+1}(y') = \frac{\sigma_n}{\sigma_{n+1}} \left[\frac{p_{n+1}(y'')\, p_n(y') - p_n(y'')\, p_{n+1}(y')}{y'' - y'} \right]$$

and, therefore,

$$y'' - y' = \frac{1}{\omega_n} \frac{\sigma_n}{\sigma_{n+1}} \left[\frac{p_{n+1}(y'')}{p_{n+1}(y')} - \frac{p_n(y'')}{p_n(y')} \right] .$$

So,

$$\begin{aligned} m = y'' - y' = \frac{\sigma_n}{\sigma_{n+1}} \Bigg[& \frac{1}{p_{n+1}(y')} \left(\frac{\omega_{n+1}}{\omega_n} p_{n+2}(y') - p_n(y') \right) \\ & - \frac{1}{p_n(y')} \left(p_{n+1}(y') - \frac{\omega_{n-1}}{\omega_n}\, p_{n-1}(y') \right) \Bigg], \end{aligned}$$

i.e.,

$$\begin{aligned} m = y'' - y' = \frac{\sigma_n}{\sigma_{n+1}} \Bigg[& \left(\frac{\omega_{n+1}}{\omega_n} \frac{p_{n+2}(y')}{p_{n+1}(y')} + \frac{\omega_{n-1}}{\omega_n} \frac{p_{n-1}(y')}{p_n(y')} \right) \\ & - \left(\frac{p_n(y')}{p_{n+1}(y')} + \frac{p_{n+1}(y')}{p_n(y')} \right) \Bigg] . \end{aligned}$$

The reverse implication is clear. □

Theorem 3.10. Let (p, q, r) be an analytic triple through -1 such that

(i) the system p is in the Nevai class $M(0, 1)$ (see [9], Definition 3.1.6), with the support of the orthogonality measure contained in $[-1, 1]$, and

(ii) $$\lim_{n\to\infty} \frac{p_{n+1}(-1)}{p_n(-1)} = -1 = -\lim_{n\to\infty} \frac{p_{n+1}(1)}{p_n(1)}.$$

Then for $d \in \mathcal{D}$, the operator $S_{p,d}$ simultaneously preserves positive orthogonality of $\{q, r\}$ if, and only if, $d_n = p_n(1)/p_n(-1)$, $n \in \mathbb{N}_0$, and

$$\frac{p_1(1)p'_{n+1}(1)}{p_{n+1}(1)} = \frac{p_1(-1)p'_{n+1}(-1)}{p_{n+1}(-1)}, \qquad n \in \mathbb{N}.$$

Proof. We shall use Theorem 3.9 (with $y = -1$) and its notation. Sufficiency (that is, the 'if' part) is immediate from Theorem 3.9(a)(iii)(α). For necessity (that is, the 'only if' part), we consider a $d \in \mathcal{D}$ for which $S_{p,d}$ simultaneously preserves the positive orthogonality of $\{q, r\}$. Then, either Condition (a)(iii)(α) or Condition (b)(iii) of Theorem 3.9 is satisfied.

By definition, there exist sequences $a = (a_n)$ and $b = (b_n)$ satisfying

$$xp_n(x) = a_n\, p_{n+1}(x) + b_n\, p_n(x) + a_{n-1}\, p_{n-1}(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0,$$

and $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sigma_n/\sigma_{n+1} = \frac{1}{2}$. By Poincaré's theorem (see [9], Theorem 4.1.13) and by assumption (ii), we have

$$\lim_{n\to\infty} \frac{p_{n+1}(y')}{p_n(y')} = \begin{cases} y' + \sqrt{y'^2 - 1}, & \text{if } y' > 1, \\ y' - \sqrt{y'^2 - 1}, & \text{if } y' < -1, \\ 1, & \text{if } y' = 1. \end{cases}$$

Moreover, from Theorem 4.1.3 in [9], it follows that

$$\lim_{n\to\infty} \frac{(p_{n+1}(-1))^2}{\sum_{j=0}^{n} (p_j(-1))^2} = 0,$$

which implies, together with assumption (ii), that $\lim_{n\to\infty} \omega_{n+1}/\omega_n = 1$. As a consequence,

$$\lim_{n\to\infty} \frac{\sigma_n}{\sigma_{n+1}} \left[\left(\frac{\omega_n}{\omega_{n+1}} \frac{p_{n+2}(y')}{p_{n+1}(y')} + \frac{\omega_{n-1}}{\omega_n} \frac{p_{n-1}(y')}{p_n(y')} \right) - \left(\frac{p_n(y')}{p_{n+1}(y')} + \frac{p_{n+1}(y')}{p_n(y')} \right) \right] = 0.$$

Thus, Condition (b)(iii) cannot be satisfied. Therefore Condition (a)(iii)(α) must hold, which implies (a)(ii), where we can choose $y'' = y'$.

In the same way as before, we can also conclude that

$$\lim_{n\to\infty} \frac{p_n(-1)\, p_{n+1}(-1)}{p_1(-1)\sum_{j=0}^{n}(p_j(-1))^2} = 0,$$

which now gives us

$$\lim_{n\to\infty} \frac{p_n(y')\, p_{n+1}(y')}{p_1(y')\sum_{j=0}^{n}(p_j(y'))^2} = 0.$$

Assume now that $|y'| > 1$. In this case we can utilize the proof of Theorem 1 in [14] to show that

$$\inf_{n\in\mathbb{N}} \frac{\sigma_n}{\sigma_{n+1}} \frac{(p_n(y'))^2 + (p_{n+1}(y'))^2}{\sum_{j=0}^{n}(p_j(y'))^2} > 0.$$

Since in our case, $\inf_{n\in\mathbb{N}}\ \sigma_{n+1}/\sigma_n > 0$ and

$$\inf_{n\in\mathbb{N}} \min\{|p_n(y')/p_{n+1}(y')|, |p_{n+1}(y')/p_n(y')|\} > 0,$$

and since always

$$(p_n(y'))^2 + (p_{n+1}(y'))^2 \le 2\ \max\left\{\left|\frac{p_n(y')}{p_{n+1}(y')}\right|, \left|\frac{p_{n+1}(y')}{p_n(y')}\right|\right\} |p_n(y')|\, |p_{n+1}(y')|,$$

we get that

$$\inf_{n\in\mathbb{N}} \frac{p_n(y')\, p_{n+1}(y')}{p_1(y')\sum_{j=0}^{n}(p_j(y'))^2} > 0,$$

a contradiction. Thus, $|y'| = 1$, and since y' is real and not equal to $y = -1$, we get $y' = 1$. Now putting this into Theorem 3.9(iii)(α), we get the assertion. □

Remark 3.11

(a) The analogous result for analytic triples at 1 holds too.

(b) For $\alpha,\ \beta > -1$, the orthogonal Jacobi polynomial system $p = (P_n^{(\alpha,\beta)})$ satisfies the conditions of Theorem 3.10. We also have $P_{n+1}^{(\alpha,\beta)\,\prime}(x) = ((n+\alpha+\beta+2)/2)\, P_n^{(\alpha+1,\beta+1)}(x)$,

$$(1-x)^\alpha\,(1+x)^\beta P_n^{(\alpha,\beta)}(x)$$
$$= \frac{(-1)^n}{2^n\, n!}\ \frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right], \qquad x \in \mathbb{R}, \qquad \mathrm{n} \in \mathbb{N}_0.$$

Thus Condition (a)(iii)(α) of Theorem 3.9 reduces to, for $n \in \mathbb{N}_0$,

$$P_1^{(\alpha,\beta)}(1)\,\frac{P_n^{(\alpha+1,\beta+1)}(1)}{P_{n+1}^{(\alpha,\beta)}(1)} = P_1^{(\alpha,\beta)}(-1)\,\frac{P_n^{(\alpha+1,\beta+1)}(-1)}{P_{n+1}^{(\alpha,\beta)}(-1)} = P_1^{(\alpha,\beta)}(-1),$$

i.e.

$$(\alpha+1)_1\,\frac{(\alpha+2)_n}{n!}\,\frac{(n+1)!}{(\alpha+1)_{n+1}} = (-1)(\beta+1)_1\,\frac{(-1)^n\,(\beta+2)_n}{n!}\,\frac{(n+1)!}{(-1)^{n+1}\,(\beta+1)_{n+1}}$$

which is satisfied. So, the sequence

$$d = \left(\frac{p_n(1)}{p_n(-1)}\right) = \left(\frac{P_n^{(\alpha,\beta)}(1)}{P_n^{(\alpha,\beta)}(-1)}\right) = \left((-1)^n\,\frac{(\alpha+1)_n}{(\beta+1)_n}\right)$$

is the unique sequence such that $S_{p,d}$ preserves the positive orthogonality of the analytic triple $\{\,P_n^{(\alpha,\beta)},\, P_n^{(\alpha,\beta+1)},\, P_n^{(\alpha,\beta+2)}\}$. We have already noted in the Introduction that it preserves the orthogonality of $\{\,P_n^{(\alpha,\beta+t)} : t > 0\,\}$.

4. DILATIONS FOR LAGUERRE POLYNOMIALS

4.1. Discussion and Definition

Denote by (L_n^α) the generalized Laguerre polynomial system, as given in the Introduction. Clearly, $(L_n^\alpha)'(x) = -L_{n-1}^{\alpha+1}(x)\,,\ x \in \mathbb{R}\,,\ n \in \mathbb{N}_0$, where $L_{-1}^\alpha \equiv 0$. The associated monic polynomial system (l_n^α) is given by $l_n^\alpha = (-1)^n n! L_n^\alpha,\ n \in \mathbb{N}_0$, and, for $\alpha > -1$, the orthonormal polynomial system $(\widetilde{L}_n^\alpha)$ with leading coefficients σ_n positive and negative at even and odd n's, respectively, is given by

$$\widetilde{L}_n^\alpha = \left(\frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha = \left(\,L_n^\alpha(0)\,\right)^{-1/2} L_n^\alpha\,, \qquad n \in \mathbb{N}_0\,.$$

Also, $L_n^{\alpha+1} = \sum_{j=0}^n L_j^\alpha$ and, more generally, for t with $\alpha + t \in \mathbb{R} \setminus (-\mathbb{N})$,

$$L_n^{\alpha+t} = \sum_{j=0}^{n} \binom{t+n-j-1}{n-j} L_j^\alpha\ .$$

(i) Let $p = (l_n^\alpha)$. Then the associated sequences (b_n) and (c_n), as in Theorem 2.2, are given by $b_n = 2n+1+\alpha,\quad c_n = n(n+\alpha)$,

$n \in \mathbb{N}_0$. For $t \in \mathbb{R}$ with $\alpha + t \notin -\mathbb{N}$, the monic matrix for $(l_n^{\alpha+t})$ is $A^t = (a_{j,k}^t)$, where

$$a_{j,k}^t = (-1)^{j-k} \ \frac{j!}{k!} \ \frac{t(t+1)\cdots(t+j-k-1)}{(j-k)!}\,, \quad 0 \le k < j\;.$$

In particular, $a_{j,k}^t \neq 0\,,\quad 0 \le k \le j\,,\ j \in \mathbb{N}_0$ if $t \notin -\mathbb{N}_0$, and $a_{j,k}^t \neq 0$ if, and only if, $\max\{0, t+j\} \le k \le j\,,\ j \in \mathbb{N}_0$ in case $t \in -\mathbb{N}_0$. Also,

$$a_{j+1,k}^t = -\frac{(j+1)\,(t+j-k)}{j-k+1}\ a_{j,k}^t\,, \quad 0 \le k \le j\,, \quad j \in \mathbb{N}_0,$$

$$a_{j,k-1}^t = -\frac{k\,(t+j-k)}{j-k+1}\ a_{j,k}^t\,, \qquad 1 \le k \le j\,, \quad j \in \mathbb{N},$$

$$a_{j,k+1}^t = \frac{j}{(k+1)}\ a_{j-1,k}^t\,, \qquad 0 \le k \le j\,, \quad j \in \mathbb{N}.$$

Finally, $a_{k+1,k}^t = -(k+1)t$ and $a_{k+2,k}^t = ((k+2)(k+1)/2)\ t(t+1),\ k \in \mathbb{N}_0$.

(ii) The system $(L_n^{\alpha+1})$ is (L_n^{α})-analytic with constant one as the coefficient sequence. Also, $(l_n^{\alpha+1})$ is the monic kernel polynomial at 0 of (l_n^{α}). Thus, for $\alpha > -1$, $(\widetilde{L}_n^{\alpha+1})$ is $(\widetilde{L}_n^{\alpha})$-analytic with coefficient sequence $(\widetilde{L}_n^{\alpha}(0))$. Thus $((\widetilde{L}_n^{\alpha})\,,\,(\widetilde{L}_n^{\alpha+1})\,,\,(\widetilde{L}_n^{\alpha+2})\,)$ is an analytic triple through zero.

Theorem 4.2. Assume that $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$, $t \in \mathbb{R} \setminus \{0\}$ and $\alpha + t \notin -\mathbb{N}$. Let $p = (l_n^{\alpha})$ and $d \in \mathcal{D}$. Denote $u_n := (n+1)\,d_n/d_{n+1}$ and, for notational convenience, $u_{-1} := 0$.

(a) The system $(S_{p,d}(l_n^{\alpha+t}))$ is orthogonal if, and only if,

(i) $$-u_{n-1}u_n\frac{t(t-1)}{2} \neq n(n+\alpha) + 2u_{n-1}t - u_{n-1}^2t^2 + u_{n-2}u_{n-1}\frac{t(t+1)}{2} \quad \text{for } n \in \mathbb{N},$$

(ii) $u_n(t-1)(t-2)$

$$= \frac{1}{(n-j)(n-j-1)}\Bigg[2(n-j-1)(n-j+1)u_{n-1}t(t-1)$$

$$-(n-j)(n-j+1)u_{n-2}t(t+1)$$

$$+2u_{j-1}(t+n-j-1)(t+n-j)+2(n-j)(n-j+1)$$

$$\times\left((j+1+\alpha)\frac{j+1}{u_j}-(n+\alpha)\frac{n}{u_{n-1}}+2(n-j-1)\right)\Bigg]$$

$$\text{for}\quad\begin{cases}2\le j+2\le n & \text{if } t\notin -\mathbb{N},\\ 2\le j+2\le n\le j-t+1 & \text{if } t\in -\mathbb{N}.\end{cases}$$

(b) Suppose $\alpha, \alpha+t>-1$. Then the system $(S_{p,d}(l_n^{\alpha+t}))$ is positive orthogonal if, and only if, d is real and (a) above holds with the inequality in (i) strengthened to "$<$".

Proof. Writing $a_{n,j}$ instead of $a_{n,j}^t$, 4.1(i) gives

$$a_{n+1,n-1}-a_{n+1,n}a_{n,n-1} = -\frac{(n+1)n}{2}t(t-1), \qquad n\in\mathbb{N},$$

and, for $2\le j+2\le n$,

$$a_{n+1,j}-a_{n+1,n}a_{n,j}=\frac{(n+1)(n-j)}{(n-j+1)}(t-1)a_{n,j}$$

$$=-\frac{(n+1)n}{(n-j+1)}(t-1)(t+n-j-1)a_{n-1,j}$$

and

$$a_{n,j-1}=\frac{nj}{(n-j)(n-j+1)}(t+n-j)(t+n-j-1)a_{n-1,j},$$

where, for notational convenience, we take $a_{n,-1}=0$.

When applying these identities to the expressions in Theorem 2.4, we note that both sides of 2.4(a$_d$)(iii) have a common factor $a_{n-1,0}$ and those of 2.4(a$_d$)(iv) have a common factor $a_{n-1,j}$. In the case $t\notin -\mathbb{N}$, these factors are not equal to 0 and can be cancelled. In the case $t\in -\mathbb{N}$, these factors are not equal to 0 and can be cancelled if, and only if, $n-j\le -t+1$; for $n-j>-t+1$, Conditions 2.4(a$_d$)(iii) and 2.4(a$_d$)(iv) are trivially satisfied. Now combining 2.4(a$_d$)(i) and 2.4(a$_d$)(ii) into one line, as well as 2.4(a$_d$)(iii) and 2.4(a_d)(iv), we arrive at

$$-\frac{d_{n-1}}{d_{n+1}}\frac{n(n+1)}{2}t(t-1) \neq n(n+\alpha) + \frac{d_{n-1}}{d_n}\,2nt - \left(\frac{d_{n-1}}{d_n}\,nt\right)^2 + \frac{d_{n-2}}{d_n}\frac{n(n-1)}{2}t(t+1), \qquad \text{for } n \in \mathbb{N}$$

and

$$\begin{aligned}\frac{1}{d_{n+1}}(t-1)(t-2) &= \frac{1}{(n+1)(n-j)(n-j-1)d_n} \\ &\times \Bigg[2(n-j-1)(n-j+1)n\frac{d_{n-1}}{d_n}t(t-1) \\ &- (n-j)(n-j+1)(n-1)\frac{d_{n-2}}{d_{n-1}}t(t+1) \\ &+ 2j(t+n-j-1)(t+n-j)\frac{d_{j-1}}{d_j} + 2(n-j)(n-j+1) \\ &\times \left((j+1+\alpha)\frac{d_{j+1}}{d_j} - (n+\alpha)\frac{d_n}{d_{n-1}} + 2(n-j-1)\right)\Bigg]\end{aligned}$$

$$\text{for} \quad \begin{cases} 2 \le j+2 \le n & \text{if } t \notin -\mathbb{N}, \\ 2 \le j+2 \le n \le j-t+1 & \text{if } t \in -\mathbb{N} \end{cases}$$

Expressing the d_n's in terms of u_n's, we arrive at (a); Part (b) follows from Part (a). □

Remark 4.3.

(a) We note that by setting $j = n-2$ in Part (a)(ii) above and doing some simplifications, we arrive at

$$u_n(t-1)(t-2) = 3t(t-1)u_{n-1} - 3t(t+1)u_{n-2} + (t+1)(t+2)u_{n-3} + 6\left((n-1+\alpha)\frac{n-1}{u_{n-2}} - (n+\alpha)\frac{n}{u_{n-1}} + 2\right)$$

for $n \geq 2$. If $t = -1$, this is even equivalent to (a)(ii), because then $j = n-2$ is the only admissible case.

(b) If $t \neq -1$, then we can also set $j = n-3$ and get

$$u_n(t-1)(t-2) = \frac{8}{3}t(t-1)u_{n-1} - 2t(t+1)u_{n-2} + \frac{1}{3}(t+2)(t+3)u_{n-4} + 4\left((n-2+\alpha)\frac{n-2}{u_{n-3}} - (n+\alpha)\frac{n}{u_{n-1}} + 4\right)$$

for $n \geq 3$.

(c) The next result, Corollary 4.4, can be seen as a contrast to Theorem 1.1. The statement there is that for the sequence $d = ((-1)^n)$ and the system $p = (P_n^{(\gamma,\gamma)})$, the operator $S_{p,d}$ simultaneously preserves orthogonality of uncountably many systems of Jacobi polynomials. Here, for $d = (d_1^n)$ and $p = (l_n^\alpha)$, the operator $S_{p,d}$ preserves orthogonality of no other system of Laguerre polynomials.

Corollary 4.4. Let $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$, $p = (l_n^\alpha)$, and let $t \in \mathbb{R} \setminus \{0\}$ with $\alpha + t \notin -\mathbb{N}$.

(a) Let $d \in \mathcal{D}$ be given by $d_n = d_1^n$ with $d_1 \neq 1$.

(b) Let $d \in \mathcal{D}$ be given by $d_n = (\Gamma(n+1)\Gamma(\alpha+1)/\Gamma(n+\alpha+1))$ with $\alpha \neq 0$.

Then $S_{p,d}$ does not preserve orthogonality of $(l_n^{\alpha+t})$.

Proof. (a) For this sequence d, we have $u_n = (n+1)/d_1$. Putting this into 4.3(a), we arrive at

$$\frac{n+1}{d_1}(t-1)(t-2) = 3t(t-1)\frac{n}{d_1} - 3t(t+1)\frac{n-1}{d_1} + (t+1)(t+2)\frac{n-2}{d_1} + 6((n-1+\alpha)\,d_1 - (n+\alpha)\,d_1 + 2).$$

This simplifies to $(1-d_1)^2/d_1 = 0$, so that necessarily $d_1 = 1$, contrary to the assumption that $d_1 \neq 1$.

(b) Now we have $u_n = n + \alpha + 1$ (and $u_{-1} = 0$). Putting this into 4.2(a)(i) and cancelling the common factor $(n+\alpha)$, which never equals 0, we obtain

$$-(n+\alpha+1)\frac{t(t-1)}{2} \neq n + 2t - (n+\alpha)t^2 + \begin{cases} (n+\alpha-1)(t(t+1)/2) & \text{for } n > 1, \\ 0 & \text{for } n = 1. \end{cases}$$

The first line simplifies to $t + n \neq 0$ for all $n > 1$, and the second line implies that $t \neq -1$. Thus it is impossible that $t \in -\mathbb{N}$.

Now for $t \notin -\mathbb{N}$, we can use Condition 4.2(a)(ii) with $j = 0$. This leads to

$$(n+\alpha+1)(t-1)(t-2) = \frac{1}{n(n-1)}\Big[2(n-1)(n+1)(n+\alpha)t(t-1) - n(n+1)(n+\alpha-1)t(t+1) + 2n(n+1)(n-1)\Big]$$

for all $n \geq 2$. Since $\alpha \neq 0$, we can cancel a factor of 2α and arrive at the identity

$$n^2 + (2t-1)n + t(t-1) = 0 \qquad \text{for all } n \geq 2,$$

which is impossible. □

Theorem 4.5. Let $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$ and $p = (l_n^\alpha)$. Then there exists no $d \in \mathcal{D}$ such that $S_{p,d}$ simultaneously preserves orthogonality of $(l_n^{\alpha+t_1})$ and $(l_n^{\alpha+t_2})$, where $t_1, t_2 \in \mathbb{R}$ are distinct and nonzero and satisfy $\alpha + t_1, \alpha + t_2 \notin -\mathbb{N}$.

Proof. Assume that for some $d \in \mathcal{D}$, the operator $S_{p,d}$ preserves orthogonality of $(l_n^{\alpha+t_1})$ and $(l_n^{\alpha+t_2})$ with some t_1, t_2 as specified. Denote $\tau := t_1 + t_2$. Substitute $t = t_1$ and $t = t_2$ in 4.3(a), subtract, and cancel the common factor $(t_1 - t_2)$. Then we get

$$(\tau-3)\,u_n = 3(\tau-1)\,u_{n-1} - 3(\tau+1)\,u_{n-2} + (\tau+3)\,u_{n-3} \qquad \text{for } n \geq 2.$$

This is a linear recursion with constant coefficients for the sequence u_n. Using standard methods (cf. [6], Section 2.3), this recursion can be solved. Explicitly, the sequence u_n satisfies the recursion if, and only if, there exist constants γ_1, γ_2 such that, for all $n \in \mathbb{N}_0$,

$$u_n = \gamma_1\,(n+1) \qquad \text{if } \tau = 3,$$

$$u_n = \gamma_1\,(n+1) + \gamma_2\,(1-\sigma^{n+1}) \qquad \text{if } \tau \neq 3, \text{where } \sigma := \frac{\tau+3}{\tau-3},$$

and the constants γ_1, γ_2 are chosen such that $u_n \neq 0$ for all $n \in \mathbb{N}_0$. Here we used the initial condition $u_{-1} = 0$, and we set $\sigma^0 := 1$ for all $\sigma \in \mathbb{R}$. Since it is impossible that $u_n = \gamma_1(n+1)$, because that would contradict Corollary 4.4(a), we can, from now on, assume that $\tau \neq 3$ and that $\gamma_2 \neq 0$.

Now we have to distinguish two cases: either $t_1, t_2 \neq -1$ or, say, $t_1 = -1$.

If $t_1, t_2 \neq -1$, then using 4.3(b) with $n = 3$, we get in the same way as before the identity

$$(\tau - 3)\, u_3 = \tfrac{8}{3}(\tau - 1)\, u_2 - 2(\tau + 1)\, u_1.$$

Now using the above explicit formula for u_n, simplifying and cancelling the common factor $\gamma_2 \neq 0$, we get

$$(\tau - 3)\, \sigma^4 = \tfrac{8}{3}(\tau - 1)\, \sigma^3 - 2(\tau + 1)\, \sigma^2 + \tfrac{1}{3}(\tau + 5),$$

which, after remembering that $\sigma = (\tau + 3)/(\tau - 3)$, leads to a contradiction.

On the other hand, if $t_1 = -1$, then 4.3(a) implies

$$u_n - u_{n-1} = (n - 1 + \alpha)\frac{n-1}{u_{n-2}} - (n + \alpha)\frac{n}{u_{n-1}} + 2 \qquad \text{for } n \geq 2. \qquad (*)$$

Replacing n by $n - 1$ and adding that to the line above, we obtain

$$u_n - u_{n-2} = (n - 2 + \alpha)\frac{n-2}{u_{n-3}} - (n + \alpha)\frac{n}{u_{n-1}} + 4 \qquad \text{for } n \geq 3. \qquad (**)$$

Since $t_2 \neq -1$, Remark 4.3(b) is applicable for $t = t_2$ and gives now

$$\begin{aligned} u_n(t_2 - 1)(t_2 - 2) = {} & \tfrac{8}{3} t_2(t_2 - 1)u_{n-1} - 2t_2(t_2 + 1)u_{n-2} \\ & + \tfrac{1}{3}(t_2 + 2)(t_2 + 3)u_{n-4} + 4(u_n - u_{n-2}) \end{aligned}$$

for all $n \geq 3$. Next, setting $n = 3$, using $-1 + t_2 = \tau$ and simplifying, we obtain

$$(\tau^2 - \tau - 4)\, u_3 - \tfrac{8}{3}\tau(\tau + 1)\, u_2 + 2(\tau^2 + 3\tau + 4)\, u_1 = 0.$$

Again, using the above explicit formula for u_n, simplifying and cancelling the common factor $\gamma_2 \neq 0$, we get

$$(\tau^2 - \tau - 4)\,\sigma^4 - \tfrac{8}{3}\tau(\tau+1)\,\sigma^3 + 2(\tau^2 + 3\tau + 4)\,\sigma^2 - \tfrac{1}{3}(\tau+3)(\tau+4) = 0.$$

With $\sigma = (\tau+3)/(\tau-3)$, this implies $\tau = 0$ or $\tau = -3$, i.e., $\sigma = -1$ or $\sigma = 0$. In both cases, $u_n - u_{n-2} = 2\gamma_1$ for $n \geq 2$.

For $\sigma = -1$, we have

$$u_n = \left\{ \begin{array}{ll} \gamma_1(n+1) & \text{if } n \text{ is odd} \\ \gamma_1(n+1) + 2\gamma_2 & \text{if } n \text{ is even} \end{array} \right\} \text{ for } n \in \mathbb{N}_0.$$

Putting this into $(**)$ with $n = 4$, we obtain

$$2\gamma_1 = \frac{2(2+\alpha)}{2\gamma_1} - \frac{4(4+\alpha)}{4\gamma_1} + 4,$$

which simplifies to $\gamma_1 = 1$, and then for odd n,

$$2 = \frac{1+\alpha}{1+2\gamma_2} - \frac{3(3+\alpha)}{3+2\gamma_2} + 4,$$

which simplifies to $\gamma_2 = \alpha/2$, since $\gamma_2 \neq 0$. But then

$$u_n = \left\{ \begin{array}{ll} n+1 & \text{if } n \text{ is odd} \\ n+\alpha+1 & \text{if } n \text{ is even} \end{array} \right\}.$$

Putting this into $(*)$, we see that necessarily $\alpha = 0$, thus $\gamma_2 = 0$, a contradiction.

For $\sigma = 0$, we have $u_n = \gamma_1(n+1) + \gamma_2$ for $n \in \mathbb{N}_0$. Putting this into $(*)$, we obtain

$$\gamma_1 = (n-1+\alpha)\;\frac{n-1}{\gamma_1(n-1)+\gamma_2} - (n+\alpha)\;\frac{n}{\gamma_1 n + \gamma_2} + 2,$$

which simplifies to

$$\gamma_1(\gamma_1-1)^2\,n^2 + (\gamma_1-1)^2(-\gamma_1+2\gamma_2)\,n$$
$$+\,\gamma_2\,(\gamma_1\gamma_2 - 2\gamma_2 + 2\gamma_1 + \alpha - 1 - \gamma_1^2) = 0$$

for all $n \geq 2$. Comparing coefficients, we see that necessarily

$$\gamma_1(\gamma_1-1)^2 = 0, \qquad (\gamma_1-1)^2(-\gamma_1+2\gamma_2) = 0$$

and

$$\gamma_2\,(\gamma_1\gamma_2 - 2\gamma_2 + 2\gamma_1 + \alpha - 1 - \gamma_1^2) = 0.$$

Since it is impossible that $\gamma_1 = 0$, because with the second equation that would imply $\gamma_2 = 0$, which was excluded, we must have $\gamma_1 = 1$ and then $\gamma_2 = \alpha$. But that means $u_n = n + \alpha + 1$, and this contradicts Corollary 4.4(b). □

Applications 4.6. Let $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$.

(a) Theorem 3.3 and 4.1(ii) can be combined to obtain uncountably many d's for which $S_{(L_n^\alpha),d}(L_n^{\alpha+1})$ is orthogonal.

(b) Theorem 4.5 with $t_1 = 1$ and $t_2 = 2$ and Theorem 3.9 together give the following result for $\alpha > -1$.

(i) There exists no $y' < 0$ such that

$$L_1^\alpha(y')\, L_n^{\alpha+1}(y') = (n+1)\, L_{n+1}^\alpha(y')\,, \qquad n \in \mathbb{N}.$$

(ii) There exist no y', y'' with $y' < 0\,, y'' \le 0\,, y' \ne y''$ that satisfy

$$\frac{L_1^\alpha(y')}{y'' - y'} \left[\frac{L_{n+1}^\alpha(y'')}{L_{n+1}^\alpha(y')} - 1 \right] = n + 1\,, \qquad n \in \mathbb{N}.$$

To see this, we note that in Condition (a)(iii) of Theorem 3.9, we can replace (p_n) by $(u_n\, p_n)$ for any $u = (u_n) \in \mathcal{D}$. Next, we find that

$$L_1^\alpha(0)\ L_{n+1}^{\alpha\,\prime}(0)/L_{n+1}^\alpha(0) = n + 1$$

for all $n \in \mathbb{N}$ and $((\widetilde{L}_n^\alpha), (v_n \widetilde{L}_n^{\alpha+1}), (w_n \widetilde{L}_n^{\alpha+2}))$ is an analytic triple through 0 for suitable $(v_n), (w_n) \in \mathcal{D}$. So, by Theorem 4.5 above, $p = (L_n^\alpha)$ does not satisfy (a)(iii) of Theorem 3.9, i.e., (i) and (ii) above hold.

(c)We can give another proof of (c)(ii) using Theorem 3.9(b). In the notation of that theorem with $p = (\widetilde{L}_n^\alpha)\,, \quad y = 0\,, \quad y' < 0$, we have

$$\frac{\sigma_n}{\sigma_{n+1}} = \sqrt{(n+1)(n+\alpha+1)},$$

$$\omega_n = \frac{\sqrt{(\alpha+1)(n+1)(n+\alpha+1)}}{p_1(y')}.$$

So,

$$\lim_{n\to\infty} \frac{1}{n}\, \frac{\sigma_n}{\sigma_{n+1}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = 1 = \lim_{n\to\infty} \frac{\omega_{n-1}}{\omega_n}.$$

.

Also by [9],

$$\lim_{n\to\infty}\frac{p_{n+2}(y')}{p_{n+1}(y')} = -1 = \lim_{n\to\infty}\frac{p_{n-1}(y')}{p_n(y')}.$$

So,

$$\lim_{n\to\infty}\frac{\sigma_n}{\sigma_{n+1}}\left[\frac{\omega_{n+1}}{\omega_n}\frac{p_{n+2}(y')}{p_{n+1}(y')}+\frac{\omega_{n-1}}{\omega_n}\frac{p_{n-1}(y')}{p_n(y')}-\left(\frac{p_n(y')}{p_{n+1}(y')}+\frac{p_{n+1}(y')}{p_n(y')}\right)\right]=0.$$

An appeal to Theorem 3.9(b) completes the proof.

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