An Application of Markov Chain Analysis to the Game of Squash*

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ABSTRACT

If the score in a squash game is tied late in the game, one player has a choice of how many additional points (from a prespecified set of possibilities) are to be played to determine the winner. This paper constructs a Markov chain model of the situation and solves for the optimal strategy. Expressions for the optimal strategy are obtained with a symbolic algebra computer package. Results are given for both international and American scoring systems. The model and analysis are very suitable for educational purposes. The resulting Markov chain is small enough that it can be easily presented in a classroom setting, yet the model is sufficiently complex that algebraic manipulation is nearly hopeless. The final results illustrate the power of the combination of mathematical and computer modeling applied to a problem of practical interest.

Subject Areas: Computer Applications and Markov Processes.

INTRODUCTION

This paper focuses on a decision problem that often arises in the game of squash. In certain situations late in the game, one player must decide how many additional points (from a prespecified set of possibilities) are to be played. The problem is of great interest to professional squash players, and is also faced by many amateur squash players on a daily basis. In this paper, a Markov chain approach is proposed to model the situation and an optimal strategy is derived. Quantitative analyses have been applied to other decision problems in sports. Decision theory is used in [5] to decide whether to attempt a one-point or two-point conversion in college football. A Poisson process model is used in [9] and [12] to decide the optimal time to pull the goalie in a hockey game. Dynamic programming is applied to pole-vaulting strategy in [3] and to serving strategy in tennis in [10]. Simulation is used to determine win, place, and show probabilities in a round robin jai-alai tournament in [2]. Several books have been written on mathematical applications in sports [7] [8] [11].

This paper illustrates the application of Markov chain analysis to a real-world problem. The topic is very suitable for educational purposes for several reasons. The problem is a useful one to motivate Markov chain analysis because it is practical, easily understood, and interesting to many. The resulting Markov chain is small enough that it can be easily presented in a classroom setting, yet the model

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is sufficiently complex that algebraic manipulation is nearly hopeless. The problem illustrates the usefulness of software packages for numerical, algebraic, and graphical manipulations. The final results illustrate the power of the combination of mathematical and computer modeling applied to a problem of practical interest. The topic could be used as the basis for a project involving modeling and computation. Issues of parameter estimation and model validation could also be discussed.

There are two main scoring systems in squash: international scoring and American scoring. International scoring is typically used when playing with a soft ball or English ball. American scoring is more often used when playing with a hard ball. Both scoring systems share a common feature: if the score is tied near the end of the game, one player can decide the number of additional points to be played to determine the winner.

In the international scoring system for squash, the first player to score nine points is the winner. However, if the game becomes tied at 8-8, the player who reaches 8 points first has two options. The options are to play to 9 points (called "set one") or to play to 10 points (called "set two"). Set one means that the next player to score wins the game. Set two means that the first player to score two points wins the game. In this scoring system, points are only scored by the player who is serving. A player who wins a rally must serve the next rally. Thus, if the game is tied at 8-8, the player who is receiving, that is, not serving the next rally, decides. The question is whether the player should choose set one or set two.

The decision problem can be modeled naturally as a Markov chain. Determining an algebraic expression for the optimal decision requires deriving the absorption probabilities of the Markov chain. In one case, this requires the symbolic inversion of an eight-by-eight matrix. A symbolic algebra computer package is a perfect tool for this purpose. The result is a simple optimal decision rule for the player.

A MARKOV CHAIN MODEL

The game of squash can be modeled as a Markov chain. For a thorough introduction to Markov chains and their applications, see Isaacson and Madsen [4] or Kemeny and Snell [6]. The two players will be called A and B and a score of i-j means that player A has scored i points and player B has scored j points. The states of the chain are defined by the score and the identity of the server. For example, state (3, 4, B) means that player B is serving and leading by 4 points to 3. A transition from one state to another corresponds to the outcome of a rally. Define:

Player A Serving
p = probability player A wins a rally
$1-p$ = probability player A loses a rally

Player B Serving
q = probability player A wins a rally
$1-q$ = probability player A loses a rally.

We assume that p and q are constant over time and independent of the current score. This assumption seems very appropriate as a first approximation to reality. Since the decision problem only arises at the end of the game, we only need to assume that p and q are constant over that period. We also assume that p and q are independent of the choice of set one or set two.

To avoid trivial cases, we assume $0<p<1$ and $0<q<1$. The rallies between expert squash players playing soft ball tend to be very long. Service winners are almost nonexistent at the professional level. This means that there is relatively little advantage to serving, unlike tennis where the server often has a significant advantage. This observation corresponds to assuming $p \approx q$. Our analysis will proceed using p and q separately, and then will be specialized to the case $p=q$.

The parameters p and q could be estimated in a number of ways. An obvious approach is to use historical data on the outcomes of rallies between two players. If the players haven't played each other, other schemes could be devised based on the rankings of the players, or their performance against common opponents.

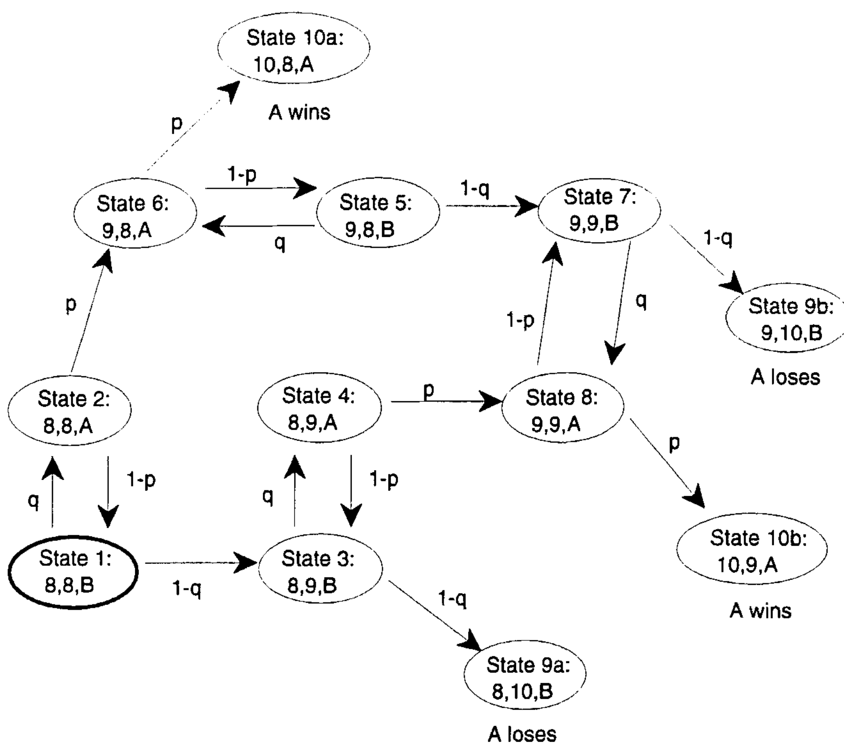
Suppose player A reaches 8 points first and chooses set two. This can only happen if player B was serving and won the previous rally to tie the score at 8-8. The Markov chain corresponding to the remainder of the game is illustrated in Figure 1.

The initial state of the Markov chain is labeled state 1 in Figure 1, corresponding to a score of 8-8 with player B to serve. If player B wins the rally, a transition is made to state 3, corresponding to a score of 8-9 in favor of B with B to serve next. If player A wins the rally from state 1, a transition is made to state 2, corresponding to a score of 8-8 with A to serve next. In Figure 1, states 9a and 9b correspond to player A losing the game. Since these states represent a loss by the same player (although by different scores), they can be combined into a single state 9. Similarly, states 10a and 10b can be combined into a single state 10. The one-step transition probability matrix P can be written as:

$$
P = \begin{array}{l|cccccccc|cc|}
 & (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) & (9) & (10) \\
 & 8,8,B & 8,8,A & 8,9,B & 8,9,A & 9,8,B & 9,8,A & 9,9,B & 9,9,A & A \text{ loses} & A \text{ wins} \\
\hline
(1)\ 8,8,B & & q & 1-q & & & & & & & \\
(2)\ 8,8,A & 1-p & & & & & p & & & & \\
(3)\ 8,9,B & & & & q & & & & & 1-q & \\
(4)\ 8,9,A & & & 1-p & & & & & p & & \\
(5)\ 9,8,B & & & & & & q & 1-q & & & \\
(6)\ 9,8,A & & & & & 1-p & & & & & p \\
(7)\ 9,9,B & & & & & & & & q & 1-q & \\
(8)\ 9,9,A & & & & & & & 1-p & & & p \\
\hline
(9)\ A \text{ loses} & & & & & & & & & 1 & \\
(10)\ A \text{ wins} & & & & & & & & & & 1
\end{array}
\qquad (1)
$$

where blank entries indicate zeros.

Figure 1: Markov chain if Player A chooses set two.



ANALYSIS AND RESULTS

States 1-8 are transient states and states 9 and 10 are absorbing states. Using this partition of the states, the matrix P can be written as

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}. \tag{2}$$

Let π_{ij} denote the probability of being absorbed in state j starting from transient state i. The matrix π of absorption probabilities is given by $\pi=(I-Q)^{-1}R$. The probability that player A wins the game (i.e., is absorbed into state 10 starting from state 1) is $\pi_{1,10}$.

The objective of player A is to pick the strategy, that is, set one or set two, that maximizes the probability of winning the game. The previous analysis illustrated how to set up an appropriate Markov chain for the set two option. If player A chooses set one, the same procedure could be applied, but this is not necessary because the work has already been done! Note that if player A chooses set one, the score is tied with player B to serve, and one more point for either player wins the game. This Markov chain is equivalent to the one in Figure 1, but starting from state (9, 9, *B*) (denoted state 7). Hence, if set one is chosen, the probability player *A* wins the game is $\pi_{7,10}$.

Player *A* should choose set two if $\pi_{1,10}-\pi_{7,10}>0$. For any specific values of p and q, π can be computed numerically. This can be done in any number of ways: using a spreadsheet package (e.g., Lotus, Excel, or Quattro); a matrix language (e.g., MATLAB, Gauss, or APL); or a programming language (e.g., C, Pascal, or Fortran). By varying p and q, tables or graphs of win probabilities could be developed. This approach is useful for solving specific problems and for developing intuition about the character of the solution.

Numerical procedures are not sufficient to provide an explicit expression for the solution to the problem nor are they sufficient to prove assertions about the solution. Deriving an explicit expression for π would involve a large amount of tedious algebra. However symbolic algebra packages, for example, Mathematica or Maple, can do these manipulations automatically. A comprehensive treatment of Mathematica is given in [13]. A reference manual for Maple is [1]. The main results are

$$\pi_{1,10} = \frac{p^2q(pq^2 - q^2 - q + 2}{(1 - q + pq)^3},$$

$$\pi_{7,10} = \frac{pq}{(1 - q + pq)},$$

$$\pi_{1,10} - \pi_{7,10} = \frac{pq(q - 1)(1 - 2p - q + pq)}{(1 - q + pq)^3}. \tag{3}$$

Let $g(p,q)=\pi_{1,10}-\pi_{7,10}$. The function g represents the increase in the probability of player *A* winning by choosing set two over set one. A graph of $g(p,q)$ is given in Figure 2 and a contour plot of the same function is shown in Figure 3. The contour $g=0$ is shown in Figure 3 as the line that intersects the horizontal axis at $p=.5$. The contour lines change in increments of .01. They decrease to the left (lower values of p and q) to -.03 and increase to the right to .20. So $g>0$ is the region to the right of the $g=0$ contour.

Player *A* should choose set two if $g(p,q)>0$. Since the denominator of g is always positive, $g(p,q)>0$ if the numerator is positive. This happens when $1-2p-q+pq>0$, or

$$p > \frac{1 - q}{2 - q}. \tag{4}$$

Equation (4) summarizes the optimal decision rule. Player *A* should choose set two when $p>(1-q)/(2-q)$. Figure 3 shows that for a large region of reasonable parameter values, the optimal decision is to choose set two.

Since rallies between expert squash players tend to be quite long in soft ball, we now specialize the results to the case $p=q$. Figure 4 shows how the probability of player *A* winning the game varies with p, when set two is chosen (i.e., $\pi_{1,10}(p,p)$) and when set one is chosen (i.e., $\pi_{7,10}(p,p)$). A graph of $g(p,p)=\pi_{1,10}-\pi_{7,10}$ is shown in Figure 5.

Figure 2: Graph of g $(p,\ q)$.

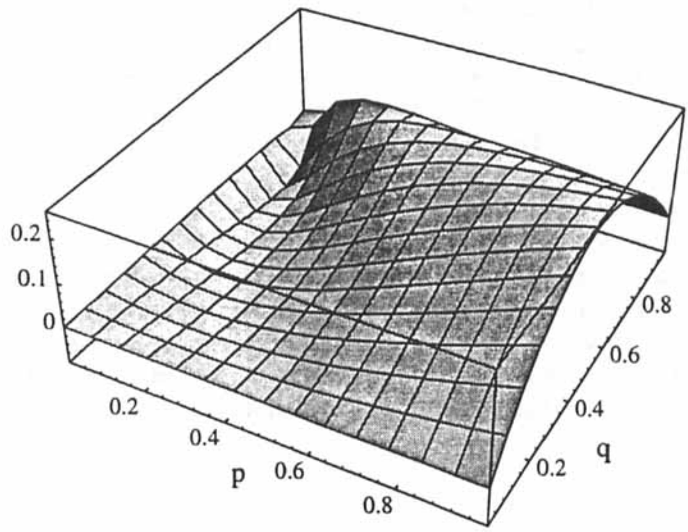


Figure 3: Contour plot of g $(p,\ q)$.

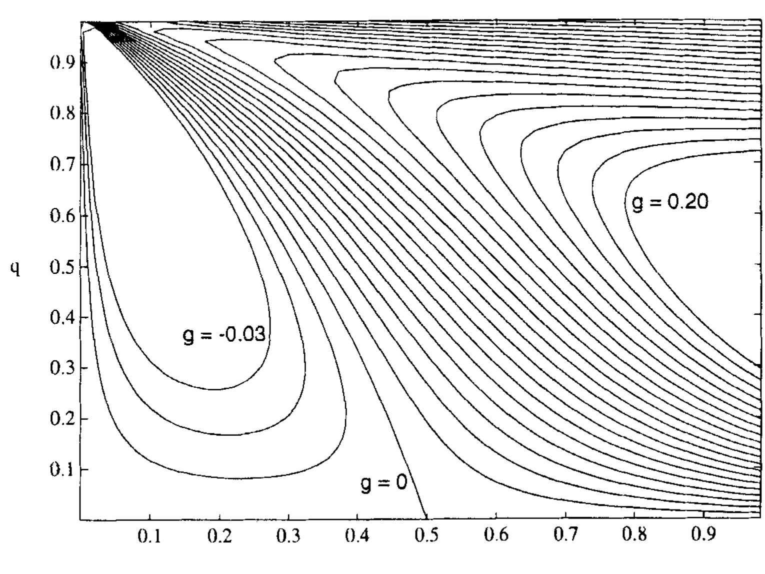
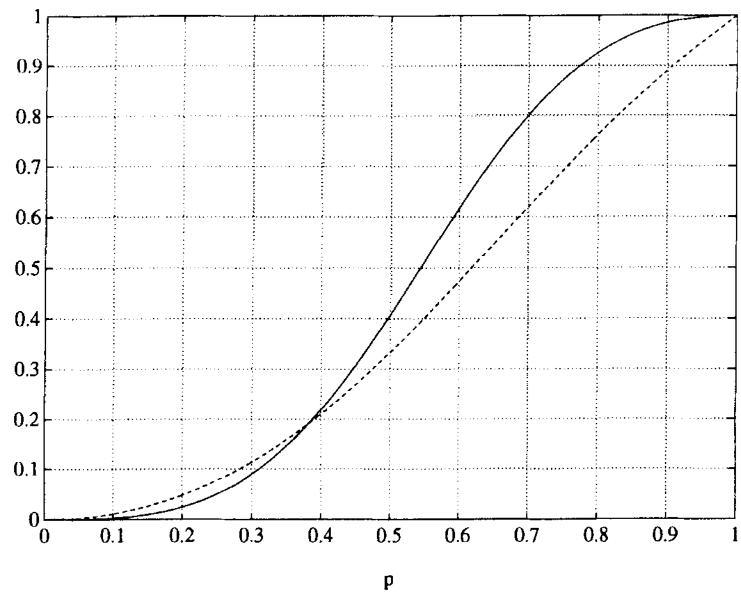


Figure 4: Probability A wins game (Set two solid, set one dashed).



The previous expression can be simplified to show that the region $g(p,p)>0$ is given by $p>(3-\sqrt{5})/2\approx.382$. If players A and B are equally matched, that is, $p=.5$, then Figure 5 shows that there is a substantial 7.4 percent gain in win probability by choosing set two (from 33.3 percent to 40.7 percent).

Figures 3 and 5 show that there is a large region where set two is optimal. Since this region includes most reasonable parameter values, the optimal decision is quite robust with respect to uncertainty in the parameters. This result squares with intuition. Since the player with the decision is not the server, that player needs to win at least two rallies to win the game, while the server would only need to win one rally if set one were chosen.

AMERICAN FORM OF SCORING

A similar choice exists in the American form of scoring, also known as point-per-rally scoring. In this system, a point is scored whenever a rally is won, no matter which player served. The winner of a rally serves to start the next rally. The game is played to 15 points. However, if the game is tied at 13-13, the player who reaches 13 points first has three choices.

1. 2 out of 3 (also known as "no set"): the player to win 2 of the next 3 points wins. Thus the first player to score 15 points is the winner.
2. 3 out of 5: the first player to reach 16 wins the game.
3. 5 out of 9: the first player to reach 18 wins the game.

However, if the game reaches 14-14 without first having reached 13-13, the player who reaches 14 points first has two choices:

Figure 5: Gain from choosing set two over set one.

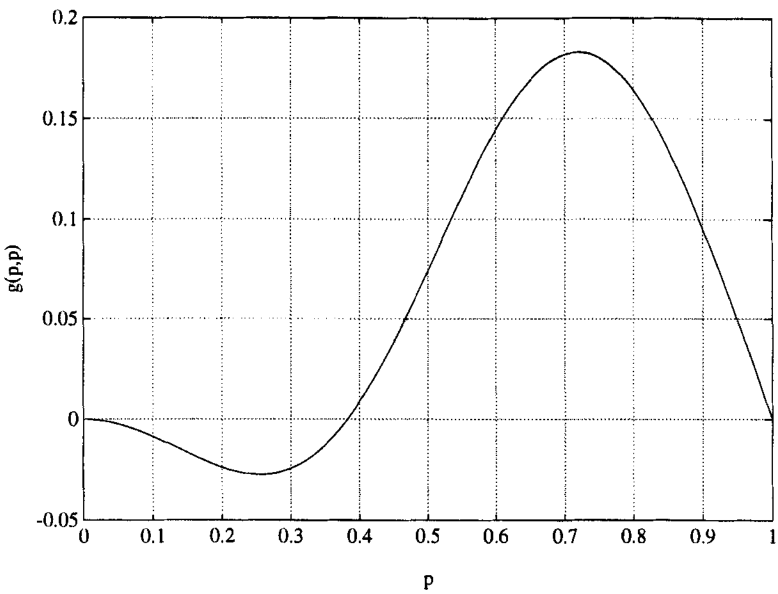
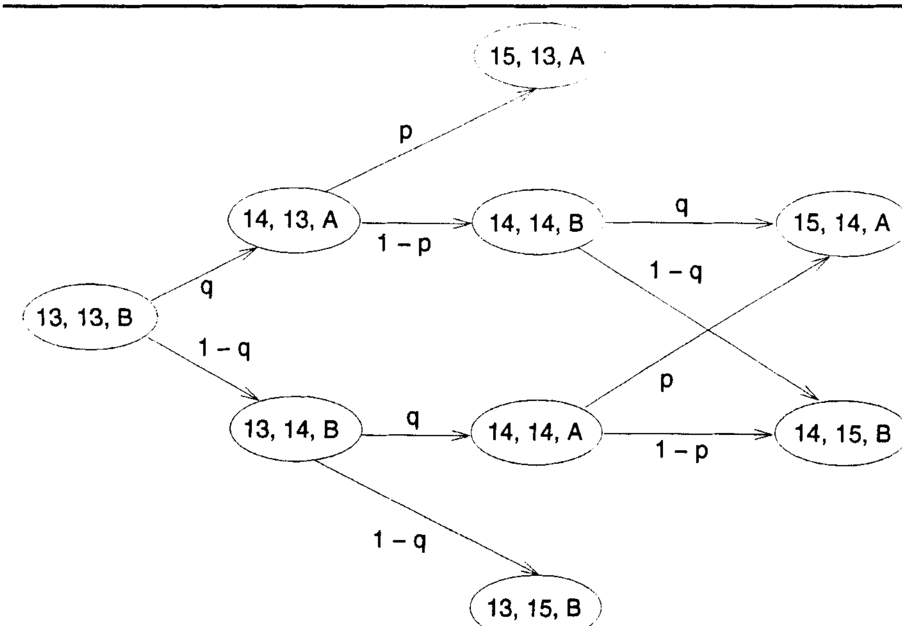


Figure 6: Markov chain for American scoring.



1. 1 point, or "no set": the player who wins the next point wins, and the game ends at 15, or
2. 3 out of 5: the first player to reach 17 wins the game.

We first consider the decision faced by player A with the score tied at 13-13 with player B to serve. We use the same notation to represent a state, namely (i, j, X) means that player A has i points, player B has j points, and player X is serving. The Markov chain if player A chooses "no set" is shown in Figure 6.

Since there are no "loops" in this Markov chain, absorption probabilities can be computed directly. For example, the probability of ending in state (15, 13, A) is pq. The probability of ending in state (15, 14, A) is $q^2(1-p)+(1-q)qp$. Thus, the probability that player A reaches 15 points first and wins, denoted π_5, is the sum of these two probabilities:

$$\pi_5 = q^2 - 2q^2p + 2pq. \tag{5}$$

A similar but larger Markov chain can be developed if player A chooses to play to 16 points. A symbolic algebra package is also useful to derive and manipulate the resulting long algebraic expressions for the absorption probabilities. In this case the probability player A wins, denoted π_6, is

$$\pi_6 = 6q^2p - 9p^2q^2 - 6q^3p + 6p^2q^3 + q^3 + 3p^2q. \tag{6}$$

Similarly, the probability player A wins when playing to 18 points is

$$\pi_8 = 5p^4q + 60p^2q^3 + 40p^3q^2 - 200p^3q^3 - 150p^2q^4 + 300p^3q^4 + 20q^4p + q^5$$
$$- 20q^5p + 90p^2q^5 - 140p^3q^5 - 50p^4q^2 + 150p^4q^3 - 175p^4q^4 + 70p^4q^5. \tag{7}$$

For any given values of p and q, player A should choose the strategy which gives the highest probability of winning. So the region in which playing to 15 points is optimal is

$$\{(p,q)\colon \pi_5 > \pi_6 \text{ and } \pi_5 > \pi_8\}. \tag{8}$$

The other regions are derived similarly. These regions are displayed in Figure 7.

The three regions intersect at the point $p=q=.5$. In this case, all choices give the same .5 probability of winning. The region where playing to 16 points is optimal is quite small. Figure 8 shows results for $p=q$. In this case, player A should choose to play to 15 if $p<.5$ and play to 18 if $p>.5$.

Since the serve carries a greater advantage in hard ball, another interesting case to consider is $p=1-q$. Here, players are equally matched, but serving matters. For example, serving a rally is advantageous when $p>.5$ (which implies $q<.5$). Table 1 shows the probability of player A winning when playing to 15, 16, or 18 points for various values of p and $q=1-p$. The table shows that player A should choose to play to 15 if $p<.5$ and play to 18 if $p>.5$. For example, if $p=.6$ and $q=.4$, the gain in probability by choosing 18 over 15 is 2.4 percent.

Figure 7: Optimal decision regions.

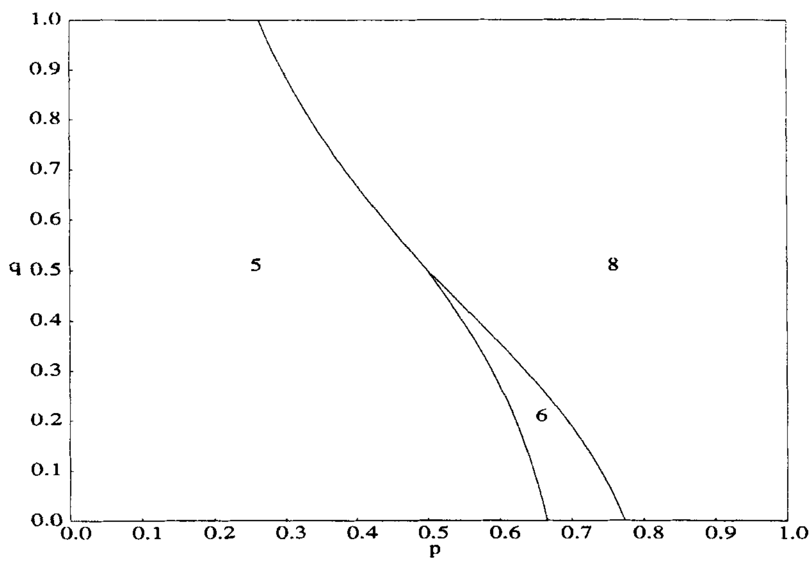


Figure 8: Probability A wins (p=q) (15 solid, 16 dashed, 18 dotted).

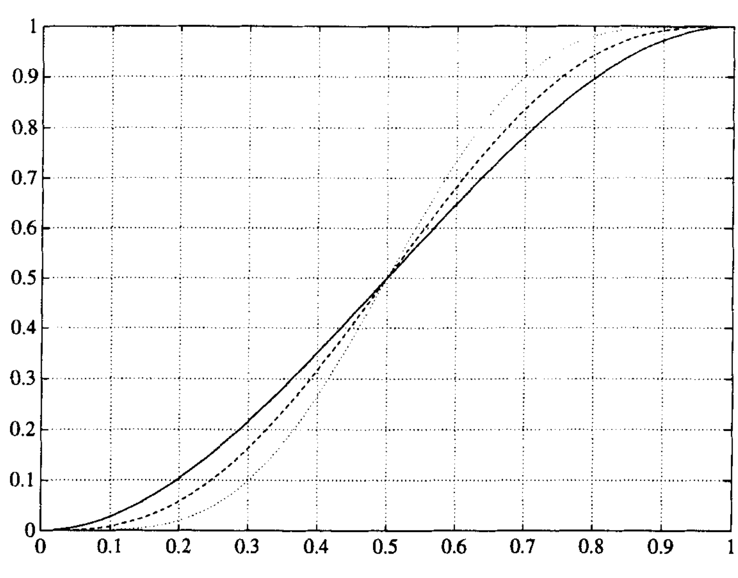


Table 1: Probability player A wins from 13-13.

p	$q=1-p$	π_5	π_6	π_8	$\pi_5-\pi_8$
.30	.70	.6160	.5849	.5606	.0554
.35	.65	.5817	.5601	.5433	.0384
.40	.60	.5520	.5386	.5280	.0240
.45	.55	.5253	.5189	.5138	.0115
.50	.50	.5000	.5000	.5000	0
.55	.45	.4748	.4811	.4862	-.0115
.60	.40	.4480	.4614	.4720	-.0240
.65	.35	.4183	.4399	.4567	-.0384
.70	.30	.3840	.4151	.4394	-.0554

Decision from 14-14

If the score is tied at 14-14, without ever reaching 13-13, a similar analysis can be done. The probability that player A wins if A chooses to play to 15 is simply q. If player A chooses to play to 17, the probability of winning is:

$$6q^2p - 9p^2q^2 - 6q^3p + 6p^2q^3 + q^3 + 3p^2q. \tag{9}$$

Winning 3 out of 5 starting from 14-14 has the same probability as winning 3 out of 5 starting from 13. Hence, this is the same as the expression for π_6 from before.

The gain in probability of winning by choosing 15 over 17 is denoted h, that is, $h(p,q)=q-\pi_6$. A three-dimensional graph of h is shown in Figure 9.

Figure 10 displays contour lines of h and thus indicates the regions of parameter values for which playing to 15 or 17 is optimal. The contours start at the left from .3 and decrease by .03 to $-.3$ at the right of the graph. The contour $h=0$ is the nearly straight line in the middle. The region to the left of $h=0$ represents parameters (p,q) for which $h>0$, that is, playing to 15 is optimal. The region to the right of $h=0$ represents parameters for which $h<0$, that is, playing to 17 is optimal. This result also squares with intuition. The smaller the probability of winning a point, the higher the gain from choosing the game to 15. Exact numerical values can be obtained directly from the formula for $h(p,q)$.

CONCLUSION

A player's decision problem in squash can be modeled as a relatively small Markov chain. However, obtaining closed form analytical results could be exceptionally tedious. Symbolic algebra packages are useful tools for quickly and correctly performing algebraic manipulations, including symbolic matrix inversion.

Using this simple but reasonable model, the optimal decision rule under international scoring was derived. Set two is the optimal decision for a wide range of reasonable parameter values. In professional tournaments, set two is almost always chosen unless a player is extremely tired. The model is consistent with players' intuition about the optimal decision. The model also quantifies the penalty or reward for choosing the other option.

Figure 9: Gain from choosing 15 over 17.

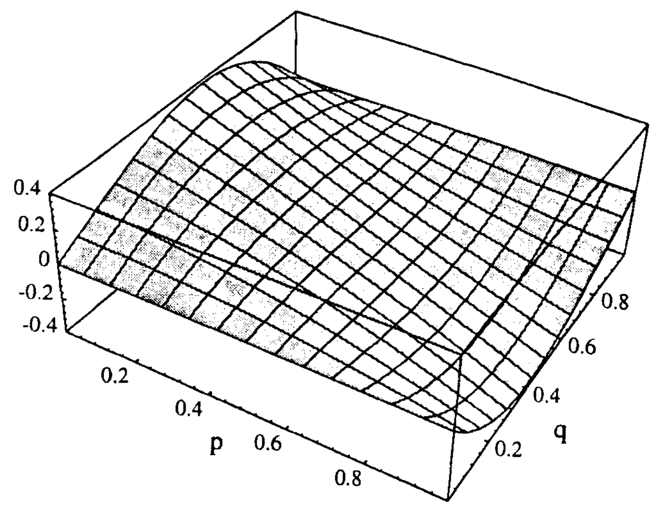
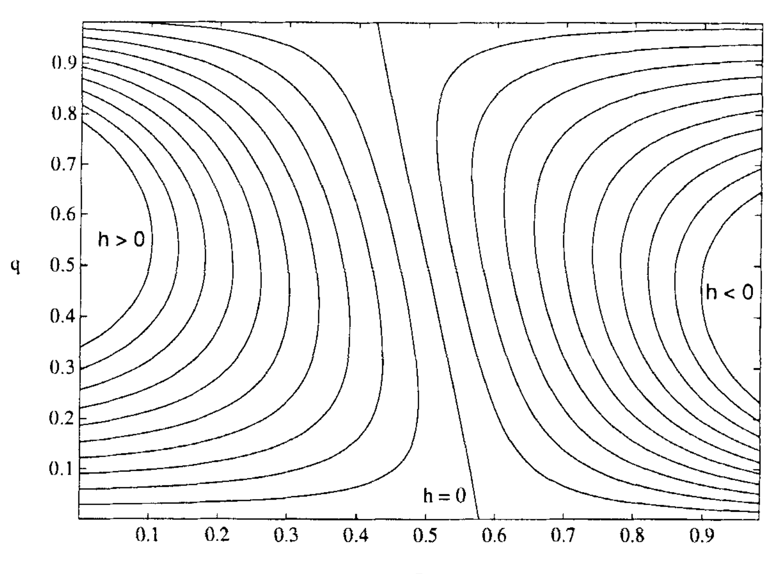


Figure 10: Contour lines of *h*.



In soft ball professional competition there is a growing trend to using a modified version of American scoring. The only modification is that the 13-13 decision is eliminated. The 14-14 decision still remains. An optimal decision rule for this situation was also derived.

The decision problem described in this paper could be used as a classroom example to illustrate several points. The problem shows the application of Markov analysis to a real-world situation. The problem is of great interest to professional squash players, and is also faced by many amateur squash players. The resulting Markov chain is small enough to be derived by students or presented on a blackboard. Even so, the subsequent analysis is sufficiently complex that algebraic manipulation is nearly impossible. This naturally leads to the use of symbolic algebra computer packages to do tedious yet simple algebraic manipulations. The final results illustrate the power of the combination of mathematical and computer modeling. Issues of parameter estimation and model validation could also be raised in the classroom or given as an additional project. [Received: November 2, 1992. Accepted: August 3, 1993.]

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