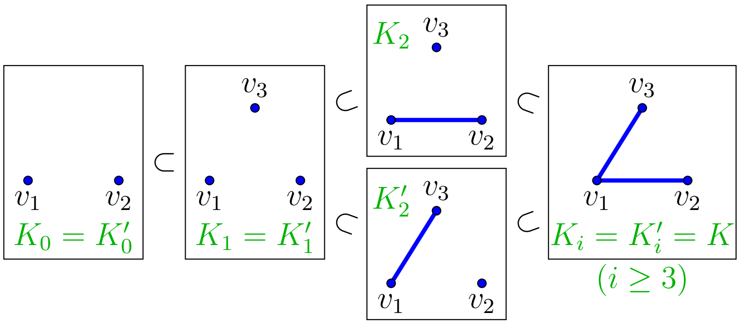
Problem List 3 (Persistent homology)

TDA, Summer Semester 2022/23, IM UWr

1. Consider the following filtrations $(\{K_i\}, K)$ and $(\{K'_i\}, K)$ from Problem List 2.



   (a) Find bases $\mathcal{B}_i$ for the homology groups $H_0(K_i;\mathbb{R})$ such that $(f_i)_*(e) \in \mathcal{B}_{i+1} \cup \{0\}$ for all $e \in \mathcal{B}_i$ and such that $e = e'$ whenever $e, e' \in \mathcal{B}_i$ satisfy $(f_i)_*(e) = (f_i)_*(e') \neq 0$; here $f_i\colon K_i \to K_{i+1}$ is the inclusion.

   (b) Do the same for $(\{K'_i\}, K)$.

   (c) Decompose the persistence modules $H_0(\{K_i\}, K;\mathbb{R})$ and $H_0(\{K'_i\}, K;\mathbb{R})$ into cyclic submodules.

   (d) Sketch the persistence diagrams and barcodes for the corresponding modules.

   [*Recall: in Problem List 2, we have computed that* $H_0(K_i;\mathbb{R}) = C_0(K_i;\mathbb{R})/V_i$ *and* $H_0(K'_i;\mathbb{R}) = C_0(K'_i;\mathbb{R})/V'_i$*, where*

$$V_0 = V'_0 = 0, \qquad V_1 = V'_1 = 0, \qquad V_2 = \operatorname{span}\{e_{v_2} - e_{v_1}\}, \qquad V'_2 = \operatorname{span}\{e_{v_3} - e_{v_1}\},$$
$$V_i = V'_i = \operatorname{span}\{e_{v_2} - e_{v_1}, e_{v_3} - e_{v_1}\} \quad \textit{for } i \geq 3,$$

   and the maps $(f_i)_*$ *send* $e_{v_j} + V_i$ *to* $e_{v_j} + V_{i+1}$ *(similarly for* $(f'_i)_*$*).*]

2. Consider the following multisets: $\rho = \{\{(0,1),(3,6),(5,8)\}\}$, $\sigma = \{\{(2,7),(6,8),(7,8)\}\}$ and $\tau = \{\{(3,7),(7,8)\}\}$.

   (a) Sketch the persistence diagrams for ρ, σ and τ.

   (b) Sketch the corresponding barcodes.

   (c) Compute the bottleneck distances $d_\infty(\rho,\sigma)$, $d_\infty(\rho,\tau)$ and $d_\infty(\sigma,\tau)$.

3. Consider the multisets $\sigma = \{\{(0,3),(0,4)\}\}$ and $\tau = \{\{(2,6),(3,6)\}\}$.

   (a) Write down all bijections that are potentially optimal for (σ,τ) (see page 28 of the lecture notes).

   (b) Compute the Wasserstein distances $d_{p,q}(\sigma,\tau)$ for $(p,q) \in \{(1,1),(2,1),(\infty,1),(2,2)\}$.

4. Consider the subset $Y := \left\{(-\frac{1}{2},-1),(\frac{1}{2},-1),(-\frac{1}{2},0),(\frac{1}{2},0),(0,\frac{\sqrt{3}}{2})\right\}$ of $\mathbb{R}^2$. Given $r \geq 0$, let K'_r be the nerve of the collection of balls $\{B_{r/2}(x) \mid y \in Y\}$, so that $\{K'_{i\varepsilon} \mid i \in \mathbb{N}\}$ is the step-ε Čech filtration for Y.

   (a) Draw the simplicial complexes $L_0 := K'_0$, $L_1 := K'_1$, $L_2 := K'_{2/\sqrt{3}}$ and $L_3 := K'_{\sqrt{2}}$.

   (b) It turns out that

   $$K'_r = \begin{cases} L_0 & \text{if } 0 \leq r < 1, \\ L_1 & \text{if } 1 \leq r < \frac{2}{\sqrt{3}}, \\ L_2 & \text{if } \frac{2}{\sqrt{3}} \leq r < \sqrt{2}. \end{cases}$$

   Use this to compute $H_1(K'_r;\mathbb{R})$ for $0 \leq r < \sqrt{2}$.

   (c) It turns out that $H_1(K'_r;\mathbb{R}) = 0$ for $r \geq \sqrt{2}$, and that the map $H_1(L_1;\mathbb{R}) \to H_1(L_2;\mathbb{R})$ induced by the (obvious) embedding $L_1 \subset L_2$ is non-zero. Use this to compute step-0.1 and step-0.01 persistence diagrams for the corresponding Čech filtrations.