# Convolution associated with the free cosh–law


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**Abstract**

In this paper we consider the deformation of conditionally free convolution, connected with the free cosh–law. Because that measure is freely infinitely divisible, we can define a new, associative convolution using the theory from [KW]. We calculate the central and Poisson measure for that convolution and show that the coefficients of the continued fraction form of the Cauchy transform for the central and Poisson limit measures of that convolution are equal to the respective coefficients of the underlying measure starting from the third level.


## 1 Definitions

In this paper we continue the investigations on the convolutions arising from the conditionally free convolution of Bożejko, Leinert and Speicher [BLS] through deformations of the second measure. Those investigations started with the $t$–deformation studied in the papers of Bożejko and Wysoczański [BW1, BW2] and Wojakowski [Wo]. This was generalized by Krystek and Yoshida [KY]. Further examples were provided by Oravecz [O1, O2] and Krystek and Wojakowski [KW]. In the latter paper, the authors introduced a family of deformations depending on a selected compactly supported freely infinitely divisible measure, and proved limit theorems in terms of properties of the $R$–transforms of the limiting measures. In this work we calculate the central and Poisson measures for the convolution driven by the free cosh–law, that is by the measure with density

$$\zeta(x) = \frac{\sqrt{8-x^2}}{2\pi(x^2+1)}\chi_{[-4,4]}\,\mathrm{d}\,x.$$

The measure $\zeta$ is infinitely divisible with respect to the free convolution and is a particular case of the central measures for conditionally free convolution [BLS]. Those measures were also considered by [BB] and called free Meixner laws. It can be calculated that Voiculescu's $R^{\boxplus}$–transform and Cauchy transform of


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the measure $\zeta$ are equal to

$$R^{\boxplus}_{\zeta}(z) = \frac{2z}{1+\sqrt{1-4z^2}} = \int \frac{z}{1-xz}\,d\omega(x)$$
$$G_{\zeta}(z) = \int_{\mathbb{R}} \frac{d\zeta(x)}{z-x} = \frac{3z-\sqrt{z^2-8}}{2(z^2+1)},$$

where $\omega$ is the Wigner law.

Let us start with the definition of the infinitely divisible deformation and of the respective convolution connected with the free $\cosh$–law, see [KW] for details, definitions and theorems in more general cases:

**Definition 1.** *Let $\zeta$ be the free $\cosh$–law and $\mu$ any measure with compact support. Let us consider the following map $\mu \mapsto \mathcal{D}^{\zeta}_t\mu$:*

$$\mathcal{D}^{\zeta}_t\mu = \zeta_{t\cdot R^{\boxplus}_{\mu}(2)},$$

*which depends on the second free cumulant of the measure $\mu$ and on the nonnegative parameter $t$. For $s \geq 0$ we understand by $\zeta_s$ the s–fold free convolution power of $\zeta$ having the following $R^{\boxplus}$–transform:*

$$R^{\boxplus}_{\zeta_s}(z) = s\cdot R^{\boxplus}_{\zeta}(z).$$

**Definition 2.** *For compactly supported probability measures $\mu, \nu$ we define their $\boxed{\zeta}$–convolution by*

$$\mu\boxed{\zeta}\nu = (\mu, \mathcal{D}^{\zeta}_t\mu)\boxed{c}(\nu, \mathcal{D}^{\zeta}_t\nu).$$

By a result of [KW] we have that the convolution $\boxed{\zeta}$ is associative.

## 2 Central limit theorem

**Theorem 1.** *Let $\mu$ be a compactly supported probability measure on the real line with mean zero and variance equal to 1. Let $\zeta$ be the free cosh–law. Then the sequence*

$$\begin{aligned}&\mathbb{D}_{1/\sqrt{N}}\mu\boxed{\zeta}\ldots\boxed{\zeta}\mathbb{D}_{1/\sqrt{N}}\mu\\ &= (\mathbb{D}_{1/\sqrt{N}}\mu, \mathcal{D}^{\zeta}_t\mathbb{D}_{1/\sqrt{N}}\mu)\boxed{c}\ldots\boxed{c}(\mathbb{D}_{1/\sqrt{N}}\mu, \mathcal{D}^{\zeta}_t\mathbb{D}_{1/\sqrt{N}}\mu)\end{aligned}$$

*is $*$ - weakly convergent as $N \to \infty$ to the measure $\xi_t$, such that $\zeta_t = \mathcal{D}^{\zeta}_t\xi_t$ and*

$$R^{\boxed{c}}_{(\xi_t,\zeta_t)}(z) = z.$$

*The measure $\xi_t$ is absolutely continuous with density $f_{\xi_t}(x)$*

$$f_{\xi_t}(x) = \frac{1}{2\pi}\frac{t\sqrt{(4+4t-x^2)}}{x^4+(t^2-t-2)x^2+t+1} \quad \text{for } x \in \left[-2\sqrt{t+1}, 2\sqrt{t+1}\right].$$

*Proof.* Using the general central limit theorem from [KW] we obtain that the limiting measure $\xi_t$ satisfies the relation

$$R^{\boxed{c}}_{(\xi_t,\zeta_t)}(z) = z.$$

Because of

$$\frac{1}{G_{\zeta_t}(z)} = z - R^{\boxplus}_{\zeta_t}(G_{\zeta_t}(z)) = z - t\cdot R^{\boxplus}_{\zeta}(G_{\zeta_t}(z)),$$
$$\frac{1}{G_{\zeta_t}(z)} = z - \frac{2t\,G_{\zeta_t}(z)}{1+\sqrt{1-4\,G_{\zeta_t}(z)^2}},$$

we have the following expression for the Cauchy transform of the measure $\zeta_t$

$$G_{\zeta_t}(z) = \frac{(2+t)\,z - \sqrt{t^2\,(-4-4\,t+z^2)}}{2\,(t^2+z^2)}.$$

Using the relation between conditional transform and Cauchy transforms we obtain

$$\begin{aligned}
\frac{1}{G_{\xi_t}(z)} &= z - R^{\boxed{c}}_{(\xi_t,\zeta_t)}(G_{\zeta_t}(z)) = z - \frac{(2+t)\,z - \sqrt{t^2\,(z^2-4\,t-4)}}{2\,(t^2+z^2)} \\
G_{\xi_t}(z) &= \frac{2\left(t^2+z^2\right)}{(2\,t^2-t-2)\,z + 2\,z^3 + \sqrt{t^2\,(z^2-4\,t-4)}} \\
&= \frac{\left(2\,t^2-t-2\right)\,z + 2\,z^3 - \sqrt{t^2\,(z^2-4\,t-4)}}{2\,(z^4+(t^2-t-2)\,z^2+1+t)}.
\end{aligned}$$

The appropriate choice of the branch of a square root gives in limit for real $z = x$

$$\sqrt{t^2\,(z^2-4\,t-4)} = \begin{cases} t\,\sqrt{x^2-4\,t-4} & \text{for } x \geq 2\,\sqrt{t+1}, \\ -t\,\sqrt{x^2-4\,t-4} & \text{for } x \leq -2\,\sqrt{t+1}, \\ i\,t\,\sqrt{4\,t+4-x^2} & \text{for } -2\,\sqrt{t+1} < x < 2\,\sqrt{t+1}. \end{cases}$$

We would like to find the atoms of the measure $\xi_t$, that is the zeros of the denominator of the Cauchy transform of the measure $\xi_t$ . There is no atoms if either

$$\left(t^2-t-2\right)^2 - 4\,(1+t) < 0, \quad \text{that is for} \quad t \in (0,3)$$

or

$$\left(t^2-t-2\right)^2 - 4\,(1+t) \geq 0, \quad \text{and} \quad \left(t^2-t-2\right) > 0 \quad \text{and} \quad (1+t) > 0,$$

that is when

$$t \geq 3.$$

Thus we have atoms only for $t = 0$.

By Proposition in chapter XIII.6 in [RS4] one can find that there is no singular parts of the measure $\xi_t$. It follows from regularity properties of $G_{\xi_t}(z)$.

The density $f_{\xi_t}(x),\ x \in \mathbb{R}$ of the absolute continuous part of the measure $\xi_t$ can now be calculated by the Stieltjes formula

$$f_{\xi_t}(x) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+} \Im G_{\xi_t}(x+i\,\epsilon)\,.$$

Thus we obtain

$$f_{\xi_t}(x) = \frac{1}{2\,\pi}\frac{t\,\sqrt{(4+4\,t-x^2)}}{x^4+(t^2-t-2)\,x^2+t+1} \quad \text{for } x \in \left[-2\,\sqrt{t+1}, 2\,\sqrt{t+1}\right].$$

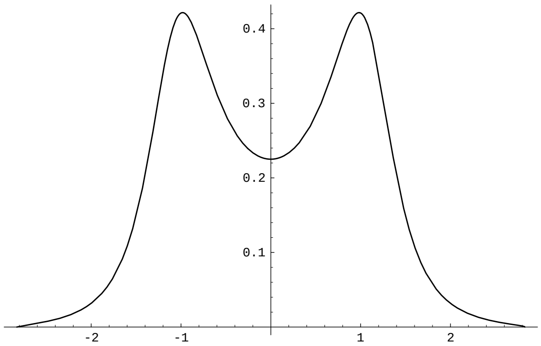
Moreover we can calculate that the measure $\xi_t$ has the following continued fraction expansion

$$G_{\xi_t}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{1+t}{z - \cfrac{1+t}{z - \ddots}}}}}.$$

□

A diagram of this measure for $t = 1$ is presented on the following figure.

Figure 1: Density of the central limit measure for the cosh–convolution




# 3 Poisson limit theorem

**Theorem 2.** *For* $\lambda > 0$ *define for all* $N$

$$\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1, \quad N \geq 1.$$

*Then we have*

$$w^* - \lim_{N\to\infty} (\mu_N, \mathcal{D}_t^{\zeta}\mu_N) \boxed{c} \dots \boxed{c} (\mu_N, \mathcal{D}_t^{\zeta}\mu_N) = (p_\lambda, \zeta_{\lambda\, t}),$$

*where*

$$R^{\boxed{c}}_{(p_\lambda, \zeta_{\lambda\, t})}(z) = \frac{\lambda}{1-z}.$$

*The absolutely continuous part of the measure* $p_\lambda$ *is given by*

$$f_{p_\lambda}(x) = \frac{\lambda\sqrt{t^2\,\lambda^2\,\left(4 + 4\,t\,\lambda - x^2\right)}}{\pi\, Q(x)} \quad \textit{for } x \in \left[-2\sqrt{1+\lambda\, t}, 2\sqrt{1+\lambda\, t}\right]$$

*and where*

$$\begin{aligned} Q(x) &= 2\,x^4 - x^3\,\left(4 + 4\,\lambda + 2\,t\,\lambda\right) \\ &\quad + x^2\,\left(2 + 4\,\lambda + 2\,t\,\lambda + 2\,\lambda^2 + 2\,t\,\lambda^2 + 2\,t^2\,\lambda^2\right) - 4\,t^2\,\lambda^3\,x + 2\,t^2\,\lambda^4. \end{aligned}$$

*The measure* $p_\lambda$ *may have at most four atoms.*

*Proof.* Using the fact that the limiting measure $p_\lambda$ is determined by the relation [KW]

$$R^{\boxed{c}}_{(p_\lambda, \zeta_{\lambda\, t})}(z) = \frac{\lambda}{1-z},$$

and because of

$$G_{\zeta_{\lambda\, t}}(z) = \frac{2\,z + t\,z\,\lambda - t\,\lambda\sqrt{z^2 - 4 - 4\,t\,\lambda}}{2\,z^2 + 2\,t^2\,\lambda^2},$$

we obtain

$$\frac{1}{G_{p_\lambda}(z)} = z - \frac{2\,\lambda\,\left(z^2 + t^2\,\lambda^2\right)}{2\,z^2 - z\,\left(2 + t\,\lambda\right) + t\,\lambda\,\left(2\,t\,\lambda + \sqrt{z^2 - 4 - 4\,t\,\lambda}\right)}.$$

Hence

$$G_{p_\lambda}(z) = \frac{2z^2 - z\,(2+t\lambda) + t\lambda\,\left(2t\lambda + \sqrt{-4+z^2-4t\lambda}\right)}{2z^3 - 2t^2\lambda^3 - z^2\,(2+(2+t)\,\lambda) + tz\lambda\,\left(2t\lambda+\sqrt{z^2-4-4t\lambda}\right)}$$
$$= \frac{2z^3 - 2z^2\,(2+\lambda+t\lambda) + z\,\left(2+2\lambda+2t\lambda+t\lambda^2+2t^2\lambda^2\right)}{Q(z)}$$
$$- \frac{t\lambda^2\,\left(2t\lambda+\sqrt{z^2-4-4t\lambda}\right)}{Q(z)},$$

where

$$Q(z) = 2z^4 - z^3\,(4+4\lambda+2t\lambda)$$
$$+ z^2\,\left(2+4\lambda+2t\lambda+2\lambda^2+2t\lambda^2+2t^2\lambda^2\right) - 4t^2\lambda^3 z + 2t^2\lambda^4.$$

The denominator of the Cauchy transform of the measure $p_\lambda$ may have four simple real roots $x_j$, thus the measure $p_\lambda$ has at most four atoms $\delta_{x_j}$ with respective weights. The actual dependence of atoms and weights on $\lambda$ and $t$ is rather complicated.

The density $f_{p_\lambda}(x),\ x\in\mathbb{R}$ of the absolutely continuous part of the measure $p_\lambda$ can now be calculated by the Stieltjes formula

$$f_{p_\lambda}(x) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\Im G_{p_\lambda}(x+i\epsilon)$$
$$= \frac{\lambda\sqrt{t^2\lambda^2\,(4+4t\lambda-x^2)}}{\pi Q(x)}$$

for $x \in \left[-2\sqrt{1+\lambda t}, 2\sqrt{1+\lambda t}\right]$.

Moreover, we obtain the following continued fraction form of the Cauchy transform of the measure $p_\lambda$:

$$G_{p_\lambda}(z) = \cfrac{1}{z-\lambda-\cfrac{\lambda}{z-1-\cfrac{t\lambda}{z-\cfrac{1+t\lambda}{z-\cfrac{1+t\lambda}{z-\ddots}}}}}.$$

□

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