Convolution and central limit theorem arising from addition of field operators in one mode type Interacting Fock Spaces

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1 One–mode type interacting Fock spaces

Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and $\mathcal{H}$ its complexification. We will use the scalar product symbol $\langle\,,\,\rangle$ in both cases. One can think of them as of real and complex $\mathbf{L}^2(\mathbb{R}_+)$. Let $\omega_i,\ i=0,1,\ldots$ be a sequence of positive numbers such that if $\omega_j = 0$ then $\omega_k = 0$ for all $k > j$. The corresponding one–mode type interacting Fock space is a Hilbert space

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}.$$

with scalar product

$$\langle \Omega, \Omega \rangle_\omega = 1$$
$$\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_m \rangle_\omega = \delta_{n,m} \prod_{j=1}^{n} \omega_{j-1} \langle x_j, y_j \rangle.$$

On it we define the creation operator $c^*(f)$ to be the same as the free creation:

$$c^*(f)\Omega = f$$
$$c^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n.$$

The annihilation operator $c(f)$ is the adjoint of the creation $c^*(f)$ and has the form

$$c(f)\Omega = 0$$
$$c(f) f_1 \otimes \cdots \otimes f_n = \omega_{n-1} \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n.$$

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In [ACL] the authors use the operators Λ_α defined by the following relations:

$$\Lambda_\alpha \Omega = \alpha_0 \Omega$$
$$\Lambda_\alpha f_1 \otimes \cdots \otimes f_n = \alpha_n f_1 \otimes \cdots \otimes f_n.$$

where $\alpha = \{\alpha_j\}_{j=0}^{\infty}$ can be any sequence of real numbers. They need them to construct the field operators $Q_\alpha(f)$:

$$Q_\alpha(f) = c^*(f) + \Lambda_\alpha + c(f) \tag{1}$$

which are the building blocks of their central limit theorem. The symbol f in the above definition denotes a vector from the real Hilbert space $\mathcal{H}_\mathbb{R}$, this can be for instance the indicators $\chi_{[k,k+1)}$ when $\mathcal{H}_\mathbb{R} = \mathbf{L}^2(\mathbb{R}_+)$, as it is in [ACL]. Since the operators $Q_\alpha(f)$ are always used with the argument f such that $\|f\| = 1$, we would like to use other field operators $\mathrm{F}(f)$ that would coincide with $Q_\alpha(f)$ for $\|f\| = 1$ and would be linear also in f. The part that requires attention is the number operator component. Let $\{e_j\},\ j = 1,2,\dots$ be an orthonormal basis of the space $\mathcal{H}_\mathbb{R}$. We define the preservation operator $\Lambda\colon \mathcal{H}_\mathbb{R} \mapsto \mathcal{L}(\mathcal{F}(\mathcal{H}))$ by linear extension of

$$\Lambda(e_j)\,\Omega = \alpha_0(e_j)\,\Omega$$
$$\Lambda(e_j)\, f_1 \otimes \cdots \otimes f_n = \alpha_n(e_j)\, f_1 \otimes \cdots \otimes f_n,$$

where $\alpha_n(e_j),\ n = 0,1,2,\dots$ is a sequence of real numbers for every $j = 1,2,\dots$.

Remark 1 *The operator $\Lambda\colon \mathcal{H}_\mathbb{R} \mapsto \mathcal{L}(\mathcal{F}(\mathcal{H}))$ is in general unbounded and thus for some $f = \sum_{j=1}^{\infty} \gamma_j e_j$ the operator $\Lambda(f)$:*

$$\Lambda(f)\, f_1 \otimes \cdots \otimes f_n = \left(\sum_{j=1}^{\infty} \gamma_j\, \alpha_n(e_j)\right) f_1 \otimes \cdots \otimes f_n,$$

may not be defined. However, $\Lambda(e_j)$ is well defined. Also $f = \sum_{j=1}^{N} \gamma_j e_j,\ N < \infty$ is well defined, moreover, it is bounded whenever $\alpha_n(e_j), n = 0,1,\dots$ are bounded for each $j = 1,2,\dots,N$. For such f we denote

$$\alpha_n(f) = \sum_{j=1}^{N} \gamma_j\, \alpha_n(e_j).$$

The operator $\mathrm{F}(f)$

$$\mathrm{F}(f) = c^*(f) + \Lambda(f) + c(f) \tag{2}$$

is now linear in f:

$$\mathrm{F}(af + bg) = a c^*(f) + b c^*(g) + a\Lambda(f) + b\Lambda(g) + a c(f) + b c(g) = a\mathrm{F}(f) + b\mathrm{F}(g),$$

for $a,b \in \mathbb{R},\ f,g \in \mathcal{H}_\mathbb{R}$, moreover, for $\|f\| = 1$ we have $\mathrm{F}(f) = Q_\alpha(f)$.

In [AB] the authors prove that the operator $\mathrm{F}(f)$, where $f \in \mathcal{H}$ with $\|f\| = 1$, has spectral distribution given by a measure μ. The moments of the measure μ are equal to the vacuum state of powers of $\mathrm{F}(f)$, that is $m_\mu(k) = \left\langle (\mathrm{F}(f))^k \Omega, \Omega \right\rangle$, and its Cauchy transform in continued fraction form is:

$$G_\mu(z) = \cfrac{1}{z - \alpha_0(f) - \cfrac{\omega_0}{z - \alpha_1(f) - \cfrac{\omega_1}{z - \alpha_2(f) - \cfrac{\omega_2}{z - \alpha_3(f) - \ddots}}}}.$$

We are concerned with the problem of expressing the convolution measure μ of two measures μ_1 and μ_2 with respective Cauchy transforms

$$G_{\mu_1}(z) = \cfrac{1}{z-\alpha_{1,0}-\cfrac{\omega_0}{z-\alpha_{1,1}-\cfrac{\omega_1}{z-\alpha_{1,2}-\cfrac{\omega_2}{z-\alpha_{1,3}-\ddots}}}}, \tag{3}$$

$$G_{\mu_2}(z) = \cfrac{1}{z-\alpha_{2,0}-\cfrac{\omega_0}{z-\alpha_{2,1}-\cfrac{\omega_1}{z-\alpha_{2,2}-\cfrac{\omega_2}{z-\alpha_{2,3}-\ddots}}}}, \tag{4}$$

defined as the spectral measure of the sum of the operators $\mathrm{F}(f_1)+\mathrm{F}(f_2)$, where $\|f_1\| = \|f_2\| = 1$ and $\langle f_1, f_2\rangle = 0$, for a choice of $\alpha_n(f_j) = \alpha_{j,n}$ corresponding to μ_1 and μ_2.

Theorem 1 *Let μ_1, μ_2 be probability measures with compact support and Cauchy transforms given by* (3) *and* (4). *Let $f_1, f_2 \in \mathcal{H}_{\mathbb{R}}$ be orthonormal vectors such that the spectral measure of $\mathrm{F}(f_i)$ be μ_i for $i = 1,2$. Then the convolution measure μ defined as the spectral measure of the operator $\mathrm{F}(f_1)+\mathrm{F}(f_2)$ is the universal convolution $\mu = \mu_1 \boxplus \mu_2$ with Cauchy transform in the form*

$$G_{\mu}(z) = \cfrac{1}{z-(\alpha_{1,0}+\alpha_{2,0})-\cfrac{\omega_0+\omega_0}{z-(\alpha_{1,1}+\alpha_{2,1})-\cfrac{\omega_1+\omega_1}{z-(\alpha_{1,2}+\alpha_{2,2})-\cfrac{\omega_2+\omega_2}{z-(\alpha_{1,3}+\alpha_{2,3})-\ddots}}}}.$$

Proof: By linearity we have $\mathrm{F}(f_1)+\mathrm{F}(f_2) = \mathrm{F}(f_1+f_2)$. Since the norm $\|f_1+f_2\| = \sqrt{2}$, we cannot use directly the Accardi-Bożejko result on $\mathrm{F}(f_1+f_2)$. We write $g = \frac{f_1+f_2}{\sqrt{2}}$. Let us take a look at the explicit action of $\mathrm{F}(g)$ on elementary tensors:

$$\begin{aligned}
\mathrm{F}(g)\Omega &= \frac{f_1+f_2}{\sqrt{2}} + \frac{\alpha_0(f_1)+\alpha_0(f_2)}{\sqrt{2}}\Omega \\
\mathrm{F}(g)x_1\otimes\cdots\otimes x_n &= \frac{f_1+f_2}{\sqrt{2}}\otimes x_1\otimes\cdots\otimes x_n + \frac{\alpha_n(f_1)+\alpha_n(f_2)}{\sqrt{2}}x_1\otimes\cdots\otimes x_n + \omega_{n-1}\left\langle \frac{f_1+f_2}{\sqrt{2}}, x_1\right\rangle x_2\otimes\cdots\otimes x_n \\
&= g\otimes x_1\otimes\cdots\otimes x_n + \frac{\alpha_n(f_1)+\alpha_n(f_2)}{\sqrt{2}}x_1\otimes\cdots\otimes x_n + \omega_{n-1}\langle g, x_1\rangle x_2\otimes\cdots\otimes x_n.
\end{aligned}$$

We can apply the Accardi-Bożejko result to the above and we get that the measure ν corresponding to the operator $\mathrm{F}(g)$ has the following Cauchy transform:

$$G_{\nu}(z) = \cfrac{1}{z-\frac{\alpha_0(f_1)+\alpha_0(f_2)}{\sqrt{2}}-\cfrac{\omega_0}{z-\frac{\alpha_1(f_1)+\alpha_1(f_2)}{\sqrt{2}}-\cfrac{\omega_1}{z-\frac{\alpha_2(f_1)+\alpha_2(f_2)}{\sqrt{2}}-\cfrac{\omega_2}{z-\frac{\alpha_3(f_1)+\alpha_3(f_2)}{\sqrt{2}}-\ddots}}}}.$$

Getting back to the operator $\mathrm{F}(f_1+f_2)$ we see that

$$(\mathrm{F}(f_1+f_2))^k = \left(\sqrt{2}\right)^k (\mathrm{F}(g))^k .$$

Hence, the spectral measure μ of the operator $\mathrm{F}(f_1+f_2)$ will be a dilation of the measure ν by a factor $\sqrt{2}$. This translates into Cauchy transforms as follows

$$G_\mu(z) = \cfrac{1}{z-(\alpha_0(f_1)+\alpha_0(f_2)) - \cfrac{2\,\omega_0}{z-(\alpha_1(f_1)+\alpha_1(f_2)) - \cfrac{2\,\omega_1}{z-(\alpha_2(f_1)+\alpha_2(f_2)) - \cfrac{2\,\omega_2}{z-(\alpha_3(f_1)+\alpha_3(f_2)) - \ddots}}}}.$$

This means that the convolution under study is nothing else than the universal convolution of Accardi–Bożejko, which consists in addition of the corresponding Jacobi coefficients. □

Remark 2 *The above theorem can be extended to unbounded measures with all moments. However, the resulting Jacobi coefficients will correspond to a unique measure only if the convolved measures are uniquely determined by their moments. Otherwise, the result of the convolution is not a single measure but a whole class of measures having the same Jacobi coefficients. The proof goes along the same lines, the only difference being that some of the continued fractions may need to be considered as formal ones [C].*

2 Another proof of the Central Limit Theorem of [ACL].

In the paper [ACL] the authors prove a theorem (Theorem 5.1 together with Corollary 5.1), to which we shall present another proof using the properties of the relating convolution.

Theorem 2 *Let μ be a mean–zero probability measure on $(\mathbb{R},\mathscr{B})$ with moments of all orders and Jacobi coefficients $\{\omega_n\},\{\alpha_n\}$, $n=0,1,\ldots$. Let $\mathcal{F}\left(\mathbf{L}^2(\mathbb{R}_+,\mathbb{C}),\{\gamma_j\}\right)$ be a one–mode type complex interacting Fock space with parameters γ_j such that $\omega_j=\frac{\gamma_j}{\gamma_{j-1}}$. Let $\{c_j\}$ be a bounded sequence with the property that*

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{N} c_j \to 1.$$

Let $f_j=\chi_{[j,j+1)}$, $f_j\in\mathbf{L}^2(\mathbb{R}_+,\mathbb{R})\subset\mathbf{L}^2(\mathbb{R}_+,\mathbb{C})$, and

$$Q_\alpha(f_j) = c^*(f_j) + c_j\Lambda_\alpha + c(f_j).$$

Then for any $k\in\mathbb{N}$

$$\lim_{N\to\infty}\left\langle \left(\frac{1}{\sqrt{N}}\sum_{j=1}^{N} Q_\alpha(f_j)\right)^k \Omega,\Omega\right\rangle = m_\mu(k).$$

Proof: We first note that $\langle f_j, f_k\rangle=\delta_{j,k}$. Moreover, we could require just this condition and drop the assumption on the specific form of the f_j's. We shall also drop the assumption on μ having mean zero. We then look at the action of the operator $c_j\Lambda_\alpha$:

$$c_j\Lambda_\alpha\Omega = c_j\,\alpha_0\,\Omega$$
$$c_j\Lambda_\alpha x_1\otimes\cdots\otimes x_n = c_j\,\alpha_n x_1\otimes\cdots\otimes x_n.$$

From the above we see that we can find an operator $\Lambda\colon \mathbf{L}^2(\mathbb{R}_+,\mathbb{R}) \mapsto \mathcal{L}\left(\mathcal{F}\left(\mathbf{L}^2(\mathbb{R}_+,\mathbb{C}),\{\gamma_j\}\right)\right)$ such that $\Lambda(f_j) = c_j\,\Lambda_\alpha$. It corresponds to the choice $\alpha_0(f_j) = c_j\,\alpha_n$, since the $f_j,\ j = 1,2,\ldots$ are orthonormal.

The moments of the spectral measure of the operator $\frac{1}{\sqrt{N}}\sum_{j=1}^N Q_\alpha(f_j)$ will thus be the same as the moments of the spectral measure μ_N of the operator $\frac{1}{\sqrt{N}}\sum_{j=1}^N \mathrm{F}(f_j)$, where $\mathrm{F}(f_j)$ is defined as in equation (2). We can now use the result on the Cauchy transform of convolutions to the measure μ_N:

$$G_{\mu_N}(z) = \cfrac{1}{z - \frac{1}{\sqrt{N}}\sum_{j=1}^N \alpha_0(f_j) - \cfrac{\omega_0}{z - \frac{1}{\sqrt{N}}\sum_{j=1}^N \alpha_0(f_j) - \cfrac{\omega_1}{z - \frac{1}{\sqrt{N}}\sum_{j=1}^N \alpha_0(f_j) - \cfrac{\omega_2}{z - \frac{1}{\sqrt{N}}\sum_{j=1}^N \alpha_0(f_j) - \ddots}}}}.$$

By the definition of $\alpha_0(f_j) = c_j\,\alpha_n$ we get

$$G_{\mu_N}(z) = \cfrac{1}{z - \alpha_0\frac{1}{\sqrt{N}}\sum_{j=1}^N c_j - \cfrac{\omega_0}{z - \alpha_1\frac{1}{\sqrt{N}}\sum_{j=1}^N c_j - \cfrac{\omega_1}{z - \alpha_2\frac{1}{\sqrt{N}}\sum_{j=1}^N c_j - \cfrac{\omega_2}{z - \alpha_3\frac{1}{\sqrt{N}}\sum_{j=1}^N c_j - \ddots}}}}.$$

We are now interested in letting $N \to \infty$. The limit we consider is in the sense of converging moments. Since moments are polynomially expressible in terms of Jacobi coefficients of order not higher than the order of the moment [AB], convergence of Jacobi coefficients implies convergence of moments. We write μ_∞ for the measure with limiting Jacobi coefficients. By the assumption on c_j we have

$$G_{\mu_\infty}(z) = \cfrac{1}{z - \alpha_0 - \cfrac{\omega_0}{z - \alpha_1 - \cfrac{\omega_1}{z - \alpha_2 - \cfrac{\omega_2}{z - \alpha_3 - \ddots}}}},$$

hence $\mu_\infty = \mu$. □

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