Problem List 2 (Homology)

TDA, Summer Semester 2022/23, IM UWr

1. For each of the following sequences of vector spaces (with the rightmost vector space being C_{-1}) and maps between them (represented by the given matrices) below, decide if the sequence is a chain complex, and give a reason why (or why not).

   (a) $\cdots \to 0 \to 0 \to \mathbb{R} \xrightarrow{\begin{pmatrix}3\\2\end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix}-2 & 3\end{pmatrix}} \mathbb{R} \to 0 \to 0;$

   (b) $\cdots \to 0 \to 0 \to \mathbb{R} \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \mathbb{R} \to 0 \to 0;$

   (c) $\cdots \to 0 \to 0 \to \mathbb{R} \xrightarrow{\begin{pmatrix}1\\1\\0\end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix}1 & 1 & 0\\1 & 1 & 0\end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{R} \to 0;$

   (d) $\cdots \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\begin{pmatrix}1\\1\\0\end{pmatrix}} (\mathbb{Z}/2\mathbb{Z})^3 \xrightarrow{\begin{pmatrix}1 & 1 & 0\\1 & 1 & 0\end{pmatrix}} (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}/2\mathbb{Z} \to 0;$

   (e) $\cdots \to 0 \to 0 \to \mathbb{R} \xrightarrow{\begin{pmatrix}2\\-1\end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix}1 & 2\\2 & 4\\3 & 6\end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix}1 & 1 & -1\end{pmatrix}} \mathbb{R} \to 0.$

   (f) $\cdots \to 0 \to 0 \to \mathbb{R} \xrightarrow{\begin{pmatrix}0\\0\end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix}0 & 0 & 0\end{pmatrix}} \mathbb{R} \to 0.$

2. For each of the sequences in the previous question that *is* a chain complex, compute the homology groups of that chain complex.

3. Let K be a *square*, i.e. the simplicial complex $K = \{a, b, c, d, ab, bc, cd, ad\}$.

   (a) Write down the chain complex $(C_\bullet(K; \mathbb{R}), \partial_\bullet)$.

   (b) Compute the homology groups $H_n(K; \mathbb{R})$.

   (c) Let $K' = K \setminus \{ad\}$. Compute the homology groups $H_n(K'; \mathbb{R})$.

   (d) Let $K'' = K \cup \{ac\}$. Compute the homology groups $H_n(K''; \mathbb{R})$.

4. Let K be a simplicial complex, and let $v \in V(K)$. Define the simplicial complex $K' = K \cup \{u, uv\}$, where u is a "new" vertex (i.e. $u \notin V(K)$). Let $\mathbb{F}$ be a field, and denote by $(C_\bullet(K;\mathbb{F}), \partial_\bullet^K)$ and $(C_\bullet(K';\mathbb{F}), \partial_\bullet^{K'})$ the corresponding chain complexes.

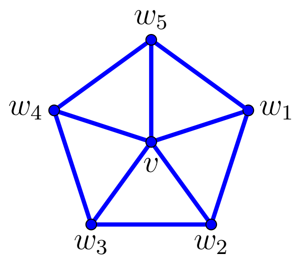
   (a) Express the dimensions $\dim_\mathbb{F} C_n(K';\mathbb{F})$ in terms of $\dim_\mathbb{F} C_n(K';\mathbb{F})$.

   (b) Write down the maps $\partial_n^{K'}$ in terms of ∂_n^K (using the matrices representing those maps).

   (c) Show that $\dim_\mathbb{F} \ker(\partial_n^K) = \dim_\mathbb{F} \ker(\partial_n^{K'})$ and $\dim_\mathbb{F} \operatorname{im}(\partial_{n+1}^K) = \dim_\mathbb{F} \operatorname{im}(\partial_{n+1}^{K'})$ for every $n \geq 1$. Deduce that $H_n(K;\mathbb{F}) \cong H_n(K';\mathbb{F})$ for all $n \geq 1$.

   (d) Show that $\dim_\mathbb{F} \ker(\partial_0^K) = \dim_\mathbb{F} \ker(\partial_0^{K'}) - 1$ and $\dim_\mathbb{F} \operatorname{im}(\partial_1^K) = \dim_\mathbb{F} \operatorname{im}(\partial_1^{K'}) - 1$, and deduce that $H_0(K;\mathbb{F}) \cong H_0(K';\mathbb{F})$.

5. Let $m \geq 3$, and let K_m be an *m-wheel*, defined as the simplicial complex



$$K_m = \{v, w_1, \dots, w_m, vw_1, \dots, vw_m, w_1w_2, \dots, w_{m-1}w_m, w_1w_m\};$$

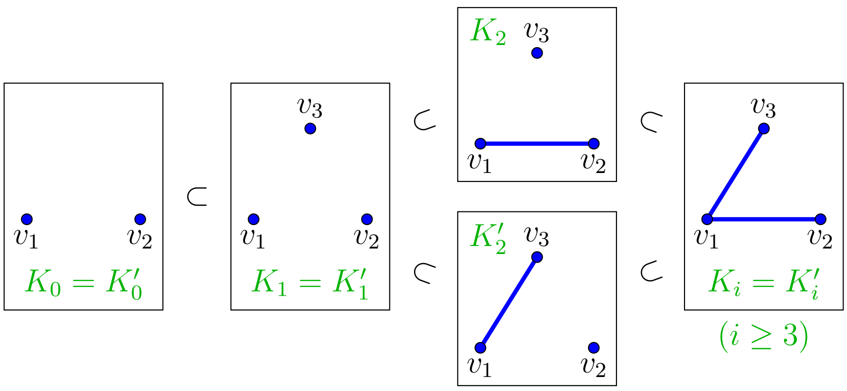
   see the picture on the right for the case $m = 5$. Let $\mathbb{F}$ be a field, and let $(C_\bullet(K_m;\mathbb{F}), \partial_\bullet)$ be the corresponding chain complex.

   (a) Show that $\operatorname{im}(\partial_1)$ consists precisely of expressions of the form $\alpha e_v + \sum_{i=1}^m \beta_i e_{w_i}$, where $\alpha, \beta_1, \dots, \beta_m \in \mathbb{F}$ satisfy $\alpha + \sum_{i=1}^m \beta_i = 0$. Use this to compute $H_0(K_m;\mathbb{F})$.

   (b) Compute $H_n(K_m;\mathbb{F})$ for all $n \geq 2$.

   (c) Use the Euler characteristic of K_m to compute $H_1(K_m;\mathbb{F})$.

6. Consider the following two filtrations of simplicial complexes, denoted by $(\{K_i\}, \{f_i\})$ and $(\{K'_i\}, \{f'_i\})$, where $f_i \colon K_i \to K_{i+1}$ and $f'_i \colon K'_i \to K'_{i+1}$ are the inclusions.



   (a) Compute the homology groups $H_0(K_i;\mathbb{R})$ and $H_0(K'_i;\mathbb{R})$ for all $i \in \mathbb{N}$, explicitly writing down their bases (as $\mathbb{R}$-vector spaces).

   (b) Write down matrices representing the maps $(f_i)_* \colon H_0(K_i;\mathbb{R}) \to H_0(K_{i+1};\mathbb{R})$ and $(f'_i)_* \colon H_0(K'_i;\mathbb{R}) \to H_0(K'_{i+1};\mathbb{R})$ induced in homology (with respect to the bases you chose in the previous part).

   (c) Show that $(f'_1)_* \circ (f'_0)_*$ is an isomorphism, but $(f_1)_* \circ (f_0)_*$ is not.