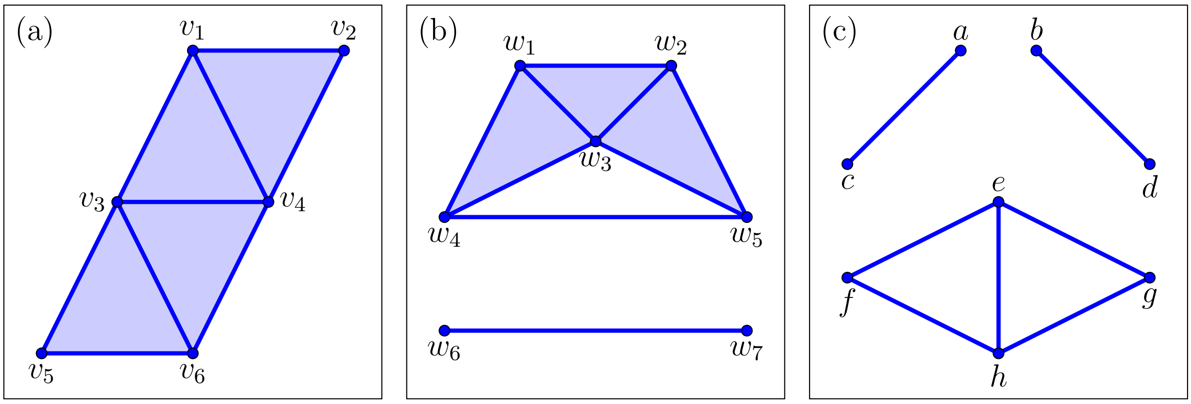
Problem List 1 (Filtrations of simplicial complexes)

TDA, Summer Semester 2022/23, IM UWr

1. Describe each of the given geometric simplicial complexes as an abstract simplicial complex (that is, write down abstract simplicial complexes whose geometric realisations are shown). In each case, give the dimension of the simplicial complex and the number of n-simplices for each $n \geq 0$.



2. Decide if the following abstract simplicial complexes have geometric realisations in $\mathbb{R}^2$. For each complex that does have a geometric realisation in $\mathbb{R}^2$, exhibit it (by drawing the geometric simplicial complex on paper / blackboard). For each one that does not, give a reason why.

   (a) $K_a = \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd\}$;

   (b) $K_b = \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}$;

   (c) $K_c = \{u_1, u_2, v_1, v_2, v_3, u_1v_1, u_1v_2, u_1v_3, u_2v_1, u_2v_2, u_2v_3\}$;

   (d) $K_d = \{x_1x_2x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_6, x_3x_5x_6$ and all their faces$\}$;

   (e) $K_e = \{x_1x_2x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_6, x_3x_5x_6, x_1x_4x_6$ and all their faces$\}$.

3. Given a finite metric space (X, d), we say that a real number $\alpha \geq 0$ is a *Rips-critical value* for X if $d(x, y) = \alpha$ for some $x, y \in X$.

   (a) Let (X, d) be a finite metric space, and let $\{K_i \mid i \in \mathbb{N}\}$ be the step-ε Rips filtration for X. Given any natural numbers $i < j$, explain why $K_i = K_j$ if and only if there are no Rips-critical values for X in the interval $(i\varepsilon, j\varepsilon]$.

   (b) Consider the subset $Y = \{y_0, y_1, \ldots, y_6\} \subset \mathbb{R}^2$, where $y_0 = (0, 0)$ and $y_j = (\cos(\pi j/3), \sin(\pi j/3))$ for $1 \leq j \leq 6$ (so that Y consists of vertices and the centre of a regular hexagon). Find the Rips-critical values for Y.

   (c) Find the step-$\frac{1}{10}$ Rips filtration for Y (either drawing each complex in the filtration or writing down its vertices and edges is enough). Find the dimension of each complex appearing in the filtration.

4. Given a finite subset $X \subseteq \mathbb{R}^m$, we say that a real number $\alpha \geq 0$ is a *Čech-critical value* for X if the nerve of $\{B_{\alpha/2}(x) \mid x \in X\}$ is not the same as the nerve of $\{B_{(\alpha-\delta)/2}(x) \mid x \in X\}$ for any $\delta > 0$.

   (a) Let $X \subseteq \mathbb{R}^m$ be a finite subset. Show that any Rips-critical value for X (equipped with the "usual", i.e. Euclidean, metric) is also Čech-critical.

   (b) Let $Y = \{y_0, \dots, y_6\} \subset \mathbb{R}^2$ be the subset considered in the previous exercise. Find a Čech-critical value for Y that is not Rips-critical.

5. Consider the metric space (X, d), where $X = \{x_1, \dots, x_5\}$, defined by the following *distance matrix* D: that is, we have $d(x_i, x_j) = D_{ij}$.

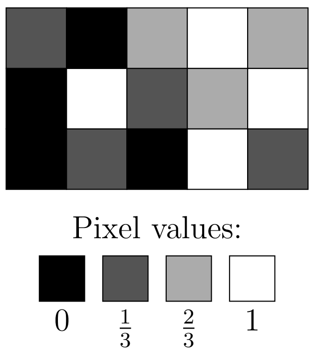
$$D := \begin{pmatrix} 0 & 2 & 2 & 3 & 3 \\ 2 & 0 & 3 & 4 & 5 \\ 2 & 3 & 0 & 5 & 4 \\ 3 & 4 & 5 & 0 & 6 \\ 3 & 5 & 4 & 6 & 0 \end{pmatrix}$$

   (a) Find the step-1 Rips filtration for X, and write down the dimension of each complex appearing in the filtration.

   (b) We say a triple $\{x, y, z\}$ of distinct points in a metric space (X', d') form a *degenerate triangle* if either $d'(x, z) = d'(x, y) + d'(y, z)$ or $d'(x, z) = |d'(x, y) - d'(y, z)|$. Find all degenerate triangles in (X, d).

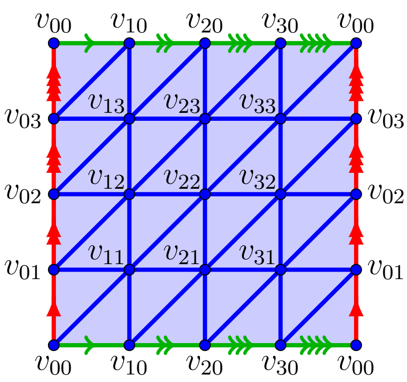
   (c) By considering how degenerate triangles may appear in $\mathbb{R}^m$, show that (X, d) is not isometric to any subspace of $\mathbb{R}^m$ (for any m).

6. Consider the greyscale picture displayed on the right.



   (a) Construct the combinatorial Morse function F corresponding to this picture (by drawing the corresponding simplicial complex and labelling each simplex σ with the value of $F(\sigma)$, for instance).

   (b) Let $\{K_i \mid i \in \mathbb{N}\}$ be the step-$\frac{1}{3}$ Morse filtration for this picture. Draw K_i for each i (as a geometric simplicial complex).



7. Consider a "triangulation of a torus": a simplicial complex K represented by the picture on the left, with the edges and vertices identified as shown.

   (a) Define $\widetilde{F}: V(K) \to \mathbb{N}$ by $\widetilde{F}(v_{ij}) = |i-2| + 3|j-2|$. Construct the combinatorial Morse function F induced by $\widetilde{F}$ (by labelling each simplex σ in the picture on the left by the value of $F(\sigma)$, for instance).

   (b) Let $\{K_i \mid i \in \mathbb{N}\}$ be the step-1 Morse filtration for F. Draw (in three dimensions, if necessary) the complexes K_0, K_2, K_8, K_9 and K_{11}.