Positivity of Turán determinants for orthogonal polynomials

Ryszard Szwarc*

Abstract

The orthogonal polynomials p_n satisfy Turán's inequality if $p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0$ for $n \geq 1$ and for all x in the interval of orthogonality. We give general criteria for orthogonal polynomials to satisfy Turán's inequality. This yields the known results for classical orthogonal polynomials as well as new results, for example, for the q–ultraspherical polynomials.

1 Introduction

In the 1940's, while studying the zeros of Legendre polynomials $P_n(x)$, Turán [T] discovered that

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad -1 \leq x \leq 1 \tag{1}$$

with equality only for $x = \pm 1$. Szegö [Sz1] gave four different proofs of (1). Shortly after that, analogous results were obtained for other classical orthogonal polynomials such as ultraspherical polynomials [Sk, S], Laguerre and Hermite polynomials [MN], and Bessel functions [Sk, S].

In [KS] Karlin and Szegö raised the question of determining the range of parameters (α, β) for which (1) holds for Jacobi polynomials of order (α, β); i.e. denoting $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$,

$$[R_n^{(\alpha,\beta)}(x)]^2 - R_{n-1}^{(\alpha,\beta)}(x)R_{n+1}^{(\alpha,\beta)}(x) \geq 0, \qquad -1 \leq x \leq 1. \tag{2}$$

*This work has been partially supported by KBN (Poland) under grant 2 P03A 030 09.

[0]1991 *Mathematics Subject Classification.* Primary 42C05, 47B39

[0]*Key words and phrases*: orthogonal polynomials, Turán's inequality, recurrence formula.

In 1962 Szegö [Sz2] proved (2) for $\beta \geq |\alpha|,\ \alpha > -1$. In a series of two papers [G1, G2] Gasper extended Szegö's result by showing that (2) holds if and only if $\beta \geq \alpha > -1$.

More recently, attention has also turned to the q-analogues of the classical polynomials [BI1].

All the results mentioned above were proved using differential equations, that the classical orthogonal polynomials satisfy. Therefore the methods cannot be used to extend (1) to more general orthogonal polynomials. In 1970 Askey [A, Thm. 3] gave a general criterion for monic symmetric orthogonal polynomials to satisfy the Turán type inequality on the entire real line. His result, however, does not imply (1) for the Legendre polynomials because the latter are not monic in the standard normalization, and they do not satisfy Askey's assumptions in the monic normalization. In this paper we give general criteria for orthogonal polynomials implying (1) holds for x in the support of corresponding orthogonality measure. The assumptions are stated in terms of the coefficients of the recurrence relation that the orthogonal polynomials satisfy. They admit a very simple form in the case of symmetric orthogonal polynomials; i.e. the case $p_n(-x) = (-1)^n p_n(x)$. In particular, the results apply to all the ultraspherical polynomials, giving yet another proof of Turán's inequality for the Legendre polynomials.

It turns out that the way we normalize the polynomials is essential for the Turán inequality to hold. The results concerning the classical orthogonal polynomials used the normalization at one endpoint of the interval of orthogonality, e.g. at $x = 1$ for the Jacobi polynomials and at $x = 0$ for the Laguerre polynomials. We will also use this normalization and will show that this choice is optimal (Proposition 1). However, the recurrence relation for the polynomials normalized in this way may not be available explicitly. This is the case of the q–ultraspherical polynomials. We give a way of overcoming this obstacle (Corollary 1). In particular, we prove the Turán inequality for all q-ultraspherical polynomials with $q > 0$. These polynomials have been studied by Bustoz and Ismail [BI1] but with a normalization other than at $x = 1$. The same method is applied to the symmetric Pollaczek polynomials, studied in [BI2], again with different normalization.

In Section 6 we prove results for nonsymmetric orthogonal polynomials (Thm. 4). The assumptions again are given in terms of the coefficients in a three term recurrence relation but they are much more involved.

In Section 7 we state results concerning polynomials orthogonal on the positive half axis. In particular they can be applied to the Laguerre polynomials of any order α.

2 Basic formulas.

Let p_n be polynomials orthogonal with respect to a probability measure on $\mathbb{R}$. The expressions

$$\Delta_n(x) = p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \quad n = 0, 1, \ldots, \tag{3}$$

are called the Turán determinants. Our goal is to give conditions implying the nonnegativity of $\Delta_n(x)$ for x in the support of the orthogonality measure.

The first problem we encounter is that the orthogonality determines the polynomials p_n up to a nonzero multiple. The sign of $\Delta_n(x)$ may change if we multiply each p_n by different nonzero constants. We will normalize the polynomials p_n to obtain the sharpest results possible. Namely, we will assume that

$$p_n(a) = 1$$

at a point a in the support of the orthogonality measure. In this way the Turán determinant vanishes at $x = a$.

Our main interest is focused on the case when the orthogonality measure is supported in a bounded interval. By an affine change of variables we can assume that this interval is $[-1, 1]$. In that case we set $a = 1$. Since the polynomials p_n do not change sign in the interval $[1, +\infty)$ they have positive leading coefficients.

Assume that the polynomials p_n are orthogonal, with positive leading coefficients and $p_n(1) = 1$. Then they satisfy the three term recurrence relation

$$xp_n(x) = \gamma_n p_{n+1}(x) + \beta_n p_n(x) + \alpha_n p_{n-1}(x) \quad n = 0, 1, \ldots, \tag{4}$$

with initial conditions $p_{-1} = 0$, $p_0 = 1$, where α_n, β_n, and γ_n are given sequences of real valued coefficients such that

$$\alpha_0 = 0, \quad \alpha_{n+1} > 0, \quad \gamma_n > 0 \quad \text{for } n = 0, 1, \ldots.$$

Plugging $x = 1$ into (4) gives

$$\alpha_n + \beta_n + \gamma_n = 1 \qquad n = 0, 1, \ldots. \tag{5}$$

Proposition 1 *Let the polynomials p_n satisfy (4) and (5). Then*

$$\gamma_n\Delta_n = \gamma_n p_n^2 + \alpha_n p_{n-1}^2 - (x-\beta_n)p_{n-1}p_n, \tag{6}$$
$$\gamma_n\Delta_n = (p_{n-1}-p_n)[(\gamma_{n-1}-\gamma_n)p_n + (\alpha_n-\alpha_{n-1})p_{n-1}] + \alpha_{n-1}\Delta_{n-1}, \tag{7}$$
$$\gamma_n\Delta_n = (p_n-p_{n-1})(\gamma_n p_n - \alpha_n p_{n-1}) + (1-x)p_{n-1}p_n, \tag{8}$$

for $n = 1, 2, \ldots$.

Proof. By (4) we get

$$\begin{aligned}\gamma_n\Delta_n &= \gamma_n p_n^2 - \gamma_n p_{n-1}[(x-\beta_n) - \alpha_n p_{n-1}]\\ &= \gamma_n p_n^2 + \alpha_n p_{n-1}^2 - (x-\beta_n)p_{n-1}p_n\\ &= \gamma_n p_n^2 + \alpha_n p_{n-1}^2 - (\beta_{n-1}-\beta_n)p_{n-1}p_n - (x-\beta_{n-1})p_{n-1}p_n.\end{aligned}$$

Now applying (4), with n replaced by $n-1$, to the last term yields

$$\gamma_n\Delta_n = (\gamma_n-\gamma_{n-1})p_n^2 + (\alpha_n-\alpha_{n-1})p_{n-1}^2 - (\beta_{n-1}-\beta_n)p_{n-1}p_n + \alpha_{n-1}\Delta_{n-1}.$$

The use of

$$\beta_{n-1}-\beta_n = (\gamma_n-\gamma_{n-1}) + (\alpha_n-\alpha_{n-1})$$

concludes the proof of (7). In order to get (8) replace β_n with $1-\alpha_n-\gamma_n$ in (6). □

3 Symmetric polynomials

We will consider first the symmetric orthogonal polynomials, i.e. the orthogonal polynomials satisfying

$$p_n(-x) = (-1)^n p_n(x). \tag{9}$$

Theorem 1 *Let the polynomials p_n satisfy*

$$xp_n(x) = \gamma_n p_{n+1}(x) + \alpha_n p_{n-1}(x) \qquad n = 0, 1, \ldots. \tag{10}$$

with $p_{-1}=0$, $p_0=1$, where $\alpha_0=0$, $\alpha_{n+1}>0$, $\gamma_n>0$, and

$$\alpha_n + \gamma_n = a \qquad n = 0, 1, \ldots.$$

Assume that either (i) or (ii) is satisfied where *(i)* α_n *is nondecreasing and* $\alpha_n \leq \frac{a}{2}$ *for* $n = 1, 2, \ldots$.

(ii) α_n *is nonincreasing and* $\alpha_n \geq \frac{a}{2}$ *for* $n = 1, 2, \ldots$.

Then

$$\Delta_n(x) \geq 0, \qquad \text{for } -a \leq x \leq a, \quad n = 0, 1, \ldots,$$

and the equality holds if and only if $n \geq 1$ *and* $x = \pm a$.

Moreover if (i) is satisfied then

$$\Delta_n(x) < 0, \qquad \text{for } |x| > a, \quad n = 1, 2, \ldots.$$

Proof. By changing variable $x \to ax$ we can restrict ourselves to the case $a = 1$. We prove part (i) only, because the proof of (ii) can be obtained from that of (i) by obvious modifications.

By assumption we have $p_n(1) = 1$ and $p_n(-1) = (-1)^n$. Hence $\Delta_n(\pm 1) = 0$. Assume now that $|x| < 1$. By (9) it suffices to consider $0 \leq x < 1$. The proof will go by induction. We have $\gamma_1 \Delta_1(x) = \alpha_1(1 - x^2) \geq 0$. Now assume $\Delta_{n-1}(x) > 0$.

In view of $\beta_n = 0$ and $\alpha_n - \alpha_{n-1} = \gamma_{n-1} - \gamma_n$, Proposition 1 implies

$$\gamma_n \Delta_n = \gamma_n p_n^2 + \alpha_n p_{n-1}^2 - x p_{n-1} p_n, \qquad (11)$$

$$\gamma_n \Delta_n = (\alpha_n - \alpha_{n-1})(p_{n-1}^2 - p_n^2) + \alpha_{n-1} \Delta_{n-1}. \qquad (12)$$

By (11) and the positivity of x we may restrict ourselves to the case $p_{n-1}(x) p_n(x) > 0$. We will assume that $p_{n-1}(x) > 0$ and $p_n(x) > 0$ (the case $p_{n-1}(x) < 0$ and $p_n(x) < 0$ can be dealt with similarly). By (12) and by the induction hypothesis it suffices to consider the case $p_{n-1}(x) < p_n(x)$, since by assumption (i) we have $\alpha_{n-1} \leq \alpha_n$. In that case since $\gamma_n = 1 - \alpha_n \geq \frac{1}{2} \geq \alpha_n$ we get

$$\gamma_n p_n(x) - \alpha_n p_{n-1}(x) \geq \alpha_n [p_n(x) - p_{n-1}(x)] \geq 0.$$

Now we apply (8) and obtain

$$\gamma_n \Delta_n \geq (1 - x) p_{n-1}(x) p_n(x) > 0.$$

The proof of part (i) is thus complete.

We turn to the last part of the statement. Let (i) be satisfied and $|x| > 1$. By symmetry we can assume $x > 1$. As before we proceed by induction. We have

$$\gamma_1 \Delta_1(x) = \alpha_1(1 - x^2) < 1.$$

Assume now that $\Delta_m(x) < 0$ for $1 \le m \le n-1$. Since $p_n(1) = 1$ and the leading coefficients of p_n's are positive, the polynomials p_n are positive for $x > 1$. Thus

$$0 > \frac{\Delta_m(x)}{p_{m-1}(x)p_m(x)} = \frac{p_m(x)}{p_{m-1}(x)} - \frac{p_{m+1}(x)}{p_m(x)}.$$

for $1 \le m \le n-1$. Hence

$$\frac{p_n(x)}{p_{n-1}(x)} \ge \ldots \ge \frac{p_1(x)}{p_0(x)} = x > 1.$$

Now by (12) we get

$$\gamma_n \Delta_n \le \alpha_{n-1}\Delta_{n-1} < 0.$$

□

Remark. The second part of Theorem 1 is not true under assumption (ii). Indeed, by (10), the leading coefficient of the Turán determinant $\gamma_n\Delta_n(x)$ is equal to $\gamma_1^{-2}\ldots\gamma_{n-1}^{-2}(\alpha_{n-1}-\alpha_n)$. Thus $\Delta_n(x)$ is positive at infinity for $n \ge 2$. One might expect that in this case $\Delta_n(x)$ is nonnegative on the whole real axis, but this is not true either. Indeed, it can be computed that

$$\gamma_1^2\gamma_2\Delta_2(x) = (x^2-1)[(\gamma_2-\gamma_1)x^2 - \alpha_1^2\gamma_2].$$

One can verify that under assumption (ii) we have

$$r := \frac{\alpha_1^2\gamma_2}{\gamma_2-\gamma_1} > 1.$$

(Actually $r \ge 1$ follows from Theorem 1 (ii).) Hence $\Delta_2(x) < 0$ for $1 < x < r$.

Sometimes we have to deal with polynomials which are orthogonal in the interval $[-1, 1]$ and normalized at $x = 1$, but the three term recurrence relation is not available in explicit form. In such cases the following will be useful.

Corollary 1 *Let the polynomials p_n satisfy*

$$xp_n = \gamma_n p_{n+1} + \alpha_n p_{n-1}, \qquad n = 0, 1, \ldots,$$

with $p_{-1} = 0$, $p_0 = 1$ and $\alpha_0 = 0$. Assume that the sequences α_n and $\alpha_n + \gamma_n$ are nondecreasing and

$$\lim_{n\to\infty}\alpha_n = \frac{1}{2}a \qquad \lim_{n\to\infty}\gamma_n = \frac{1}{2}a^{-1},$$

where $0 < a < 1$. *Then the orthogonality measure for* p_n *is supported in the interval* $[-1, 1]$.

Assume that in addition at least one of the following holds

(i) γ_n *is nondecreasing,*

(ii) $\gamma_0 \geq 1$.

Then

$$\Delta_n(x) = \widetilde{p}_n^2(x) - \widetilde{p}_{n-1}(x)\widetilde{p}_{n+1}(x) \geq 0 \iff -1 \leq x \leq 1,$$

where $\widetilde{p}_n(x) = p_n(x)/p_n(1)$.

Proof. First we will show that $p_n(1) > 0$. In view of symmetry of the polynomials this will imply that the support of the orthogonality measure is contained in $[-1, 1]$.

We will show by induction that $p_n(1)/p_{n-1}(1) \geq a > 0$. We have

$$\frac{p_0(1)}{p_1(1)} = \gamma_0 \leq \frac{1}{2}(a + a^{-1}) \leq a^{-1}.$$

Assume that $p_n(1)/p_{n-1}(1) \geq a$. Then from recurrence relation we get

$$\frac{p_{n+1}(1)}{p_n(1)} = \frac{1}{\gamma_n}\left(1 - \alpha_n \frac{p_{n-1}(1)}{p_n(1)}\right) \geq \frac{1}{\gamma_n}\left(1 - a^{-1}\alpha_n\right).$$

On the other hand

$$\begin{aligned}
\gamma_n &= (\alpha_n + \gamma_n) - \alpha_n \leq \frac{1}{2}(a + a^{-1}) - \alpha_n \\
&\leq \frac{1}{2}(a + a^{-1}) - a^{-2}\alpha_n + (a^{-2} - 1)\alpha_n \\
&\leq \frac{1}{2}(a + a^{-1}) - a^{-2}\alpha_n + (a^{-2} - 1)\frac{1}{2}a \\
&= a^{-1}(1 - a^{-1}\alpha_n).
\end{aligned}$$

Therefore

$$\frac{p_{n+1}(1)}{p_n(1)} \geq a.$$

Now we show that $c_n = p_n^2(1) - p_{n-1}(1)p_{n+1}(1) > 0$ by induction. Assume (i). Similarly to the proof of Proposition 1 we obtain

$$\gamma_n c_n = (\gamma_n - \gamma_{n-1})p_n^2(1) + (\alpha_n - \alpha_{n-1})p_{n-1}^2 + \alpha_{n-1}c_{n-1}. \tag{13}$$

This implies $c_n > 0$ for every n.

Assume now that (ii) holds and that $c_m > 0$ for $m \leq n-1$. Hence the sequence $p_{m+1}(1)/p_m(1)$ is positive and nonincreasing for $m \leq n-1$. In particular $p_n(1)/p_{n-1}(1) \leq p_1(1)/p_0(1) = \gamma_0^{-1} \leq 1$. Therefore $p_n(1) \leq p_{n-1}(1)$. Rewrite (13) in the form

$$\begin{aligned}\gamma_n c_n &= [(\alpha_n + \gamma_n) - (\alpha_{n-1} + \gamma_{n-1})]p_n^2(1) \\ &\quad + (\alpha_n - \alpha_{n-1})[p_{n-1}^2(1) - p_n^2(1)] + \alpha_{n-1}c_{n-1}.\end{aligned}$$

Thus $c_n > 0$.

We have shown that, in both cases (i) and (ii), we have $c_n > 0$ and hence the sequence $p_{n-1}(1)/p_n(1)$ is nondecreasing. Denote its limit by r. Now plugging $x = 1$ into the recurrence relation for p_n, dividing both sides by $p_n(1)$ and taking the limits gives

$$1 = \frac{1}{2}[ar + (ar)^{-1}].$$

Thus $r = a^{-1}$. Let

$$\widetilde{p}_n(x) = \frac{p_n(x)}{p_n(1)}.$$

From the recurrence relation for p_n we obtain

$$x\widetilde{p}_n = \widetilde{\gamma}_n \widetilde{p}_{n+1} + \widetilde{\alpha}_n \widetilde{p}_{n-1}, \tag{14}$$

where

$$\widetilde{\alpha}_n = \frac{p_{n-1}(1)}{p_n(1)}\alpha_n, \qquad \widetilde{\gamma}_n = \frac{p_{n+1}(1)}{p_n(1)}\gamma_n.$$

By plugging $x = 1$ into (14) we get

$$\widetilde{\alpha}_n + \widetilde{\gamma}_n = 1.$$

Since both $p_{n-1}(1)/p_n(1)$ and α_n are nondecreasing, so is $\widetilde{\alpha}_n$. Moreover it tends to $1/2$ at infinity because the first of its factors tends to a^{-1} while the second tends to $a/2$. Therefore the polynomials $\widetilde{p}_n$ satisfy the assumptions of Theorem 1 (i). This completes the proof. $\square$

4 The best normalization.

Assume that the polynomials p_n satisfy (4) and (5). By multiplying each p_n by a positive constant σ_n we obtain polynomials $p_n^{(\sigma_n)}(x) = \sigma_n p_n(x)$. The positivity of Turán's determinant for the polynomials p_n is not equivalent to that for the polynomials $p_n^{(\sigma_n)}$. However, it is possible that the positivity of Turan's determinants in one normalization implies the positivity in other normalizations. It turns out that the normalization at the right most end of the interval of orthogonality has this feature.

Proposition 2 *Let the polynomials p_n satisfy (4) and (5). Assume that*

$$p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0, \quad -1 \leq x \leq 1, \quad n \geq 1.$$

Let $p_n^{(\sigma)}(x) = \sigma_n p_n(x)$, where σ_n is a sequence of positive constants. Then

$$\{p_n^{(\sigma)}(x)\}^2 - p_{n-1}^{(\sigma)}(x)p_{n+1}^{(\sigma)}(x) \geq 0, \quad -1 \leq x \leq 1, \quad n \geq 1$$

if and only if

$$\sigma_n^2 - \sigma_{n-1}\sigma_{n+1} \geq 0, \quad n \geq 1.$$

Proof. We have

$$\{p_n^{(\sigma)}(x)\}^2 - p_{n-1}^{(\sigma)}(x)p_{n+1}^{(\sigma)}(x)$$
$$= (\sigma_n^2 - \sigma_{n-1}\sigma_{n+1})p_n^2(x) + \sigma_{n-1}\sigma_{n+1}(p_n^2(x) - p_{n-1}(x)p_{n+1}(x)).$$

This shows the "if" part. On the other hand, since (3) is equivalent to $p_n(1) = 1$ for $n \geq 0$, we obtain

$$\{p_n^{(\sigma_n)}(1)\}^2 - p_{n-1}^{(\sigma_n)}(1)p_{n+1}^{(\sigma_n)}(1) = \sigma_n^2 - \sigma_{n-1}\sigma_{n+1}.$$

This shows the "only if" part. □

Remark. Proposition 2 says that if the Turán inequality holds for the polynomials normalized at $x = 1$ then it remains true for any other normalization if and only if it holds only at the point $x = 1$, because $p_n^{(\sigma)}(1) = \sigma_n$.

5 Applications to special symmetric polynomials.

We will test Theorem 1 on three classes of polynomials: ultraspherical, q – ultraspherical and symmetric Pollaczek polynomials.

The positivity of Turán's determinants for the first case is well known (see [E, p. 209]). The ultraspherical polynomials $C_n^{(\lambda)}$ are orthogonal in the interval $(-1,1)$ with respect to the measure $(1 - x^2)^{\lambda - (1/2)}\, dx$, where $\lambda > -\frac{1}{2}$. When normalized at $x = 1$ they satisfy the recurrence relation

$$x\widetilde{C}_n^{(\lambda)} = \frac{n+2\lambda}{2n+2\lambda}\widetilde{C}_{n+1}^{(\lambda)} + \frac{n}{2n+2\lambda}\widetilde{C}_{n-1}^{(\lambda)}.$$

It can be checked easily that Theorem 1 (i) or (ii) applies according to $\lambda \geq 0$ or $\lambda \leq 0$.

Let us turn to the q–ultraspherical polynomials. They have been studied by Bustoz and Ismail [BI1] but with a normalization other than the one at the right end of the interval of orthogonality. We will exhibit that our normalization is sharper in the sense that we can derive the results of [BI1] from ours. Moreover, we will have no restrictions on the parameters other than that q be positive.

In standard normalization the q–ultraspherical polynomials are denoted by $C_n(x;\beta|q)$ and they satisfy the recurrence relation

$$2xC_n(x;\beta|q) = \frac{1-q^{n+1}}{1-\beta q^n}C_{n+1}(x;\beta|q) + \frac{1-\beta^2 q^{n-1}}{1-\beta q^n}C_{n-1}(x;\beta|q). \qquad (15)$$

The orthogonality measure is known explicitly (see [AI], [AW, Thm. 2.2 and Sect. 4] or [GR, Sect. 7.4]). When $|\beta|$, $|q| < 1$ it is absolutely continuous with respect to the Lebesgue measure on the interval $[-1,1]$.

Theorem 2 *Let $0 < q < 1$ and $|\beta| < 1$. Let $\widetilde{C}_n(x;\beta|q)$ denote the q–ultraspherical polynomials normalized at $x = 1$, i.e.*

$$\widetilde{C}_n(x;\beta|q) = \frac{C_n(x;\beta|q)}{C_n(1;\beta|q)}.$$

Let

$$\Delta_n(x;\beta|q) = \widetilde{C}_n^2(x;\beta|q) - \widetilde{C}_{n-1}(x;\beta|q)\widetilde{C}_{n+1}(x;\beta|q).$$

Then

$$\Delta_n(x;\beta|q) \geq 0 \quad \textit{if and only if} \quad -1 \leq x \leq 1,$$

with equality only for $x = \pm 1$.

Proof. The main obstacle in applying Theorem 1 lies in the fact that the values $C_n(1;\beta|q)$ are not given explicitly. Therefore, we cannot give explicitly the recurrence relation for $\widetilde{C}_n(x;\beta|q)$.

We will break the proof into two subcases.

(i) $0 < \beta < 1$.
Introduce the polynomials

$$p_n(x) = \beta^{n/2} \prod_{m=1}^{n} \frac{1-\beta^2 q^{m-1}}{1-q^m} C_n^2(x;\beta|q).$$

Then by (15) we obtain

$$xp_n = \gamma_n p_{n+1} + \alpha_n p_{n-1},$$

$$\alpha_n = \beta^{1/2}\frac{1-q^n}{2(1-\beta q^n)} \qquad \gamma_n = \beta^{-1/2}\frac{1-\beta^2 q^n}{2(1-\beta q^n)}.$$

Observe that

$$\alpha_n + \gamma_n = \frac{1}{2}(\beta^{1/2} + \beta^{-1/2}).$$

Moreover α_n is nondecreasing and converges to $\frac{1}{2}\beta^{1/2}$. Finally

$$\gamma_0 = \frac{1}{2}(\beta^{1/2} + \beta^{-1/2}) > 1.$$

Therefore we can apply Corollary 1(ii) with $a = \beta^{1/2}$.

(ii) $-1 < \beta \le 0$.
Introduce the polynomials

$$p_n(x) = \prod_{m=1}^{n} \frac{1-\beta^2 q^{m-1}}{1-q^m} C_n^2(x;\beta|q).$$

Then by (15) we obtain

$$xp_n = \gamma_n p_{n+1} + \alpha_n p_{n-1},$$

$$\alpha_n = \frac{1-\beta^2 q^n}{2(1-\beta q^n)} \qquad \gamma_n = \frac{1-q^n}{2(1-\beta q^n)}.$$

Since both α_n and γ_n are increasing sequences convergent to 1 we can apply Corollary 1(i) with $a = 1$. □

We turn now to the symmetric Pollaczek polynomials $P_n^\lambda(x;a)$. They are orthogonal in the interval $[-1,1]$ and satisfy the recurrence relation

$$xP_n^\lambda(x;a) = \frac{n+1}{2(n+\lambda+a)}P_{n+1}^\lambda(x;a) + \frac{n+2\lambda-1}{2(n+\lambda+a)}P_{n-1}^\lambda(x;a),$$

where the parameters satisfy $a>0,\ \lambda>0$. We cannot compute the value $P_n^\lambda(1;a)$ in order to pass directly to normalization at $x=1$. Instead, we consider another auxiliary normalization. Let

$$p_n(x) = \frac{n!}{(2\lambda)_n}P_n^\lambda(x;a),$$

where $(\mu)_n = \mu(\mu+1)\dots(\mu+n-1)$. Then the polynomials p_n satisfy the recurrence relation

$$xp_n = \frac{n+2\lambda}{2(n+\lambda+a)}p_{n+1} + \frac{n}{2(n+\lambda+a)}p_{n-1}.$$

Observe that the assumptions of Corollary 1 (i) or (ii) are fulfilled according to $\lambda \geq a$ or $\lambda \leq a$. Therefore we have the following.

Theorem 3 *Let $\lambda>0$, $a>0$. Let $\widetilde{P}_n^\lambda(x;a)$ denote the Pollaczek polynomials normalized at $x=1$, i.e.*

$$\widetilde{P}_n^\lambda(x;a) = \frac{P_n^\lambda(x;a)}{P_n^\lambda(1;a)}.$$

Then

$$\{\widetilde{P}_n^\lambda(x;a)\}^2 - \widetilde{P}_{n-1}^\lambda(x;a)\widetilde{P}_{n+1}^\lambda(x;a) \geq 0 \quad \textit{if and only if} \quad -1\leq x\leq 1,$$

with equality only for $x=\pm1$.

6 Nonsymmetric polynomials orthogonal in [-1,1]

In this section we assume that polynomials p_n satisfy (4) and (5) with β_n not necessarily equal to 0 for all n.

Theorem 4 *Let polynomials p_n satisfy (4) and (5). Let*

$$|\gamma_0-\gamma_1| \le \alpha_1\gamma_0-(\gamma_0-\gamma_1)(1-\gamma_0). \tag{16}$$

Assume that for each $n \ge 2$ one of the following four conditions is satisfied.

(i)

$$\begin{aligned}&\alpha_{n-1}\le\alpha_n\le\gamma_n\le\gamma_{n-1},\\ &\frac{\beta_n+1+\sqrt{(\beta_n+1)^2-4\alpha_n\gamma_n}}{2\gamma_n}\le\frac{\alpha_n-\alpha_{n-1}}{\gamma_{n-1}-\gamma_n}\quad\text{or } (\beta_n+1)^2-4\alpha_n\gamma_n<0.\end{aligned}$$

(ii)

$$\begin{aligned}&\alpha_{n-1}\ge\alpha_n\ge\gamma_n\ge\gamma_{n-1},\\ &\frac{\beta_n+1-\sqrt{(\beta_n+1)^2-4\alpha_n\gamma_n}}{2\gamma_n}\ge\frac{\alpha_{n-1}-\alpha_n}{\gamma_n-\gamma_{n-1}}\quad\text{or } (\beta_n+1)^2-4\alpha_n\gamma_n<0.\end{aligned}$$

(iii)

$$\begin{aligned}&\alpha_{n-1}\ge\alpha_n\ge\frac{1}{2},\quad \gamma_{n-1}\ge\gamma_n\ge\frac{1}{2},\\ &\frac{\alpha_n-\alpha_{n-1}}{\gamma_n-\gamma_{n-1}}\le\frac{\alpha_n}{\gamma_n}\le 1\quad\text{or}\quad\frac{\alpha_n-\alpha_{n-1}}{\gamma_n-\gamma_{n-1}}\ge\frac{\alpha_n}{\gamma_n}\ge 1.\end{aligned}$$

(iv)

$$\begin{aligned}&\alpha_{n-1}\le\alpha_n,\quad \gamma_{n-1}\le\gamma_n,\\ &\begin{cases}\alpha_n\le\gamma_n\\ \alpha_n-\alpha_{n-1}\ge\gamma_n-\gamma_{n-1}\end{cases}\quad\text{or}\quad\begin{cases}\alpha_n\ge\gamma_n\\ \alpha_n-\alpha_{n-1}\le\gamma_n-\gamma_{n-1}\end{cases}\end{aligned}$$

Then

$$\Delta_n(x)=p_n^2(x)-p_{n-1}(x)p_{n+1}(x)\ge 0\quad\text{for } -1\le x\le 1.$$

Proof. The proof will go by induction. Combining (8) and (4) for $n=1$ gives

$$\gamma_0^2\gamma_1\Delta_1(x)=(1-x)[(\gamma_0-\gamma_1)(x-\beta_0)+\alpha_1\gamma_0].$$

Now using (5) gives that the positivity of $\Delta_1(x)$ in the interval $[-1,1]$ is equivalent to (16).

Fix x in $[-1,1]$ and assume that $\Delta_n(x) \geq 0$. Consider two quadratic functions

$$\begin{aligned} A(t) &= (t+1)\{(\gamma_n - \gamma_{n-1})t - (\alpha_{n-1} - \alpha_n)\}, \\ B(x;t) &= \gamma_n t^2 - (\beta_n - x)t + \alpha_n. \end{aligned}$$

Set

$$t = -\frac{p_n(x)}{p_{n-1}(x)}.$$

By Proposition 1 it suffices to show that for any t the values $A(t)$ and $B(x;t)$ cannot be both negative. In order to achieve this we have to look at the roots of these functions. The roots of $A(t)$ are -1 and $(\alpha_{n-1} - \alpha_n)/(\gamma_n - \gamma_{n-1})$; hence they are independent of x. The roots of $B(t)$ have always the same sign and are equal to

$$r_n^{(1)}(x) = \frac{\beta_n - x - \sqrt{(\beta_n - x)^2 - 4\alpha_n\gamma_n}}{2\gamma_n}, \tag{17}$$

$$r_n^{(2)}(x) = \frac{\beta_n - x + \sqrt{(\beta_n - x)^2 - 4\alpha_n\gamma_n}}{2\gamma_n}. \tag{18}$$

Since the function $u \mapsto u + \sqrt{u^2 - a^2}$, $a > 0$, is decreasing for $u \leq -a$ and increasing for $u \geq a$ we have

$$r_n^{(1)}(1) \leq r_n^{(1)}(x) \leq r_n^{(2)}(x) \leq r_n^{(2)}(1) \quad \text{if} \quad \beta_n - x \leq 0, \tag{19}$$

$$r_n^{(1)}(-1) \leq r_n^{(1)}(x) \leq r_n^{(2)}(x) \leq r_n^{(2)}(-1) \quad \text{if} \quad \beta_n - x \geq 0, \tag{20}$$

provided that $(\beta_n - x)^2 - 4\alpha_n\gamma_n \geq 0$. Thus $B(x;t) < 0$ implies $B(1;t) < 0$ ($B(-1;t) < 0$ respectively) if $\beta_n - x \leq 0$ ($\beta_n - x \geq 0$ respectively). Hence it suffices to show that the values $A(t)$ and $B(1;t)$ (the values $A(t)$ and $B(-1;t)$ respectively) cannot be both negative if $\beta_n - x \leq 0$ ($\beta_n - x \geq 0$ respectively). We will break the proof into two subcases.

(a) $\beta_n - x \leq -2\sqrt{\alpha_n\gamma_n}$.

In view of (8) and (6)the roots of $B(1;t)$ are -1 and $-\frac{\alpha_n}{\gamma_n}$. By analysing the positions of these numbers with respect to the roots of $A(t)$ one can easily verify that under each of the four assumptions (i) through (iv) the values $A(t)$ and $B(1;t)$ cannot be both negative.

(b) $\beta_n - x \leq -2\sqrt{\alpha_n\gamma_n}$.

We examine the signs of $A(t)$ and $B(-1;t)$. Consider (i), (ii) and (iii). By analysing the mutual position of the roots of $B(-1;t)$ and $A(t)$ one can verify that $A(t)$ and $B(-1;t)$ cannot be both negative.

In case (iv) we have that $B(-1;t) \geq 0$ because

$$(1+\beta_n)^2 - 4\alpha_n\gamma_n = (2-\alpha_n-\gamma_n)^2 - 4\alpha_n\gamma_n \leq 0.$$

□

Remark 1. The assumption (iv) in Theorem 4 does not imply that the support of the orthogonality measure corresponding to the polynomials p_n is contained in $[-1,1]$. By (5) we have $p_n(1) = 1$ which implies that the support is located to the left of 1. However, it can extend to the left side beyond -1.

Remark 2. If we assume that $\beta_n = 0$ for $n \geq 0$, then Theorem 4 reduces to Theorem 1. Indeed, in this case we have

$$\begin{aligned} \frac{\beta_n + 1 - \sqrt{(\beta_n+1)^2 - 4\alpha_n\gamma_n}}{2\gamma_n} &= \min\left(1, \frac{\alpha_n}{\gamma_n}\right), \\ \frac{\beta_n + 1 + \sqrt{(\beta_n+1)^2 - 4\alpha_n\gamma_n}}{2\gamma_n} &= \max\left(1, \frac{\alpha_n}{\gamma_n}\right). \end{aligned}$$

Example. Set

$$\alpha_n = \frac{1}{2} - \frac{1}{n+2}, \quad \gamma_n = \frac{1}{2} + \frac{1}{2(n+2)}, \quad \beta_n = \frac{1}{2(n+2)}.$$

We can check easily that condition (16) is satisfied. We will check that also the assumptions (iii) are satisfied for every $n \geq 2$. Clearly we have $\alpha_{n-1} \leq \alpha_n \leq \gamma_n \leq \gamma_{n-1}$. Moreover

$$\begin{aligned} r_n^{(2)}(-1) &= \frac{\beta_n + 1 + \sqrt{(1+\beta_n)^2 - 4\alpha_n\gamma_n}}{2\gamma_n} \\ &\leq \frac{\beta_n + 1}{\gamma_n} \leq 2 = \frac{\alpha_n - \alpha_{n-1}}{\gamma_{n-1} - \gamma_n}. \end{aligned}$$

Let $p_n(x)$ satisfy (4). By Theorem 4(iii)

$$p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0 \quad \text{for} \quad -1 \leq x \leq 1.$$

Let us determine the interval of orthogonality. Since $\alpha_n + \beta_n + \gamma_n = 1$ we have $p_n(1) = 1$. Thus the support of the corresponding orthogonality measure is located to the left of 1. Actually the support is contained in the interval $[-1,1]$. Indeed, it suffices to show that $c_n = (-1)^n p_n(-1) > 0$. We will show that $c_n \geq c_{n-1} > 0$ by induction. We have $c_0 = 1$. Assume $c_n \geq c_{n-1} > 0$. Then by (4)

$$\begin{aligned}\gamma_n c_{n+1} &= (1+\beta_n)c_n - \alpha_n c_{n-1} \geq (1+\beta_n-\alpha_n)c_n \\ &\geq (1-\beta_n-\alpha_n)c_n = \gamma_n c_n.\end{aligned}$$

Thus $c_{n+1} \geq c_n > 0$.

7 Polynomials orthogonal in the interval $[0,+\infty)$.

Let p_n be polynomials orthogonal in the positive half axis normalized at $x=0$, i.e. $p_n(0)=1$. Then they satisfy the recurrence relation of the form

$$xp_n = -\gamma_n p_{n+1} + (\alpha_n+\gamma_n)p_n - \alpha_n p_{n-1}, \quad n = 0,\, 1,\, \ldots, \tag{21}$$

with initial conditions $p_{-1}=0$, $p_0=1$, where α_n, and γ_n are given sequences of real coefficients such that

$$\alpha_0 = 0,\ \gamma_0 = 1,\ \alpha_{n+1} > 0,\ \gamma_n > 0,\ \text{for } n = 0,\, 1,\, \ldots. \tag{22}$$

Theorem 5 *Let polynomials p_n satisfy (21) and (22), and let*

$$\alpha_{n-1} \leq \alpha_n, \quad \gamma_{n-1} \leq \gamma_n \quad \textit{for } n \geq 1.$$

Assume that one of the following two conditions is satisfied.

(i) $\alpha_n \leq \gamma_n \qquad \alpha_n - \alpha_{n-1} \geq \gamma_n - \gamma_{n-1}.$

(ii) $\alpha_n \geq \gamma_n \qquad \alpha_n - \alpha_{n-1} \leq \gamma_n - \gamma_{n-1}.$

Then

$$\Delta_n(x) = p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0 \quad \text{for } x \geq 0.$$

Proof. Let $q_n(x) = p_n(1-x)$. Then by (21) we obtain

$$xq_n = \gamma_n q_{n+1} + (1-\alpha_n-\gamma_n)p_n + \alpha_n q_{n-1}.$$

We have $q_n(1) = 1$. Thus the assumptions (iv) of Theorem 4 are satisfied for every n. From the proof of Theorem 4 (iv) it follows that $q_n^2(x) - q_{n-1}(x)q_{n+1}(x) \;\geq\; 0$

for $x \leq 1$ (the assumption $x \geq -1$ is inessential). Taking into account the relation between p_n and q_n gives the conclusion. □

A special case of Theorem 5 is when $\alpha_n - \alpha_{n-1} = \gamma_n - \gamma_{n-1}$ for every n. In this case, applying (8) gives the following.

Proposition 3 *Let polynomials p_n satisfy (21) and (22), and let*

$$\alpha_n - \alpha_{n-1} = \gamma_n - \gamma_{n-1}, \qquad n \geq 1.$$

Then

$$p_n^2(x) - p_{n-1}(x)p_{n+1}(x) = \sum_{k=1}^{n} \frac{(\alpha_k - \alpha_{k-1})\alpha_k \alpha_{k+1} \ldots \alpha_{n-1}}{\gamma_k \gamma_{k+1} \cdots \gamma_n} (p_k(x) - p_{k-1}(x))^2.$$

In particular, if $\alpha_n \geq \alpha_{n-1}$ for $n \geq 1$, then

$$p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0 \quad \textit{for } -\infty < x < \infty,$$

where equality holds only for $x = 0$.

Example.
Let $p_n(x) = L_n^\alpha(x)/L_n^\alpha(1)$, where $L_n^\alpha(x)$ denote the Laguerre polynomials of order $\alpha > -1$. Then the polynomials p_n satisfy

$$xp_n = -(n + \alpha + 1)p_{n+1} + (2n + \alpha + 1)p_n - np_n.$$

Then

$$\alpha_n - \alpha_{n-1} = \gamma_n - \gamma_{n-1} = 1, \quad n \geq 1.$$

Thus Proposition 3 applies. The formula for $p_n^2 - p_{n-1}p_{n+1}$ in this case is not new. It has been discovered by V. R. Thiruvenkatachar and T. S. Nanjundiah [TN] (see also [AC, 4.7].

Acknowledgement. I am grateful to J. Bustoz and M. E. H. Ismail for kindly sending me a preprint of [BI2]. I thank George Gasper for pointing out the references [AC, TN].

References

[AC] Al-Salam W.; Carlitz, L.: General Turán expressions for certain hypergeometric series, Portugal. Math. **16**, 119–127 (1957)

[A] Askey, R.: Linearization of the product of orthogonal polynomials, Problems in Analysis, R. Gunning, ed., Princeton University Press, Princeton, N.J., 223–228 (1970)

[AI] Askey, R.,Ismail, M.E.H.: A generalization of ultraspherical polynomials, in Studies in Pure Mathematics, P. Erdös, ed., Birkhäuser, Basel, 55–78 (1983)

[AW] Askey, R., Wilson, J.A.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc. **54** (1985)

[BI1] Bustoz, J., Ismail, M.E.H.: Turaán inequalities for ultraspherical and continuous q–ultraspherical polynomials, SIAM J. Math. Anal. **14**, 807–818 (1983)

[BI2] Bustoz, J., Ismail, M.E.H.: Turán inequalities for symmetric orthogonal polynomials, preprint (1995)

[E] Erdélyi, A.: "Higher transcendental functions," vol. 2, New York, 1953

[G1] Gasper, G.: On the extension of Turán's inequality to Jacobi polynomials, Duke. Math. J. **38**, 415–428 (1971)

[G2] Gasper, G.: An inequality of Turán type for Jacobi polynomials, Proc. Amer. Math. Soc. **32**, 435–439 (1972)

[GR] Gasper, G., Rahman, M.: "Basic Hypergeometric Series," Vol. 35, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 1990

[KS] Karlin, S.; Szegö, G.: On certain determinants whose elements are orthogonal polynomials, J. d'Analyse Math. **8**, 1–157 (1960/61)

[MN] Mukherjee, B.N; Nanjundiah, T.S.: On an inequality relating to Laguerre and Hermite polynomials, Math. Student. **19**, 47–48 (1951)

[Sk] Skovgaard, H.: On inequalities of the Turán type, Math. Scand. **2**, 65–73 (1954)

[S] Szász, O.: Identities and inequalities concerning orthogonal polynomials and Bessel functions, J. d'Analyse Math. **1**, 116–134 (1951)

[Sz1] Szegö, G.: On an inequality of P. Turán concerning Legendre polynomials, Bull. Amer. Math. Soc. **54**, 401–405 (1948)

[Sz2] Szegö, G.: An inequality for Jacobi polynomials, Studies in Math. Anal. and Related Topics, Stanford Univ. Press, Stanford, Calif., 392–398 (1962)

[TN] Thiruvenkatachar, V.R.; Nanjundiah, T.S.: Inequalities concerning Bessel functions and orthogonal polynomials, Proc. Indian Acad. Sci. **33**, 373–384 (1951)

[T] Turán, P.: On the zeros of the polynomials of Legendre, Časopis Pěst. Mat. **75**, 113–122 (1950)

Institute of Mathematics
Polish Academy of Sciences
ul. Kopernika
00–950 Wrocław, Poland

Current address:
Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50–384 Wrocław, Poland