# STRONG NONNEGATIVE LINEARIZATION OF ORTHOGONAL POLYNOMIALS

Ryszard Szwarc *
*Institute of Mathematics*
*Wroclaw University*
*pl. Grunwaldzki 2/4*
*50–384 Wroclaw*
*POLAND*

*and*

*Institute of Mathematics*
*Polish Academy of Science*
*ul. Śniadeckich 8*
*00–950 Warszawa*
*POLAND*

**Abstract** A stronger notion of nonnegative linearization of orthogonal polynomials is introduced. It requires that also the associated polynomials of any order have nonnegative linearization property. This turns out to be equivalent to a maximal principle of a discrete boundary value problem associated with orthogonal polynomials through the three term recurrence relation. The property is stable for certain perturbations of the recurrence relation. Criteria for the strong nonnegative linearization are derived. The range of parameters for the Jacobi polynomials satisfying this new property is determined.

**Keywords:** Orthogonal polynomials, recurrence relation, nonnegative linearization, discrete boundary value problem.

## 1. Introduction

One of the main problems in the theory of orthogonal polynomials is to determine whether the expansion of the product of two orthogonal

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polynomials in terms of these polynomials has nonnegative coefficients. We want to decide which orthogonal systems $\{p_n\}_{n=0}^{\infty}$ have the property

$$p_n(x)p_m(x) = \sum c(n,m,k)p_k(x)$$

with nonnegative coefficients $c(n,m,k)$ for every $n, m$ and $k$.

Numerous classical orthogonal polynomials as well as their $q$-analogues satisfy nonnegative linearization property (Gasper, 1970a; Gasper, 1970b; Gasper, 1983), (Gasper and Rahman, 1990), (Ramis, 1992), (Rogers, 1894), (Szwarc, 1992b; Szwarc, 1995). There are many criteria for nonnegative linearization given in terms of the coefficients of the recurrence relation the orthogonal polynomials satisfy (Askey, 1970), (Młotkowski and Szwarc, 2001), (Szwarc, 1992a; Szwarc, 1992b; Szwarc, 2003), that can be applied to general orthogonal polynomials systems. These criteria are based on the connection between the linearization property and a certain discrete boundary value problem of hyperbolic type.

In this paper we are going to show that many polynomials systems satisfy even a stronger version of nonnegative linearization. Namely let $\{p_n\}_{n=0}^{\infty}$ be an orthogonal polynomial system. Let $\{p_n^{[l]}\}_{n=0}^{\infty}$ denote the associated polynomials of order $l$. We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy *the strong nonnegative linearization property* if

$$p_n(x)p_m(x) = \sum c(n,m,k)p_k(x),$$
$$p_n^{[l]}(x)p_m^{[l]}(x) = \sum c_l(n,m,k)p_k^{[l]}(x).$$

with nonnegative coefficients $c(n,m,k)$ and $c_l(n,m,k)$ for any $n$, $m$, $k$ and $l$.

The interesting feature of this property is the fact that it is equivalent to a maximum principle of the associated boundary value problem (see Theorem 2). Also this property is invariant for certain transformations of the recurrence relation (see Proposition 2), unlike the usual nonnegative linearization property.

In the last part of this work we are going to show that the Jacobi polynomials have the strong linearization property if and only if either $\alpha = \beta \geq -1/2$ or $\alpha > \beta > -1$ and $\alpha + \beta \geq 0$.

## 2. Strong nonnegative linearization

Let $p_n$ denote a sequence of orthogonal polynomials, relative to a measure $\mu$, satisfying the recurrence relation

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1}, \qquad n \geq 0, \tag{2.1}$$

where $\gamma_n, \alpha_{n+1} > 0$ and $\beta_n \in \mathbb{R}$. We use the convention that $p_0 = 1$ and $\alpha_0 = p_{-1} = 0$. For any nonnegative integer $l$ let $p_n^{[l]}$ denote the sequence of polynomials satisfying

$$xp_n^{[l]} = \gamma_n p_{n+1}^{[l]} + \beta_n p_n^{[l]} + \alpha_n p_{n-1}^{[l]}, \qquad n \geq l+1, \tag{2.2}$$

$$p_0^{[l]} = p_1^{[l]} = \ldots = p_l^{[l]} = 0, \; p_{l+1}^{[l]} = \frac{1}{\gamma_l}. \tag{2.3}$$

For $n \geq l+1$ the polynomial $p_n^{[l]}$ is of degree $n-l-1$. The polynomials $p_n^{[l]}$ are called the *associated polynomial of order* $l+1$. These polynomials are orthogonal, as well. Let $\mu_l$ denote any orthogonality measure associated with $\{p_n^{[l]}\}_{n=l+1}^{\infty}$.

For $n \geq m \geq l+1 \geq 0$ consider the polynomials $p_n(x)p_m(x)$ and $p_n^{[l]}(x)p_m^{[l]}(x)$. We can express these products in terms of $p_k(x)$ or $p_k^{[l]}(x)$ to obtain the following.

$$p_n(x)p_m(x) = \sum_{k=0}^{\infty} c(n,m,k)p_k(x),$$

$$p_n^{[l]}(x)p_m^{[l]}(x) = \sum_{k=0}^{\infty} c_l(n,m,k)p_k^{[l]}(x).$$

The polynomial $p_n(x)p_m(x)$ has degree $n+m$ while $p_n^{[l]}(x)p_m^{[l]}(x)$ has degree $n+m-2l-2$. Hence the expansions have finite ranges and by the recurrence relation we obtain expansions of the form

$$p_n(x)p_m(x) = \sum_{k=|n-m|}^{n+m} c(n,m,k)p_k(x), \tag{2.4}$$

$$p_n^{[l]}(x)p_m^{[l]}(x) = \sum_{k=|n-m|+l+1}^{n+m-l-1} c_l(n,m,k)p_k^{[l]}(x). \tag{2.5}$$

**Definition 2.1.** *The system of orthogonal polynomials $p_n$ satisfies* ***the strong nonnegative linearization property*** (SNLP) *if*

$$c(n,m,k) \geq 0, \tag{2.6}$$

$$c_l(n,m,k) \geq 0, \tag{2.7}$$

*for any $n, m, k \geq 0$ and $l \geq 0$.*

The form of recurrence relation used in (2.1) and (2.2) is suitable for applications. For technical reasons we will work with the renormalized polynomials $P_n$ and $P_n^{[l]}$ defined as

$$P_n(x) = \frac{\gamma_0\gamma_1\dots\gamma_{n-1}}{\alpha_1\alpha_2\dots\alpha_n}p_n(x), \quad n \geq 1,$$
$$P_n^{[l]}(x) = \frac{\gamma_l\gamma_{l+1}\dots\gamma_{n-1}}{\alpha_{l+1}\alpha_{l+2}\dots\alpha_n}p_n^{[l]}(x), \quad n \geq l+1.$$

Clearly the property of strong nonnegative linearization is equivalent for the systems $\{p_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$, so we can work with the latter system from now on.

The polynomials $P_n$ satisfy

$$xP_n = \alpha_{n+1}P_{n+1} + \beta_n P_n + \gamma_{n-1}P_{n-1}, \qquad n \geq 0, \tag{2.8}$$

where $\gamma_{-1} = 0$. On the other hand the polynomials $P_n^{[l]}$ satisfy

$$xP_n^{[l]} = \alpha_{n+1}P_{n+1}^{[l]} + \beta_n P_n^{[l]} + \gamma_{n-1}P_{n-1}^{[l]}, \qquad n \geq l+1. \tag{2.9}$$

Moreover by (2.4) and (2.5) we have

$$P_n(x)P_m(x) = \sum_{k=|n-m|}^{n+m} C(n,m,k)P_k(x), \tag{2.10}$$

$$P_n^{[l]}(x)P_m^{[l]}(x) = \sum_{k=|n-m|+l+1}^{n+m-l-1} C_l(n,m,k)P_k^{[l]}(x). \tag{2.11}$$

Let $L$ denote a linear operator acting on sequences $a = \{a_n\}_{n=0}^{\infty}$ by the rule

$$(La)_n = \alpha_{n+1}a_{n+1} + \beta_n a_n + \gamma_{n-1}a_{n-1}, \qquad n \geq 0. \tag{2.12}$$

For any real number $x$ set

$$P(x) = \{P_n(x)\}_{n=0}^{\infty},$$
$$P^{[l]}(x) = \{P_n^{[l]}(x)\}_{n=0}^{\infty}.$$

Let $\delta_l$ denote the sequence whose terms are equal to zero except for the $l$th term which is equal to 1. The formulas (2.8), (2.9) and the fact that $P_{l+1}^{[l]} = \alpha_{l+1}^{-1}$ immediately imply that

$$LP(x) = xP(x), \tag{2.13}$$
$$LP^{[l]}(x) = xP^{[l]}(x) + \delta_l. \tag{2.14}$$

## 3. Hyperbolic boundary value problem and basic solutions

Let $u(n,m)$ be a matrix defined for $n \geq m \geq 0$. We introduce the operator $H$ acting on the matrices by the rule

$$\begin{aligned}(Hu)(n,m) &= \alpha_{n+1}u(n+1,m) + \beta_n u(n,m) + \gamma_{n-1}u(n-1,m) \\ &\quad - \alpha_{m+1}u(n,m+1) - \beta_m u(n,m) - \gamma_{m-1}u(n,m-1),\end{aligned} \tag{3.1}$$

for $n > m \geq 0$. By (2.13), if we take $u(n,m) = P_n(x)P_m(x)$ for some $x$, then

$$(Hu)(n,m) = 0. \tag{3.2}$$

Similarly by (2.14), if we take $u(n,m) = P_n^{[l]}(x)P_m^{[l]}(x)$, then

$$(Hu)(n,m) = P_m^{[l]}(x)\delta_l(n) - P_n^{[l]}(x)\delta_l(m).$$

Assume $n > m$. Then $n = l$ implies $P_m^{[l]}(x) = 0$. Hence

$$(Hu)(n,m) = -P_n^{[l]}(x)\delta_l(m), \text{ for } n > m \geq 0. \tag{3.3}$$

**Proposition 3.1.** *Given a matrix $v = \{v(n,m)\}_{n>m\geq 0}$ and a sequence $f = \{f(n)\}_{n\geq 0}$. Let $u = \{u(n,m)\}_{n\geq m\geq 0}$ satisfy*

$$\begin{aligned} Hu(n,m) &= v(n,m), \text{ for } n > m \geq 0, \\ u(n,0) &= f(n), \text{ for } n \geq 0. \end{aligned}$$

*Then*

$$u(n,m) = -\sum_{k>l\geq 0} v(k,l)C_l(n,m,k) + \sum_{k\geq 0} f(k)C(n,m,k).$$

*Proof.* The formula (3.1) and the fact that $\alpha_m > 0$ imply that $u$ is uniquely determined.

Let $u_k(n,m) = C(n,m,k)$. By (2.10) we have

$$u_k(n,m) = \left(\int_{\mathbb{R}} P_k^2(x)\,d\mu(x)\right)^{-1} \int_{\mathbb{R}} P_n(x)P_m(x)P_k(x)\,d\mu(x).$$

Therefore by (3.2) we obtain

$$\begin{aligned} (Hu_k)(n,m) &= 0, \text{ for } n > m \geq 0, \\ u_k(n,0) &= \delta_k(n), \text{ for } n \geq 0. \end{aligned}$$

For $k > l \geq 0$ let $u_{k,l}(n,m) = C_l(n,m,k)$. By (2.11) we have

$$u_{k,l}(n,m) = \left( \int_{\mathbb{R}} \left\{ P_k^{[l]}(x) \right\}^2 d\mu_l(x) \right)^{-1} \int_{\mathbb{R}} P_n^{[l]}(x) P_m^{[l]}(x) P_k^{[l]}(x)\, d\mu_l(x).$$

Thus by (3.3) we get

$$\begin{aligned} (Hu_{k,l})(n,m) &= -\delta_{(k,l)}(n,m), \text{ for } n > m \geq 0, \\ u_{k,l}(n,0) &= 0, \text{ for } n \geq 0. \end{aligned}$$

Hence the matrix

$$u(n,m) = -\sum_{k>l\geq 0} v(k,l) u_{k,l}(n,m) + \sum_{k\geq 0} f(k) u_k(n,m)$$

satisfies the assumptions of Proposition 1. By uniqueness we have $u = u$. □

Let $H^*$ denote the adjoint operator to $H$ with respect to the inner product of matrices

$$\langle u, v \rangle = \sum_{n>m\geq 0}^{\infty} u(n,m)\overline{v(n,m)}.$$

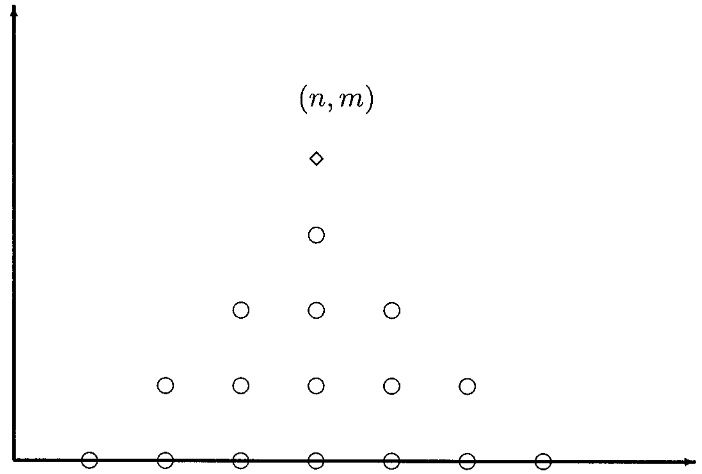
The explicit action of this operator is given by the following.

$$\begin{aligned} (H^*v)(n,m) =& \gamma_n v(n+1,m) + \beta_n v(n,m) + \alpha_n v(n-1,m) \\ & - \gamma_m v(n,m+1) - \beta_m v(n,m) - \alpha_m v(n,m-1). \end{aligned}$$

For each point $(n,m)$ with $n \geq m \geq 0$, let $\Delta_{n,m}$ denote the set of lattice points located in the triangle with vertices in $(n-m+1,0)$, $(n+m-1,0)$ nad $(n,m-1)$, i.e.

$$\Delta_{n,m} = \{(i,j) \,|\, 0 \leq j \leq i, |n-i| < m-j\}.$$

The points of $\Delta_{n,m}$ are marked in the picture below with empty circles.




By (Szwarc, 2003, Theorem 1) nonnegative linearization is equivalent to the fact that for every $(n, m)$ with $n \geq m \geq 0$ there exists a matrix $v$ such that

$$\operatorname{supp} v \subset \Delta_{n,m}, \tag{3.4}$$

$$(H^* v)(n, m) < 0, \tag{3.5}$$

$$(H^* v)(i, j) \geq 0, \text{ for } (i, j) \neq (n, m). \tag{3.6}$$

**Definition 3.2.** *Any matrix $v$ satisfying* (3.4) *and* (3.5) *will be called **a triangle function**.*

**Definition 3.3.** *Let $v_{n,m}$ denote a matrix satisfying*

$$\operatorname{supp} v_{n,m} \subset \Delta_{n,m}, \tag{3.7}$$

$$(H^* v_{n,m})(n, m) = -1, \tag{3.8}$$

$$(H^* v_{n,m})(i, j) = 0, \text{ for } 0 < j < m \tag{3.9}$$

*The matrix $v_{n,m}$ will be called **the basic triangle function**.*

The main result of this section relates the values of $v_{n,m}(k, l)$ to the coefficients $C_l(n, m, k)$.

**Theorem 3.4.** *For any $n \geq m \geq 0$ and $k > l \geq 0$ we have*

$$v_{n,m}(k, l) = C_l(n, m, k).$$

*Moreover*

$$H^* v_{n,m} = -\delta_{(n,m)} + \sum_{k=n-m}^{n+m} C(n, m, k)\delta_{(k,0)}.$$

*Proof.* Let $u(n,m) = P_n^{[l]}(x)P_m^{[l]}(x)$. We have $P_0^{[l]} = 0$, hence by (3.3), (3.8) and (3.9) we obtain

$$\begin{aligned} -P_n^{[l]}(x)P_m^{[l]}(x) &= -u(n,m) = \langle H^* v_{n,m}, u\rangle = \langle v_{n,m}, Hu\rangle \\ &= \sum_{k,j} v_{n,m}(k,j)(Hu)(k,j) = -\sum_k v_{n,m}(k,l)P_k^{[l]}(x). \end{aligned}$$

Thus by (2.11) we get $v_{n,m}(k,l) = C_l(n,m,k)$. The second part of the statement follows from (Szwarc, 2003, Lemma), but we will recapitulate the proof here for completeness. By (3.8) and (3.9) we have

$$H^* v_{n,m} = -\delta_{(n,m)} + \sum_k d_k \delta_{(k,0)}.$$

Let $u(n,m) = P_n(x)P_m(x)$. Since $Hu = 0$, we have

$$\begin{aligned} P_n(x)P_m(x) = u(n,m) &= -\langle H^* v_{n,m}, u\rangle + \sum_k d_k u(k,0) \\ &= \langle v_{n,m}, Hu\rangle + \sum_k d_k P_k(x) = \sum_k d_k P_k(x). \end{aligned}$$

Hence $d_k = C(n,m,k)$. □

## 4. Main results

The main result of this paper is the following.

**Theorem 4.1.** *Let $p_n$ be a system of orthogonal polynomials satisfying the recurrence relation*

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1},$$

*where $p_{-1} = 0$ and $p_0 = 1$. Then the following four conditions are equivalent.*

(a) *The polynomials $p_n$ satisfy the strong nonnegative linearization property.*

(b) *Let $u = \{u(n,m)\}_{n\geq m\geq 0}$ satisfy*

$$\begin{cases} (Hu)(n,m) \leq 0, & \text{for } n > m \geq 0, \\ u(n,0) \geq 0. \end{cases}$$

*Then $u(n,m) \geq 0$ for every $n \geq m \geq 0$.*

(c) *For every $n \geq m \geq 0$ there exists a triangle function $v$, satisfying*

(i) $\operatorname{supp} v \subset \Delta_{n,m}$.

(ii) $(H^*v)(n,m) < 0$.

(iii) $(H^*v)(i,j) \geq 0$ *for* $(i,j) \neq (n,m)$.

(iv) $v \geq 0$.

(d) *The basic triangle functions* $v_{n,m}$ *(see* (3.7), (3.8), (3.9)*) satisfy*

(i) $(H^*v_{n,m})(i,0) \geq 0$.

(ii) $v_{n,m} \geq 0$.

*Proof.*
(b)⇒(a)
By the proof of Proposition 1 we have that if $u_k(n,m) = C(n,m,k)$ and $u_{k,l}(n,m) = C_l(n,m,k)$ then

$$\begin{aligned} (Hu_k)(n,m) &= 0, & \qquad (Hu_{k,l})(n,m) &= -\delta_{(k,l)}(n,m) \\ u_k(n,0) &= \delta_k(n), & u_{k,l}(n,0) &= 0. \end{aligned}$$

for $n > m \geq 0$. Thus $C(n,m,k) \geq 0$ and $C_l(n,m,k) \geq 0$ for $n \geq m \geq 0$.
(a)⇒(d)
This follows immediately by Theorem 1.
(d)⇒(c)
This is clear by definition.
(c)⇒(b)
Let $u = \{u(n,m)\}_{n \geq m \geq 0}$ satisfy $(Hu)(n,m) \leq 0$, for $n > m \geq 0$ and $u(n,0) \geq 0$. We will show that $u(n,m) \geq 0$, by induction on $m$. Assume that $u(i,j) \geq 0$ for $j < m$. Let $v$ be a triangle function satisfying the assumptions (c). Then

$$0 \geq \langle Hu, v \rangle = \langle u, H^*v \rangle = u(n,m)(H^*v)(n,m) + \sum_{\substack{i \geq j \geq 0 \\ j < m}} u(i,j)(H^*v)(i,j)$$

Therefore

$$-u(n,m)(H^*v)(n,m) \geq \sum_{\substack{i \geq j \geq 0 \\ j < m}} u(i,j)(H^*v)(i,j),$$

and the conclusion follows. □

**Remark 4.2.** *Theorem 2 should be juxtaposed with the following result which can be derived from (Szwarc, 2003, Theorem 1).*

**Theorem 4.3.** *Let $p_n$ be a system of orthogonal polynomials satisfying the recurrence relation*

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1},$$

*where $p_{-1} = 0$ and $p_0 = 1$. Then the following four conditions are equivalent.*

(a) *The polynomials $p_n$ satisfy nonnegative linearization property.*

(b) *Let $u = \{u(n,m)\}_{n\geq m\geq 0}$ satisfy*

$$\begin{cases} (Hu)(n,m) &= 0, \text{ for } n > m \geq 0, \\ u(n,0) &\geq 0. \end{cases}$$

*Then $u(n,m) \geq 0$ for every $n \geq m \geq 0$.*

(c) *For every $n \geq m \geq 0$ there exists a triangle function $v$, satisfying*

(i) $\operatorname{supp} v \subset \Delta_{n,m}$.

(ii) $(H^*v)(n,m) < 0$.

(iii) $(H^*v)(i,j) \geq 0$ *for* $(i,j) \neq (n,m)$.

(d) *The basic triangle functions $v_{n,m}$ (see* (3.7), (3.8), (3.9)) *satisfy*

(i) $(H^*v_{n,m})(i,0) \geq 0$.

One of the advantages of the strong nonnegative linearization property is its stability for a certain perturbation of the coefficients in the recurrence relation. Namely the following holds.

**Proposition 4.4.** *Assume orthogonal polynomial system $\{p_n\}_{n=0}^{\infty}$ satisfies* (SNLP). *Let $\varepsilon_n$ be a nondecreasing sequence. Let $q_n$ be a sequence of polynomials satisfying the perturbed recurrence relation*

$$xq_n = \gamma_n q_{n+1} + (\beta_n + \varepsilon_n)q_n + \alpha_n q_{n-1},$$

*for $n \geq 0$. Then the system $\{q_n\}_{n=0}^{\infty}$ satisfies* (SNLP).

*Proof.* We will make use of Theorem 4.1(c). Let $H$ and $H_\varepsilon$ denote the hyperbolic operators corresponding to the unperturbed and perturbed system, respectively. For any matrix $v(i,j)$ we have

$$(H_\varepsilon^* v)(i,j) = (H^*v)(i,j) + (\varepsilon_i - \varepsilon_j)v(i,j). \tag{4.1}$$

By assumptions for any $n \geq m \geq 0$, there exists a triangle function $v$ satisfying the assumptions of Theorem 4.1(c) with respect to $H$. By

(4.1) the same matrix $v$ satisfies these assumptions with respect to $H_\varepsilon$. Indeed, the assumptions (i) and (iv) do not depend on the perturbation. Since $v(n,m) = 0$ the assumption (ii) is not affected, as well. Concerning (iii), since $v \geq 0$ and $\varepsilon_n$ is nondecreasing we have

$$(H_\varepsilon^* v)(i,j) \geq (H^* v)(i,j) \geq 0,$$

for $i \geq j \geq 0$ and $j < m$. Hence the perturbed system of polynomials satisfies (SNLP). □

## 5. Some necessary and sufficient conditions

We begin with the following generalization of Theorem 1 of (Szwarc, 1992a).

**Theorem 5.1.** *Let orthogonal polynomials* $\{p_n\}_{n=0}^\infty$ *satisfy* (2.1). *Let* $\{c_n\}_{n=0}^\infty$ *be a fixed sequence of positive numbers with* $c_0 = 1$ *and*

$$\alpha_n' = \frac{c_{n-1}}{c_n}\alpha_n, \qquad \gamma_n' = \frac{c_{n+1}}{c_n}\gamma_n, \qquad \text{for } n \geq 1.$$

*Assume that*

(i) $\beta_m \leq \beta_n$ *for* $m \leq n$.

(ii) $\alpha_m \leq \alpha_n'$ *for* $m < n$.

(iii) $\alpha_m + \gamma_m \leq \alpha_n' + \gamma_n'$ *for* $m < n$.

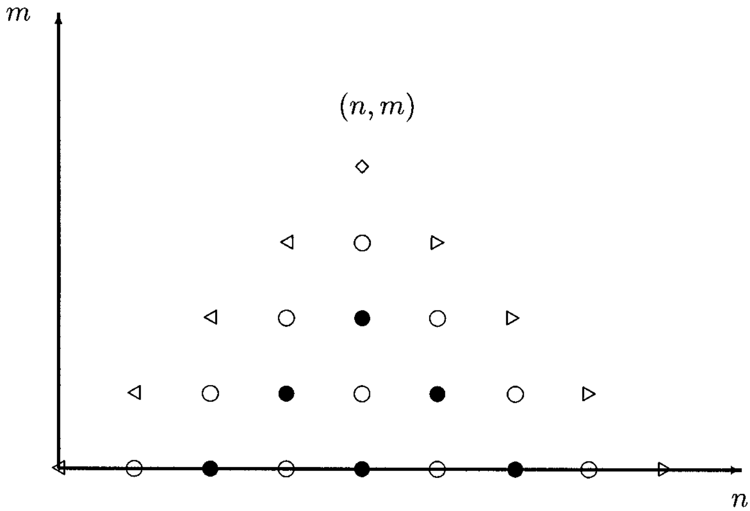
(iv) $\alpha_m \leq \gamma_n'$ *for* $m \leq n$.

*Then the system* $\{p_n\}_{n=0}^\infty$ *satisfies the strong nonnegative linearization property.*

*Proof.* It suffices to construct a suitable triangle function for every $(n,m)$, with $n \geq m$, i.e., a matrix $v$ satisfying the assumptions of Theorem 4.1(c). Fix $(n,m)$. Define the matrix $v$ according to the following.

$$v(i,j) = \begin{cases} c_i & (i,j) \in \Delta_{n,m},\ (n+m) - (i+j) \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

The points in the support of $v_{n,m}$ are marked by empty circles in the picture below.




Then supp $H^*v$ consists of the points marked by $\circ, \bullet, \triangleleft, \triangleright$ and $\diamond$. A straightforward computation gives

$$(H^*v)(i,j) = \begin{cases} -\alpha_m c_n & (i,j) = (n,m) \\ (\beta_i - \beta_j)c_i & (i,j) \; - \; \circ \\ \alpha_i c_{i-1} + \gamma_i c_{i+1} - \alpha_j c_i - \gamma_j c_i & (i,j) \; - \; \bullet \\ \alpha_i c_{i-1} - \alpha_j c_i & (i,j) \; - \; \triangleright \\ \gamma_i c_{i+1} - \alpha_j c_i & (i,j) \; - \; \triangleleft \end{cases}$$

Hence $H^*v$ satisfies the assumptions of Theorem 4.1(c). □

Applying Theorem 5.1 to the sequences

$$c_n = 1 \quad \text{or} \quad c_n = \frac{\alpha_1 \alpha_2 \ldots \alpha_n}{\gamma_0 \gamma_1 \cdots \gamma_{n-1}}$$

gives the following.

**Corollary 5.2.** *Let orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy* (2.1). *If the sequences $\alpha_n$, $\beta_n$, $\alpha_n + \gamma_n$ are nondecreasing and $\alpha_n \le \gamma_n$ for all $n$, then the system $\{p_n\}_{n=0}^{\infty}$ satisfies the strong nonnegative linearization property.*

**Corollary 5.3.** *Let orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy* (2.1). *Assume that*

(i) $\beta_m \le \beta_n$ *for* $m \le n$

(ii) $\alpha_m \le \gamma_n$ *for* $m \le n$

(iii) $\alpha_m + \gamma_m \leq \alpha_{n-1} + \gamma_{n+1}$ *for* $m < n$

(iv) $\alpha_m \leq \alpha_n$ *for* $m \leq n$

*Then the system* $\{p_n\}_{n=0}^{\infty}$ *satisfies the strong nonnegative linearization property.*

Now we turn to necessary conditions for (SNLP).

**Proposition 5.4.** *Assume a system* $\{p_n\}_{n=0}^{\infty}$ *satisfies the strong nonnegative linearization property. Then the sequence* $\beta_n$ *is nondecreasing.*

*Proof.* By (2.2) we can compute that for $n \geq 2$ we have

$$p_n^{[n-2]}(x) = \frac{1}{\gamma_{n-2}\gamma_{n-1}}(x - \beta_{n-1}).$$

But by (2.1) we have

$$(x - \beta_{n-1})p_n^{[n-2]} = \gamma_n p_{n+1}^{[n-2]} + (\beta_n - \beta_{n-1})p_n^{[n-2]} + \alpha_n p_{n-1}^{[n-2]}.$$

Thus $\beta_n \geq \beta_{n-1}$ for $n \geq 2$. On the other hand

$$p_1(x) = \frac{1}{\gamma_0}(x - \beta_0)$$

and

$$(x - \beta_0)p_1 = \gamma_1 p_2 + (\beta_1 - \beta_0)p_1 + \alpha_1 p_0.$$

Hence $\beta_1 \geq \beta_0$. □

## 6. Jacobi polynomials

The Jacobi polynomials $J_n^{(\alpha,\beta)}$ satisfy the recurrence relation

$$\begin{aligned} xJ_n^{(\alpha,\beta)} = {} & \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} J_{n+1}^{(\alpha,\beta)} \\ & + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} J_n^{(\alpha,\beta)} \\ & + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} J_{n-1}^{(\alpha,\beta)}. \end{aligned} \tag{6.1}$$

**Theorem 6.1.** *The Jacobi polynomials satisfy the strong nonnegative linearization property if and only if either* $\alpha > \beta > -1$ *and* $\alpha + \beta \geq 0$ *or* $\alpha = \beta \geq -\frac{1}{2}$.

*Proof.* Assume the Jacobi polynomials satisfy (SNLP). In particular they have nonnegative linearization property. By (Gasper, 1970a) we know that the condition $\alpha \geq \beta$ is necessary for nonnegative linearization to hold. Also if $\alpha = \beta$ then the condition $\alpha \geq -\frac{1}{2}$ is necessary (see (Askey, 1975)). Let $\alpha > \beta$. By Proposition 5.4 the sequence

$$\beta_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$$

should be nondecreasing, which holds only if $\alpha + \beta \geq 0$. Hence the conditions on $\alpha$ and $\beta$ are necessary for (SNLP).

Now we are going to show that the conditions on the parameters are also sufficient for (SNLP). Assume first that $\alpha = \beta \geq -1/2$. Let

$$R_n(x) = \frac{J_n^{(\alpha,\alpha)}(x)}{J_n^{(\alpha,\alpha)}(1)}.$$

Then by (Koekeok and Swarttouw, 1998, (1.8.1), (1.8.3)) the polynomials satisfy

$$xR_n(x) = \frac{n+2\alpha+1}{2n+2\alpha+1}R_{n+1}(x) + \frac{n}{2n+2\alpha+1}R_{n-1}(x).$$

Hence by Corollary 5.2 the polynomials satisfy (SNLP).

Assume now that $\alpha > \beta > -1$ and $\alpha + \beta \geq 0$. Let $p_n(x)$ denote the monic version of Jacobi polynomials, i.e., let

$$p_n(x) = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n} J_n^{(\alpha,\beta)}(x).$$

By (Askey, 1970) the polynomials $p_n$ satisfy the assumptions of Corollary 5.2 if $\alpha + \beta \geq 1$. Hence they satisfy (SNLP).

We have to consider the remaining case when $\alpha > \beta > -1$ and $0 \leq \alpha + \beta < 1$. By (6.1) we have

$$\alpha_n = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \quad n > 0, \tag{6.2}$$

$$\beta_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{6.3}$$

$$\gamma_n = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{6.4}$$

These numbers satisfy the assumptions of Corollary 5.3 for $\alpha \geq \beta$ and $0 \leq \alpha + \beta \leq 1$. Indeed, observe that for $n > 0$ we have

$$2\alpha_n - 1 = -\frac{(\alpha-\beta)^2}{2n+\alpha+\beta} + \frac{(\alpha-\beta)^2 - 1}{2n+\alpha+\beta+1},$$

$$2\gamma_{n-1} - 1 = -\frac{(\alpha+\beta)^2 - 1}{2n+\alpha+\beta-1} + \frac{(\alpha+\beta)^2}{2n+\alpha+\beta},$$

and

$$\begin{aligned}
&2(\alpha_n+\gamma_n)-2\\
&=\frac{-4\alpha\beta}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}-\frac{2(\alpha-\beta)^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\\
&=\frac{4\alpha\beta}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}-\frac{2(\alpha+\beta)^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}.
\end{aligned} \tag{6.5}$$

These calculations are valid only for $n > 0$, because $\alpha_0 = 0$ does not coincide with (6.2). The formulas (6.2) and (6.4) show that $\alpha_n$ is nondecreasing and $\gamma_n$ is nonincreasing when $\alpha + \beta \le 1$. Both sequences tend to $\frac{1}{2}$. This gives the conditions (ii) and (iv) of Corollary 5.3. The formula (6.5) shows that $\alpha_n + \gamma_n$ is nondecreasing for $n > 0$, regardless the sign of $\alpha\beta$. This and the fact that $\alpha_n$ is nondecreasing imply

$$\alpha_m+\gamma_m \le \alpha_{n-1}+\gamma_{n-1} \le \alpha_{n+1}+\gamma_{n-1}, \quad 0<m<n-1.$$

Thus the condition (iii) of Corollary 5.3 is satisfied for $0 < m < n-1$. It remains to show the condition (iii) for $m = 0$, i.e.

$$\alpha_0+\gamma_0=\gamma_0=\frac{2}{2+\alpha+\beta}\le \alpha_{n+1}+\gamma_{n-1}.$$

By (6.2) and (6.4) the above inequality is equivalent to the following.

$$\begin{aligned}
-\frac{(\alpha-\beta)^2}{2n+\alpha+\beta+2}&+\frac{(\alpha-\beta)^2-1}{2n+\alpha+\beta+3}\\
&-\frac{(\alpha+\beta)^2-1}{2n+\alpha+\beta-1}+\frac{(\alpha+\beta)^2}{2n+\alpha+\beta}\ge -\frac{2(\alpha+\beta)}{2+\alpha+\beta}.
\end{aligned} \tag{6.6}$$

Observe that the left hand side of (6.6) is a decreasing function of $\alpha-\beta$. Therefore we can assume that $\alpha-\beta$ attains the maximal possible value, i.e., $\beta=-1$. Let $\beta=-1$ and $x = 2n+\alpha+\beta+1$. Then $x \ge 2+\alpha+\beta+1 \ge 3$. The left hand side of (6.6) can be now written as follows.

$$\begin{aligned}
&-\frac{(\alpha+1)^2}{x+1}+\frac{(\alpha+1)^2-1}{x+2}-\frac{(\alpha-1)^2-1}{x-2}+\frac{(\alpha-1)^2-1}{x-1}\\
&=\frac{4}{(x-2)(x+2)}-\frac{(\alpha+1)^2}{(x+1)(x+2)}-\frac{(\alpha-1)^2}{(x-1)(x-2)}\\
&=\frac{4}{(x-2)(x+2)}-\frac{4}{(x+1)(x+2)}+\frac{4-(\alpha+1)^2}{(x+1)(x+2)}-\frac{(\alpha-1)^2}{(x-1)(x-2)}.
\end{aligned}$$

The first two terms of the last expression give a positive contribution to the sum because $x > 2$. Hence it suffices to show that

$$\frac{4-(\alpha+1)^2}{(x+1)(x+2)} - \frac{(\alpha-1)^2}{(x-1)(x-2)} \geq -\frac{2(\alpha-1)}{\alpha+1}. \tag{6.7}$$

Note that $\alpha - 1 \geq 0$ (as $\beta = -1$). Thus $\alpha + 1 \geq 2$ and $4 - (\alpha+1)^2 \leq 0$. Hence the left hand side of (6.7) is a nondecreasing function of $x$. Therefore we can verify (6.7) only for the smallest value of $x$, that is for $x = 2 + \alpha + \beta + 1 = 2 + \alpha$. Under substitution $x = 2 + \alpha$ the inequality (6.7) takes the form

$$\frac{4-(\alpha+1)^2}{(\alpha+3)(\alpha+4)} - \frac{(\alpha-1)^2}{(\alpha+1)\alpha} \geq -\frac{2(\alpha-1)}{\alpha+1}.$$

After simple transformations it reduces to

$$\frac{1}{\alpha+4} \leq \frac{1}{\alpha},$$

which is true because $\alpha$ is nonnegative. Summarizing, Corollary 5.3 yields that for $\alpha \geq \beta$ and $0 \leq \alpha + \beta \leq 1$ we get (SNLP). □

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