Remarks on the r and Δ convolutions*

Marek Bożejko, Anna Krystek, Łukasz Wojakowski

Mathematical Institute, University of Wrocław,
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
Fax: (+48) (71) 375 74 29
bozejko@math.uni.wroc.pl
Anna.Krystek@math.uni.wroc.pl
Lukasz.Wojakowski@math.uni.wroc.pl

March 29, 2005

Abstract

In this paper we study the properties of the r–deformation introduced in [B1]. We observe that the associated convolution coming from the conditionally free convolution is associative only for $r = 1$ and $r = 0$. We give the realization of some r–Gaussian random variables and obtain Haagerup–Pisier–Buchholz type inequalities. We also study another convolution defined with the use of the r–deformation through a moment–cumulant formula [KY1] and show that it is associative and in general not positive.

Primary: 46L53, 46L54, Secondary: 60E10

keywords: Convolution, conditionally free, free convolution, boolean convolution, deformations, moment–cumulant formulae, limit theorems, operator spaces

1 Introduction

In the conference paper [B1] Bożejko introduced for any real number $0 \leq r \leq 1$ the r–free product of states, using the conditionally free product of states, see [BS], [BLS]. For $r = 1$ it reduces to the free product of states of Voiculescu [VDN] and Avitzour [A], for $r = 0$ to the Boolean product [BGS], [F], called regular free product in the paper [B2].

In Section 2 we recall the necessary notions of noncommutative probability along with the definitions of the free, conditionally free and boolean products and convolutions.

*Partially sponsored with KBN grant no 2P03A00723 and RTN HPRN-CT-2002-00279

In Section 3 we study the properties of the r–deformation of Bożejko in the context of the convolutions and products. We observe that the r–free convolution of probability measures on the real line is not associative. This leads to the definition of the N–fold r–free product and convolution and of the iterated twofold r–free product and convolution, which can be iterated from the left or from the right. Using the combinatorial theory of cumulants, we also study another convolution interpolating between the free convolution and Boolean convolution, called r^*–convolution [KY1]. However, we show that for $0 < r < 1$ this is an associative convolution of probability measures which is not positive, i.e. in general its result can be a signed measure, see Section 6.

In Section 4.1 we prove the central limit theorem and Poisson limit theorem for the N–fold r–convolution and show that the central limit measure is the Kesten measure κ_r. Also an explicit formula of the Poisson measure is given.

In Section 4.2 we give the realization of N–fold r–Gaussian random variables for $r \geq 0$ and we obtain the Haagerup-Pisier-Buchholz type inequalities, see [B], [HP]. In the case $r = 0$ we obtain that the Boolean Gaussian random variables are bounded operators on a suitable Fock space and are completely isomorphic to the row and column operator space, i.e.

$$\left\| \sum_{i=1}^{N} a_i \otimes \omega_0\left(e_i\right) \right\| = \max\left(\left\| \sum_{i=1}^{N} a_i a_i^* \right\|^{1/2}, \left\| \sum_{i=1}^{N} a_i^* a_i \right\|^{1/2} \right).$$

In Section 4.3 we study the N–fold r–free product of states. We show that it has the Voiculescu property: if $\phi\left(a_j\right) = 0$ for all j and $a_j \in \mathcal{A}_{i_j}$, $i_1 \neq i_2 \neq \ldots \neq i_n$ then $\phi\left(a_1 a_2 \ldots a_n\right) = 0$.

In Section 5 we calculate the counterexamples which show that the twofold iterated convolution and product are not associative.

In Section 6 we show that the combinatorial approach gives a different convolution, which we call r^*–convolution and denote $\boxed{r*}$. It is associative but is not positive. The central and Poisson limit theorems for the r^*-convolution were proven in [B1] and [KY1].

In Section 7 we also present some remarks concerning the Δ–convolution introduced by Bożejko [B1] and Yoshida [Y2]. This convolution depends on positive definite sequences $\Delta = (\Delta_n)$. The r–free convolution corresponds to the sequence $\Delta_n = r$. Again we show that the Δ–convolution is associative only if $\Delta_n = 0$ or $\Delta_n = 1$, that is when it reduces to the Boolean or free case.

2 Conditionally free convolution and product of states

In the papers [BLS], [BS] the authors present the construction of the conditionally free products and convolutions. We recall their definitions and essential properties after some basic facts on noncommutative probability theory [VDN], [CO]. Let $\mathcal{A}$ be a C*–algebra. By a state we understand a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1, which is positive, i.e. satisfies $\phi(xx^*) \geq |\phi(x)|^2$. We shall call any $X \in \mathcal{A}$ a random variable, and by the distribution of a selfadjoint variable $X = X^*$ we shall understand equivalently either the sequence of moments $\varphi(X^n)$, $n = 0, 1, \ldots$ or the corresponding compactly supported measure μ_X such that

$$\varphi(X^n) = \int_{-\infty}^{+\infty} x^n \, d\mu_X(x).$$

Conversely, given a probability measure μ on the real line with compact support, we can consider a unital algebra $\mathcal{A} = \mathbb{C}(X)$ of polynomials in one variable X and a state φ_μ given by linear extension of

$$\varphi_\mu(X^n) = \int_{-\infty}^{+\infty} x^n \, d\mu(x).$$

Definition 1 *Let $(\mathcal{A}_i, \phi_i, \psi_i)$, $i = 1, \ldots, N$, $N = 2, 3, \ldots \infty$ be a sequence of unital* $\mathrm{C}^\star$*–algebras equipped with two states. Let $\mathcal{A}$ be the algebraic free product of the algebras $\mathcal{A}_i$. The conditionally free product state*

$$\Phi = \mathop{\boxed{\mathrm{C}_N}}_{i=1}^{N} (\phi_i, \psi_i)$$

is uniquely defined on $\mathcal{A}$ by the requirement that for any $a_{i_1}, \ldots, a_{i_m}$ such that $a_{i_k} \in \mathcal{A}_{i_k}$ and $i_1 \neq i_2 \neq \ldots \neq i_m$ and also $\psi_{i_k}(a_{i_k}) = 0$ we have

$$\Phi(a_{i_1} \ldots a_{i_m}) = \Phi(a_{i_1}) \ldots \Phi(a_{i_m}).$$

Remark 1 *The above definition has two special cases.*

1. *For $\psi = \phi$ it reduces to the definition of the free product: for $a_{i_1}, \ldots, a_{i_m}$ such that $a_{i_k} \in \mathcal{A}_{i_k}$ and $i_1 \neq i_2 \neq \ldots \neq i_m$ and $\phi_{i_k}(a_{i_k}) = 0$ we have $\Phi(a_{i_1} \ldots a_{i_m}) = 0$.*
2. *For $\psi(\mathbf{1}) = 1$, $\psi(a) = 0$, $a \neq \mathbf{1}$ it reduces to the definition of the Boolean product: for any $a_{i_1}, \ldots, a_{i_m}$ such that $a_{i_k} \in \mathcal{A}_{i_k}$, $a_{i_k} \neq \mathbf{1}$ and $i_1 \neq i_2 \neq \ldots \neq i_m$ we have $\Phi(a_{i_1} \ldots a_{i_m}) = \Phi(a_{i_1}) \ldots \Phi(a_{i_m})$.*

Remark 2 *For another model interpolating between those two special cases see [Len].*

We shall use the following associativity lemma proved by Młotkowski [M]

Lemma 1 *Assume that $I = \bigcup_{j \in J} I_j$ is a partition of I. Then*

$$\mathop{\boxed{\mathrm{C}_N}}_{j \in J} \left(\mathop{\boxed{\mathrm{C}_N}}_{i \in I_j} (\mu_i, \nu_i) \right) = \mathop{\boxed{\mathrm{C}_N}}_{i \in I} (\mu_i, \nu_i).$$

To define the conditionally free convolution of an N-tuple of pairs of probability measures $(\mu_1, \nu_1), \ldots, (\mu_N, \nu_N)$ consider random variables $X_1, \ldots, X_N$ such that $X_i \in (\mathcal{A}_i, \varphi_i, \psi_i)$ and X_i has distribution μ_i with respect to φ_i and ν_i with respect to ψ_i. The equation for moments of the conditionally free convolution $\mu = \boxed{c_N}_{i=1}^{N} (\mu_i, \nu_i)$ of that N-tuple is the following

$$m_{\boxed{c_N}_{i=1}^{N}(\mu_i, \nu_i)}(n) = m_\mu(n) = \Phi\left((X_1 + X_2 + \ldots + X_N)^n\right).$$

When $N=2$ we shall use the notation $(\mu_1,\nu_1) \boxed{c} (\mu_2,\nu_2)$.

In analogy to the free convolution $\mu_1 \boxplus \mu_2$, which is linearized by the series of free cumulants $R^{\boxplus}_{\mu_1}(n)$ and $R^{\boxplus}_{\mu_2}(n)$, the conditionally free convolution $(\mu_1,\nu_1) \boxed{c} (\mu_2,\nu_2)$ is linearized by pairs of series $(R^{\boxed{c}}_{\mu_1,\nu_1}(n), R^{\boxplus}_{\nu_1}(n))$ and $(R^{\boxed{c}}_{\mu_2,\nu_2}(n), R^{\boxplus}_{\nu_2}(n))$. These series are related to the respective measures by the moment–cumulant formulae

$$m_{\nu_i}(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi=(\pi_1,\ldots,\pi_k)}} \prod_{l=1}^{k} R^{\boxplus}_{\nu_i}(|\pi_l|), \tag{1}$$

$$m_{\mu_i}(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi=(\pi_1,\ldots,\pi_k)}} \prod_{\pi_l \text{outer}} R^{\boxed{c}}_{\mu_i,\nu_i}(|\pi_l|) \prod_{\pi_l \text{inner}} R^{\boxplus}_{\nu_i}(|\pi_l|), \tag{2}$$

where a block π_l is called inner when there exist some other block π_j with $a,b \in \pi_j$ with the property that $a<p<b$ for all $p \in \pi_l$. All blocks which are not enveloped in such a way are called outer. Equivalently one can say that outer blocks π_j have depth $d(\pi_j)=1$ and inner π_l have $d(\pi_l)>1$.

The cumulants are associated to respective R–transforms, which are functions analytic in a neighbourhood of zero through the relations

$$R^{\boxplus}_{\nu_i}(z) = \sum_{n=1}^{\infty} R^{\boxplus}_{\nu_i}(n) z^{n-1},$$

$$R^{\boxed{c}}_{\mu_i,\nu_i}(z) = \sum_{n=1}^{\infty} R^{\boxed{c}}_{\mu_i,\nu_i}(n) z^{n-1}.$$

In terms of the R–transforms the relations (1) and (2) correspond to the following relations

$$\frac{1}{G_{\mu_i}(z)} = z - R^{\boxed{c}}_{(\mu_i,\nu_i)}(G_{\nu_i}(z)), \tag{3}$$

$$\frac{1}{G_{\nu_i}(z)} = z - R^{\boxplus}_{\nu_i}(G_{\nu_i}(z)), \tag{4}$$

where $G_\mu(z)$, $z \in \mathbb{C}$, $\Im(z)>0$ is the Cauchy transform of the measure μ given by

$$G_\mu(z) = \int \frac{1}{z-x}\, d\mu(x).$$

The moments of the N–fold conditionally free convolution can be calculated from the formula

$$m_\mu(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi=(\pi_1,\ldots,\pi_k)}} \prod_{\pi_l \text{outer}} \left[\sum_{i=1}^{N} R^{\boxed{c}}_{\mu_i,\nu_i}(|\pi_l|)\right] \prod_{\pi_l \text{inner}} \left[\sum_{i=1}^{N} R^{\boxplus}_{\nu_i}(|\pi_l|)\right].$$

Let us also note, that for a pair of measures (μ_i,ν_i) formula (2) is equivalent to the recursive definition of the conditionally free cumulants $R^{\boxed{c}}_{\mu_i,\nu_i}(k)$:

$$m_{\mu_i}(n) = \sum_{k=1}^{n} \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ l_1+\ldots+l_k=n-k}} R^{\boxed{c}}_{\mu_i,\nu_i}(k) m_{\nu_i}(l_1)\ldots m_{\nu_i}(l_{k-1}) m_{\mu_i}(l_k). \tag{5}$$

The conditionally free convolution of pairs of measures (μ_i, ν_i) is a single measure μ, however, as noted in [BLS], if we associate to it the free convolution $\nu = \nu_1 \boxplus \cdots \boxplus \nu_N$, we obtain an associative convolution of pairs of measures. The associativity property allows us to reduce the N–fold convolution to the binary convolution, thus

$$(\mu, \nu) = (\boxed{c_N}(\mu_i, \nu_i), \boxplus \nu_i) = (\mu_1, \nu_1) \boxed{c} (\mu_2, \nu_2) \boxed{c} \cdots \boxed{c} (\mu_N, \nu_N).$$

3 Deformations

One application of the conditionally free convolution of pairs of measures (μ_i, ν_i) (and also of the corresponding product) is the construction of convolutions of single measures μ_i with the use of some transformation $T : \mu_i \mapsto \nu_i$. We call $T\mu_i$ the T–deformation of μ_i. We can thus define

$$\mu = \underset{i=1}{\overset{N}{\boxed{T_N}}} \mu_i = \underset{i=1}{\overset{N}{\boxed{c_N}}} (\mu_i, T\mu_i).$$

This N–fold convolution is well described by the corresponding pairs of series

$$(R^{\boxed{c}}_{\mu_i, T\mu_i}(n), R^{\boxplus}_{T\mu_i}(n)),$$

thus

$$m_\mu(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi = (\pi_1, \ldots, \pi_k)}} \prod_{\pi_l \,\mathrm{outer}} \left[\sum_{i=1}^{N} R^{\boxed{c}}_{\mu_i, T\mu_i}(|\pi_l|) \right] \prod_{\pi_l \,\mathrm{inner}} \left[\sum_{i=1}^{N} R^{\boxplus}_{T\mu_i}(|\pi_l|) \right].$$

However, the possibility of reducing the N–fold convolution $\boxed{T_N}$ to an iteration of the pair convolution $\boxed{T}$ is a more delicate issue. Let us take $N = 3$. We would like to be able to get $\boxed{T_N}_{i=1}^{3} \mu_i$ by first calculating $\eta = \mu_1 \boxed{T} \mu_2$ and then by calculating $\eta \boxed{T} \mu_3$. Thus

$$\eta = \mu_1 \boxed{T} \mu_2 = (\mu_1, T\mu_1) \boxed{c} (\mu_2, T\mu_2)$$

and

$$(\eta, \xi) = (\mu_1 \boxed{T} \mu_2, T\mu_1 \boxplus T\mu_2).$$

Now, to proceed with the calculation of $\eta \boxed{T} \mu_3$ we have to perform the conditionally free convolution

$$(\eta, T\eta) \boxed{c} (\mu_3, T\mu_3).$$

But the measure $T\eta$ can be different than $\xi = T\mu_1 \boxplus T\mu_2$, which is the second part of the result of $(\mu_1, T\mu_1) \boxed{c} (\mu_2, T\mu_2)$. In this way we see that such a reduction is possible if $T\eta = T\mu_1 \boxplus T\mu_2$.

Example In the case of the t–deformation U_t of Bożejko and Wysoczański [BW1], [BW2] (see also [W], [KW], [KY2], [Or]) this condition is satisfied and the corresponding convolutions of measures are associative.

Definition 2 *In [B1] Bożejko introduced the r–deformation of measures and of states*

$$V_r\,\mu_i = r\,\mu_i + (1-r)\delta_0,$$

with the corresponding convolution

$$\mu_1 \boxed{r} \mu_2 = (\mu_1, V_r\,\mu_1) \boxed{c} (\mu_2, V_r\,\mu_2)$$

called r-free convolution.

In contrast to the t–deformation U_t of the previous example the r–deformation V_r does not behave well in this application. Indeed, we have the following

Lemma 2 *In general, for* $0 < r < 1$ *there exist measures* μ_1, μ_2 *such that*

$$V_r\,(\mu_1 \boxed{r} \mu_2) \neq V_r\mu_1 \boxplus V_r\,\mu_2.$$

Proof: It is sufficient to show that if for $n = 1, 2, \ldots$

$$m_{V_r\,(\mu \boxed{r} \mu)}(n) = m_{V_r\,\mu \boxplus V_r\,\mu}(n)$$

holds then μ is the Dirac mass at zero.

For $n = 2$ we have

$$m_{V_r\,(\mu \boxed{r} \mu)}(2) = 2r\left[m_\mu(1)^2 + m_\mu(2)\right]$$
$$m_{V_r\,\mu \boxplus V_r\,\mu}(2) = 2r\left[r\,m_\mu(1)^2 + m_\mu(2)\right],$$

hence

$$m_{V_r\,(\mu \boxed{r} \mu)}(2) - m_{V_r\,\mu \boxplus V_r\,\mu}(2) = 2r(1-r)m_\mu(1)^2,$$

which implies $m_\mu(1) = 0$. Now, for $n = 4$ we get

$$\begin{aligned}
m_{V_r\,(\mu \boxed{r} \mu)}(4) &= \\
&= 2\,r\Big[(1-2\,r)\,m_\mu(1)^4 - (-3+r)\,m_\mu(1)^2\,m_\mu(2) + (1+r)\,m_\mu(2)^2 + \\
&\quad + 2\,(1+r)\,m_\mu(1)\,m_\mu(3) + m_\mu(4)\Big] \\
&= 2\,r\left[(1+r)\,m_\mu(2)^2 + m_\mu(4)\right] \\
m_{V_r\,\mu \boxplus V_r\,\mu}(4) &= \\
&= 2\,r\Big[-r^3\,m_\mu(1)^4 + 2\,r^2\,m_\mu(1)^2\,m_\mu(2) + 2\,r\,m_\mu(2)^2 + \\
&\quad + 4\,r\,m_\mu(1)\,m_\mu(3) + m_\mu(4)\Big] \\
&= 2\,r\left[2\,r\,m_\mu(2)^2 + m_\mu(4)\right],
\end{aligned}$$

hence

$$m_{V_r(\mu \boxed{r} \mu)}(4) - m_{V_r \mu \boxplus V_r \mu}(4) = 2r\,(1-r)\,m_\mu(2)^2.$$

Thus, we have seen that $m_\mu(1) = m_\mu(2) = 0$, which completes the proof. $\square$ $\square$

To see the consequences of the above lemma, let us consider three measures μ_i, $i = 1,2,3$ and try to convolve them. We start by doing the twofold convolution

$$(\mu_1 \boxed{r} \mu_2, V_r \mu_1 \boxplus V_r \mu_2) = (\mu_1, V_r \mu_1) \boxed{c} (\mu_2, V_r \mu_2).$$

Then, to get the threefold convolution μ, we take

$$(\mu, V_r(\mu_1 \boxed{r} \mu_2) \boxplus V_r \mu_3) = (\mu_1 \boxed{r} \mu_2, V_r(\mu_1 \boxed{r} \mu_2)) \boxed{c} (\mu_3, V_r \mu_3).$$

However, since $V_r \mu_1 \boxplus V_r \mu_2 \neq V_r(\mu_1 \boxed{r} \mu_2)$ in general, the measure μ is not the same as the result of

$$\tilde{\mu} = \underset{i=1}{\overset{3}{\boxed{c_N}}} (\mu_i, V_r \mu_i) = \underset{i=1}{\overset{3}{\boxed{r_N}}} \mu_i.$$

Moreover, we shall also see in Section 5 that the iterated pairwise convolution $\mu_1 \boxed{r} \mu_2$ is not associative, thus the need of introducing a left ($\overrightarrow{\boxed{r}}$) and right ($\overleftarrow{\boxed{r}}$) version.

In [B1] a definition of another r–convolution is given, which esentially consists in extending the following lemma to a moment–cumulant formula.

Lemma 1 (Lemma 7) *For some function $uc(\pi)$ defined on noncrossing partitions and appropriate cumulants $R^*(k)$*

$$m_{\mu_i}(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi = (\pi_1, \ldots, \pi_k)}} r^{uc(\pi)} \prod_{j=1}^{k} R^*_{\mu_i}(|\pi_j|) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi = (\pi_1, \ldots, \pi_k)}} r^{uc(\pi)} \prod_{j=1}^{k} R^{\boxed{c}}_{\mu_i, V_r \mu_i}(|\pi_j|).$$

One can then define the convolution $\mu_1 \boxed{r*} \mu_2$ by the requirement

$$R^*_{\mu_1 \boxed{r*} \mu_2}(n) = R^*_{\mu_1}(n) + R^*_{\mu_2}(n)$$

and use the above cumulant–moment formula to recover the moments of the measure $\mu_1 \boxed{r*} \mu_2$. We shall see later that this convolution is of a different kind than the three above.

4 The N–fold case

4.1 The N–fold r-free convolution

In this section we will only consider the r–free convolution of N measures as

$$\underset{i=1}{\overset{N}{\boxed{r_N}}} \mu_i = \underset{i=1}{\overset{N}{\boxed{c_N}}} (\mu_i, V_r \mu_i).$$

Theorem 1 (Central limit theorem) *Let μ be a compactly supported probability measure on the real line with mean zero and variance equal to 1. Then the sequence*

$$\boxed{r_N}_{i=1}^{N} D_{1/\sqrt{N}}\mu = \boxed{c_N}_{i=1}^{N}(D_{1/\sqrt{N}}\mu, V_r D_{1/\sqrt{N}}\mu)$$

is $*$-weakly convergent to the Kesten measure κ_r as $N \to \infty$.

Remark 3 *The Kesten distribution κ_r has been calculated for instance in [BW2]. It has a part absolutely continuous with respect to the Lebesgue measure, denoted $\tilde{\kappa}_r$ and for $r < \frac{1}{2}$ a discrete part $\hat{\kappa}_r$ with two atoms:*

$$\tilde{\kappa}_r \sim \frac{1}{2\pi} \cdot \frac{\sqrt{4r - x^2}}{1-(1-r)x^2} \chi_{[-2\sqrt{r}, 2\sqrt{r}]}(x)\, dx$$

$$\hat{\kappa}_r \sim \frac{1-2r}{2-2r}\left(\delta_{-\frac{1}{\sqrt{1-r}}} + \delta_{\frac{1}{\sqrt{1-r}}}\right) \qquad \text{for } r < \frac{1}{2}.$$

Its Cauchy transform $G_{\kappa_r}(z)$ has a continued fraction representation

$$G_{\kappa_r}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{r}{z - \cfrac{r}{z - \cfrac{r}{z - \ddots}}}}}.$$

Proof: The sequence of N-fold $\boxed{r}$-convolution of the measure μ is of the form

$$\mu_n = \boxed{r_N}_{i=1}^{N} D_{1/\sqrt{N}}\mu$$

and by $m_{V_r\mu}(1) = 0$, $m_{V_r\mu}(2) = r$ we obtain, that the sequence μ_n is $*$-weakly convergent to the pair (κ_r, ω_r), defined by a requirement $R^{\boxed{c}}_{(\kappa_r,\omega_r)}(z) = z$, where ω_r denotes the Wigner measure with variance r, i.e. the measure with density

$$d\omega_r(x) = \frac{1}{2\pi r}\sqrt{4r - x^2}\chi_{[-2\sqrt{r}, 2\sqrt{r}]}(x)dx.$$

Using the relation between the Cauchy transforms and the conditional cumulant transform (3), (4) we obtain

$$\frac{1}{G_{\kappa_r}(z)} = z - G_{\omega_r}(z),$$

$$\frac{1}{G_{\omega_r}(z)} = z - r\,G_{\omega_r}(z) = \cfrac{1}{z - \cfrac{r}{z - \cfrac{r}{z - \cfrac{r}{z - \ddots}}}}.$$

hence

$$\frac{1}{G_{\kappa_r}(z)} = \cfrac{1}{z - \cfrac{1}{z - \cfrac{r}{z - \cfrac{r}{z - \cfrac{r}{z - \ddots}}}}}$$

and κ_r is indeed the Kesten measure. □ □

Remark 4 *For $r \geq 1$ the Kesten measure is infinitely divisible with respect to the free convolution and it is the distribution of the free Lévy process with linear regressions and quadratic condition variances, see [BB].*

Problem 1 *The limit measure in the central limit theorem is well defined also for $r > 1$. However, the deformation V_r used to define the convolution is restricted to the case $0 \leq r \leq 1$. Is it possible to define also the convolution for $r > 1$?*

Theorem 2 (Poisson type limit theorem) *For $\lambda > 0$ define for all N*

$$\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1, \quad N \geq 1.$$

Then we have

$$\lim_{N\to\infty} \boxed{r_N}_{i=1}^{N} \mu_N = (\pi_{\lambda,r\lambda}, \pi_{r\lambda,r\lambda}),$$

in the weak-$*$ topology, and where

$$\pi_{\lambda,r\lambda} = a\delta_0 + b\delta_{z_0} + \tilde{\pi}_{\lambda,r\lambda}$$

with

$$\begin{aligned}
a &= \frac{1 - r\lambda}{1 + \lambda - r\lambda}\max(1 - r\lambda, 0), \\
b &= \frac{r z_0 - \lambda}{z_0(1-r)}\max(\lambda(1-r)^2 - r, 0), \\
z_0 &= \lambda + \frac{1}{1-r}, \\
\tilde{\pi}_{\lambda,r\lambda}(x) &\sim \frac{1}{2\pi}\frac{\sqrt{4r\lambda - (x - (1 + r\lambda))^2}}{x(x(r-1) + (1 - r\lambda + \lambda))}\,dx,
\end{aligned}$$

and $\pi_{\alpha,\beta}$ is the conditionally free Poisson measure.

Remark 5 *The Cauchy transform of the Poisson distribution $\pi_{\lambda,r\lambda}$ has the form*

$$G_{\pi_{\lambda,r\lambda}}(z) = \frac{1-z+2rz-r\lambda-\sqrt{-4r\lambda+(1-z+r\lambda)^2}}{2z\,(1+(r-1)\,z+\lambda-r\lambda)}$$

hence it has a continued fraction representation

$$G_{\pi_{\lambda,r\lambda}}(z) = \cfrac{1}{z-\lambda-\cfrac{\lambda}{z-1-r\lambda-\cfrac{r\lambda}{z-1-r\lambda-\cfrac{r\lambda}{z-1-r\lambda-\ddots}}}}.$$

Proof: By definition we have

$$V_r\,\mu_N = \left(1-r\frac{\lambda}{N}\right)\delta_0 + r\frac{\lambda}{N}\delta_1,$$

hence using the general theory of [BLS] in the case $\alpha=\lambda,\ \beta=r\lambda$ we obtain, that the limit measure is $(\pi_{\lambda,r\lambda},\pi_{r\lambda,r\lambda})$. □ □

Remark 6 *The orthogonal polynomials associated with the above Poisson measure have a nice recursion formula, i.e. the Jacobi coefficient are constant for $n>1$, see also [SY].*

4.2 The *r*–Fock space and *r*–Gaussian variables

In this section we are going to construct the simplest realization of *r*–Gaussian variables. In that realization we will obtain an interesting class of operator spaces. For another realization see [W].

Let $\mathcal{H}$ be a real, N–dimensional Hilbert space, $N\in\mathbb{N}$ and $\mathcal{H}_{\mathbb{C}}$ its complexification.We define the algebraic free Fock space

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\,\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{\mathbb{C}}^{\otimes n}.$$

We deform the scalar product as follows. For $r\geq 0$ and $x_n, y_n \in \mathcal{H}_{\mathbb{C}}^{\otimes n}$

$$\begin{aligned} \langle x_n\,,y_n\rangle_r &= r^{n-1}\,\langle x_n\,,y_n\rangle \qquad \text{for } n=1,2,3,\dots \\ \langle \Omega\,,\Omega\rangle_r &= \langle \Omega\,,\Omega\rangle = 1. \end{aligned}$$

The completion of $\mathcal{F}(\mathcal{H})$ with respect to the scalar product $\langle\ ,\ \rangle_r$ will be called the *r*–Fock space and denoted $\mathcal{F}_r(\mathcal{H})$. It is an example of one-mode interacting Fock spaces

[AB]. For $f \in \mathcal{H}$ we define the r–creation operator $c^*(f)$ and r–annihilation operator $c(f)$ by

$$
\begin{aligned}
c^*(f)x_1 \otimes \ldots \otimes x_n &= f \otimes x_1 \otimes \ldots \otimes x_n, \\
c(f)x_1 \otimes \ldots \otimes x_n &= r\langle f, x_1 \rangle x_2 \otimes \ldots \otimes x_n, \\
c(f)x_1 &= \langle f, x_1 \rangle \Omega, \\
c(f)\Omega &= 0.
\end{aligned}
$$

Lemma 3 *Let $f,g \in \mathcal{H}$ and let P_Ω be the projection onto the vacuum Ω. Then we have*

$$
c(f)c^*(g) = \langle f, g \rangle (P_\Omega + r(1-P_\Omega)) = \langle f, g \rangle \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & r & 0 & 0 & \ldots & 0 \\ 0 & 0 & r & 0 & \ldots & 0 \\ 0 & 0 & 0 & r & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & r \end{pmatrix}.
$$

Proof: For $f,g \in \mathcal{H}$ and $\xi \in \mathcal{H}_{\mathbb{C}}^{\otimes n}$, $n \geq 1$ we have

$$
c(f)c^*(g)\xi = r\langle f, g \rangle \xi,
$$

and

$$
c(f)c^*(g)\Omega = \langle f, g \rangle \Omega,
$$

hence

$$
c(f)c^*(g) = \langle f, g \rangle (P_\Omega + r(1-P_\Omega)).
$$

□ □

Lemma 4 *Let $f \in \mathcal{H}$, $r \geq 0$. Then we have*

$$
\|c^*(f)\| = \|c(f)\| = \|f\| \max(1, \sqrt{r}).
$$

Proof: By the previous lemma we have

$$
c(f)c^*(f) = \|f\|^2 \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & r & 0 & 0 & \ldots & 0 \\ 0 & 0 & r & 0 & \ldots & 0 \\ 0 & 0 & 0 & r & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & r \end{pmatrix},
$$

hence

$$
\|c(f)c^*(f)\| = \|f\|^2 \left\| \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & r & 0 & 0 & \ldots & 0 \\ 0 & 0 & r & 0 & \ldots & 0 \\ 0 & 0 & 0 & r & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & r \end{pmatrix} \right\| = \|f\|^2 \max(1, r).
$$

The fact that

$$\|c(f)c^*(f)\| = \|c(f)\|^2 = \|c^*(f)\|^2$$

ends the proof. □ □

Lemma 5 *Let $e_1,\dots e_N$ be an orthonormal basis of $\mathcal{H}$, $r \geq 0$. Then*

$$\left\| \sum_{i=1}^{N} c(e_i)\, c^*(e_i) \right\| = \max(1,r).$$

Proof: By the previous lemma we have

$$\sum_{i=1}^{N} c(e_i)\, c^*(e_i) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & r & 0 & 0 & \dots & 0 \\ 0 & 0 & r & 0 & \dots & 0 \\ 0 & 0 & 0 & r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & r \end{pmatrix},$$

hence

$$\left\| \sum_{i=1}^{N} c(e_i)\, c^*(e_i) \right\| = \max(1,r).$$

□ □

For any $f \in \mathcal{H}$ we define the corresponding r–Gaussian variable by $\omega_r(f) = c^*(f) + c(f)$ and for a bounded operator T on $\mathcal{F}_r(\mathcal{H})$ we define the vacuum state $\rho_r(T) = \langle T\Omega, \Omega \rangle$.

Remark 7 *For arbitrary $r \geq 0$ we have*

$$\rho_r(\omega_r(f)^n) = \int x^n d\kappa_r(x),$$

where κ_r is the Kesten measure, as proved in [BW2].

Theorem 3 *Let $e_1,\dots e_N$ be an orthonormal basis of $\mathcal{H}$ and $\omega_r(e_i) = c^*(e_i) + c(e_i)$. For arbitrary matrices $a_i \in M_{n\times n}(\mathbb{C})$ $(i = 1,\dots,N,\ n \in \mathbb{N})$ let*

$$\|(a_1,\dots a_N)\|_{\max} = \max\left(\left\| \sum_{i=1}^{N} a_i a_i^* \right\|^{1/2}, \left\| \sum_{i=1}^{N} a_i^* a_i \right\|^{1/2} \right).$$

Then

1. *For $r > 0$*

$$\left\| \sum_{i=1}^{N} a_i \otimes c^*(e_i) \right\| = \max(1,\sqrt{r}) \left\| \sum_{i=1}^{N} a_i^* a_i \right\|^{1/2}.$$

2. *For $r > 0$*

$$\|(a_1,\dots a_N)\|_{\max} \leq \left\| \sum_{i=1}^{N} a_i \otimes \omega_r(e_i) \right\| \leq 2\max(1,\sqrt{r})\,\|(a_1,\dots a_N)\|_{\max},$$

3. *For $r = 0$*

$$\left\| \sum_{i=1}^{N} a_i \otimes \omega_0(e_i) \right\| = \|(a_1,\dots a_N)\|_{\max}.$$

Proof:

1. Let us take $T = \sum_{i=1}^{N} a_i \otimes c(e_i)$. Then

$$T\,T^* = \sum_{i,j=1}^{N} a_i\,a_j^* \otimes c(e_i)\,c^*(e_j) = \sum_{i=1}^{N} a_i\,a_i^* \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & r & 0 & 0 & \dots & 0 \\ 0 & 0 & r & 0 & \dots & 0 \\ 0 & 0 & 0 & r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & r \end{pmatrix}.$$

Therefore

$$\|T\|^2 = \|T\,T^*\| = \left\| \sum_{i=1}^{N} a_i\,a_i^* \right\| \max(1,r),$$

and hence

$$\|T\| = \max(1,\sqrt{r}) \left\| \sum_{i=1}^{N} a_i\,a_i^* \right\|^{1/2},$$

and by taking adjoints

$$\left\| \sum_{i=1}^{N} a_i^* \otimes c^*(e_i) \right\| = \|T^*\| = \max(1,\sqrt{r}) \left\| \sum_{i=1}^{N} a_i^*\,a_i \right\|^{1/2}$$

which yields the inequality of our assertion.

2. Since for the vacuum state we have

$$\begin{aligned} \rho_r(\omega_r(e_i)\;\omega_r(e_j)) &= \left\langle\, \omega_r(e_i)\;\omega_r(e_j)\;\Omega\,,\Omega \,\right\rangle = \left\langle\, \omega_r(e_j)\;\Omega\,,\,\omega_r(e_i)\;\Omega \,\right\rangle \\ &= \left\langle\, c^*(e_j)\,\Omega\,,\,c^*(e_i)\,\Omega \,\right\rangle = \left\langle\, e_j\,,\,e_i \,\right\rangle = \delta_{i,j} \end{aligned}$$

we obtain

$$\left\|\sum_{i=1}^{N} a_i \otimes \omega_r(e_i)\right\|^2 \geq$$

$$\geq \sup_{\phi-\text{state on } M_n(\mathbb{C})} (\phi \otimes \rho_r)\left[\left(\sum_{i-1}^{N} a_i \otimes \omega_r(e_i)\right)^* \left(\sum_{i-1}^{N} a_i \otimes \omega_r(e_i)\right)\right]$$

$$= \sup_{\phi-\text{state on } M_n(\mathbb{C})} \phi\left(\sum_{i-1}^{N} a_i^* a_i\right) = \left\|\sum_{i=1}^{N} a_i^* a_i\right\|.$$

Therefore we get

$$\left\|\sum_{i=1}^{N} a_i \otimes \omega_r(e_i)\right\| \geq \left\|\sum_{i=1}^{N} a_i^* a_i\right\|^{1/2}.$$

On the other hand

$$\left\|\sum_{i=1}^{N} a_i \otimes \omega_r(e_i)\right\| = \left\|\sum_{i=1}^{N} a_i^* \otimes \omega_r(e_i)\right\| \geq \left\|\sum_{i=1}^{N} a_i a_i^*\right\|^{1/2}$$

and therefore

$$\left\|\sum_{i=1}^{N} a_i \otimes \omega_r(e_i)\right\| \geq \|(a_1, \ldots a_N)\|_{\max},$$

and it yields the left inequality of the assertion. The right inequality follows from the triangle inequality.

3. Now we shall consider the case $r = 0$, i.e. the Boolean case. Since $\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H}$ and

$$c^*(e_i)\,\Omega = e_i, \quad c^*(e_i)\,e_j = 0,$$

therefore for the arbitrary matrices $a_i \in M_{n\times n}(\mathbb{C})$ we have

$$\sum_{i=1}^{N} a_i \otimes c^*(e_i) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ a_1 & 0 & 0 & \ldots & 0 \\ a_2 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_N & 0 & 0 & \ldots & 0 \end{pmatrix},$$

hence

$$\sum_{i=1}^{N} a_i \otimes \omega_0(e_i) = \begin{pmatrix} 0 & a_1^* & a_2^* & \ldots & a_N^* \\ a_1 & 0 & 0 & \ldots & 0 \\ a_2 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_N & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Therefore

$$T\,T^* = \sum_{i,j=1}^{N} a_i\,a_i^* \otimes c\,(e_i)\,c^*(e_i) = \begin{pmatrix} \sum_{i=1}^{N} a_i\,a_i^* & 0 & 0 & \dots & 0 \\ 0 & a_1\,a_1^* & a_1\,a_2^* & \dots & a_1\,a_N^* \\ 0 & a_2\,a_1^* & a_2\,a_N^* & \dots & a_2\,a_N^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_N\,a_1^* & a_N\,a_2^* & \dots & a_N\,a_N^* \end{pmatrix},$$

which implies that

$$\|T\|^2 = \|T\,T^*\| = \max\left(\|A\|,\|B\|\right),$$

where

$$A = \sum_{i=1}^{N} a_i\,a_i^*$$

$$B = \begin{pmatrix} a_1\,a_1^* & a_1\,a_2^* & \dots & a_1\,a_N^* \\ a_2\,a_1^* & a_2\,a_N^* & \dots & a_2\,a_N^* \\ \vdots & \vdots & \ddots & \vdots \\ a_N\,a_1^* & a_N\,a_2^* & \dots & a_N\,a_N^* \end{pmatrix}.$$

Because

$$B = \begin{pmatrix} a_1\,a_1^* & a_1\,a_2^* & \dots & a_1\,a_N^* \\ a_2\,a_1^* & a_2\,a_N^* & \dots & a_2\,a_N^* \\ \vdots & \vdots & \ddots & \vdots \\ a_N\,a_1^* & a_N\,a_2^* & \dots & a_N\,a_N^* \end{pmatrix} = \begin{pmatrix} a_1^* & a_2^* & \dots & a_N^* \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_N & 0 & \dots & 0 \end{pmatrix},$$

we obtain

$$\|B\| = \left\|\sum_{i=1}^{N} a_i^*\,a_i\right\|,$$

which ends the proof.

□ □

Remark 8 *The estimation obtained in (2) is the best possible for* $r \geq 1$.

Problem 2 *Find the best possible estimation for* $0 < r < 1$.

4.3 The N–fold r-free product of states

Now we are going to recall a few facts about the r–free product of states.

Let $\mathcal{A}_i$ be non–unital $*$–algebras with states $\phi_i : \mathcal{A}_i \rightarrow \mathbb{C}$. We consider the unitalization $\tilde{\mathcal{A}}_i$ of $\mathcal{A}_i$, i.e. $\tilde{\mathcal{A}}_i = \mathcal{A}_i \oplus \mathbb{C}\mathbf{1}$ and extend ϕ_i as $\tilde{\phi}_i$

$$\tilde{\phi}_i\,(x + \alpha\mathbf{1}) = \phi_i\,(x) + \alpha.$$

For $0 \leq r \leq 1$ we define a new state $\psi_i = r\tilde{\phi}_i + (1-r)\,\delta_i$, where δ_i is the functional on $\tilde{\mathcal{A}}_i$ defined as

$$\delta_i\,(x + \alpha\mathbf{1}) = \alpha.$$

Then ψ_i is also a state on the unital algebra $\tilde{\mathcal{A}}_i$ and we can form a conditionally free product state $\tilde{\phi} = *\left(\tilde{\phi}_i, \psi_i\right)$ on the free product of algebras $\tilde{\mathcal{A}} = *\tilde{\mathcal{A}}_i$ as

$$\tilde{\phi}\,(\mathbf{1}) = 1$$

and

$$\tilde{\phi}\,(a_1 \cdots a_n) = \tilde{\phi}_{i_1}(a_1) \ldots \tilde{\phi}_{i_n}(a_n),$$

whenever $a_j \in \mathcal{A}_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n$ and $\psi_{i_j}(a_j) = 0$. The functional $\tilde{\phi}$ is a state on $\tilde{\mathcal{A}}$. Hence we also get a state $\phi = \tilde{\phi}|_{\mathcal{A}}$ on the free product of non–unital algebras $\mathcal{A} = *\mathcal{A}_i$.

Definition 3 *We call the state defined as above the r–free product of states and write* $\phi = *_r\phi_i$.

From the construction we have the following properties:

1. $\phi|_{\mathcal{A}_i} = \phi_i$
2. if $a_j \in \mathcal{A}_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n$, then

$$\begin{aligned}\tilde{\phi}[(a_1 - \psi_{i_1}(a_1)\,\mathbf{1})\,(a_2 - \psi_{i_2}(a_2)\,\mathbf{1})\ldots(a_n - \psi_{i_n}(a_{i_n})\,\mathbf{1})] = \\ = \tilde{\phi}[(a_1 - r\phi_{i_1}(a_1)\,\mathbf{1})\,(a_2 - r\phi_{i_2}(a_2)\,\mathbf{1})\ldots(a_n - r\phi_{i_n}(a_n)\,\mathbf{1})] \\ = (1-r)^n\,\phi_{i_1}(a_1)\cdots\phi_{i_n}(a_n),\end{aligned}$$

or equivalently for $n \geq 2$

$$\begin{aligned}\phi\,(a_1a_2\ldots a_n) = r\sum_j \phi\,(a_j)\,\phi\,(a_1\ldots\check{a}_j\ldots a_n) + \\ - r^2\sum_{i<j}\phi\,(a_i)\,\phi\,(a_j)\,\phi\,(a_1\ldots\check{a}_i\ldots\check{a}_j\cdots a_n) + \\ + \ldots + \left((-1)^{n+1}r^n + (1-r)^n\right)\phi\,(a_1)\ldots\phi\,(a_n).\end{aligned} \tag{6}$$

Remark 9 *In the case when* $r = 0$ *we get the regular free (Boolean) product of states. In the case* $r = 1$ *this construction gives the Voiculescu free product of states.*

Remark 10 *Form the formula* (6) *one can see that the r–free product of states has the Voiculescu property: if* $\phi\,(a_j) = 0$ *for all* j *and* $a_j \in \mathcal{A}_{i_j}, i_1 \neq i_2 \neq \ldots \neq i_n$*, then* $\phi\,(a_1 a_2 \ldots a_n) = 0$.

Lemma 6 *For* $a_1, a_2 \in \mathcal{A}_i, b \in \mathcal{A}_j,\ i \neq j$ *one has*

$$\phi\,(a_1ba_2) = (1-r)\,\phi\,(a_1)\,\phi\,(b)\,\phi\,(a_2) + r\phi\,(b)\,\phi\,(a_1a_2).$$

Proof: Taking $a_1, a_2 \in \mathcal{A}_i, b \in \mathcal{A}_j, i \neq j$, we obtain

$$\begin{aligned}\phi(a_1 b a_2) &= \phi(a_1(b - r\phi(b))a_2) + r\phi(b)\phi(a_1 a_2)\\ &= \phi(a_1)\phi(b - r\phi(b))\phi(a_2) + r\phi(b)\phi(a_1 a_2)\\ &= \phi(a_1)\phi(b)\phi(a_2) - r\phi(a_1)\phi(b)\phi(a_2) + r\phi(b)\phi(a_1 a_2)\\ &= (1-r)\phi(a_1)\phi(b)\phi(a_2) + r\phi(b)\phi(a_1 a_2),\end{aligned}$$

hence

$$\phi(a_1 b a_2) = (1-r)\phi(a_1)\phi(b)\phi(a_2) + r\phi(b)\phi(a_1 a_2).$$

□ □

Example 1 *Let $G_i = \mathbb{Z}$ be the additive group of integers and let $\phi_i : \mathbb{C}[G_i] \to \mathbb{C}$, $\phi_i \equiv 1$ for $i = 1, \ldots, N$. In this case*

$$*\mathbb{C}[G_i] = \mathbb{C}[\mathbb{F}_N] \text{ and } \phi = *_r\phi_i \equiv 1.$$

Indeed, let $a_1 \in \mathbb{C}[G_{i_1}], b_2 \in \mathbb{C}[G_{i_2}], c_3 \in \mathbb{C}[G_{i_3}]$ and $i_1 \neq i_2 \neq i_3$. If $i_1 = i_3$ by Lemma 6 we have

$$\phi(a_1 a_2 a_3) = (1-r) + r = 1$$

and for $i_1 \neq i_3$ we obtain $\phi(a_1 a_2 a_3) = 1$ by equation (6)*. By induction, since*

$$\#\{(i,j), 1 \le i < j \le n\} = \binom{n}{2}$$
$$\#\{(i,j,k), 1 \le i < j < k \le n\} = \binom{n}{3} \ldots$$

by equation (6) *we obtain*

$$\begin{aligned}\phi(a_1 a_2 \ldots a_n) &= r\sum_j \phi(a_j)\phi(a_1 \ldots \check{a}_j \ldots a_n) +\\ &\quad - r^2 \sum_{i<j} \phi(a_i)\phi(a_j)\phi(a_1 \ldots \check{a}_i \ldots \check{a}_j \cdots a_n) +\\ &\quad + \ldots + \left((-1)^{n+1} r^n + (1-r)^n\right)\phi(a_1)\ldots\phi(a_n)\\ &= r\sum_j 1 - r^2 \sum_{i<j} 1 + \ldots + (-1)^{n+1} r^n + (1-r)^n\\ &= r\,n - r^2\binom{n}{2} + \ldots + (-1)^{n+1} r^n + (1-r)^n\\ &= 1 - (1-r)^n + (1-r)^n = 1.\end{aligned}$$

5 Iterated case

5.1 Iterated *r*–free convolutions

Theorem 4 *The iterated* $\boxed{r}$ *convolution is associative if and only if* $r=0$ *or* $r=1$.

Proof: Below we calculate the first four moments of the measure $\mu\boxed{r}\nu$:

$$\begin{aligned}
m_{\mu\boxed{r}\nu}(2) &= 2\,m_\mu(1)m_\nu(1)+m_\mu(2)+m_\nu(2),\\
m_{\mu\boxed{r}\nu}(3) &= (1-r)\left[m_\mu(1)\,m_\nu(1)^2+m_\mu(1)^2\,m_\nu(1)\right]+\\
&\quad+(2+r)\left[m_\mu(1)\,m_\nu(2)+m_\mu(2)\,m_\nu(1)\right]+\\
&\quad+m_\mu(3)+m_\nu(3),\\
m_{\mu\boxed{r}\nu}(4) &= (2-2r)\left[m_\mu(1)\,m_\mu(2)\,m_\nu(1)+m_\mu(1)\,m_\nu(1)\,m_\nu(2)\right]+\\
&\quad+(2+2r)\left[m_\mu(1)\,m_\nu(3)+m_\mu(2)\,m_\nu(2)+m_\mu(3)\,m_\nu(1)\right]+\\
&\quad+(2-4r)\,m_\mu(1)^2\,m_\nu(1)^2+\\
&\quad+(1+r)\left[m_\mu(1)^2\,m_\nu(2)+m_\mu(2)\,m_\nu(1)^2\right]+\\
&\quad+m_\mu(4)+m_\nu(4).
\end{aligned}$$

Using the above relations we can calculate the fourth moment of the measure $\mu\,\overrightarrow{\boxed{r}}\,\nu\,\overrightarrow{\boxed{r}}\,\omega=(\mu\boxed{r}\nu)\boxed{r}\omega$:

$$\begin{aligned}
m_{(\mu\boxed{r}\nu)\boxed{r}\omega}(4) &= (2-2r)\left[\,m_\mu(1)\,m_\mu(2)\,m_\nu(1)+m_\mu(1)\,m_\mu(2)\,m_\omega(1)+\right.\\
&\qquad+\;m_\mu(1)\,m_\nu(1)\,m_\nu(2)+m_\mu(1)\,m_\omega(1)\,m_\omega(2)+\\
&\qquad\left.+\;m_\nu(1)\,m_\nu(2)\,m_\omega(1)+m_\nu(1)\,m_\omega(1)\,m_\omega(2)\right]+\\
&\quad+(6-6r)\,m_\mu(1)\,m_\nu(1)\,m_\omega(1)^2+\\
&\quad+(6+6r)\,m_\mu(1)\,m_\nu(1)\,m_\omega(2)+\\
&\quad+\left(6-4r-2r^2\right)\left[m_\mu(1)\,m_\nu(1)^2\,m_\omega(1)+m_\mu(1)^2\,m_\nu(1)\,m_\omega(1)\right]+\\
&\quad+\left(6+4r+2r^2\right)\left[m_\mu(1)\,m_\nu(2)\,m_\omega(1)+m_\mu(2)\,m_\nu(1)\,m_\omega(1)\right]+\\
&\quad+(2+2r)\left[m_\mu(1)\,m_\nu(3)+m_\mu(1)\,m_\omega(3)+m_\mu(2)\,m_\nu(2)+\right.\\
&\qquad+\;m_\mu(2)\,m_\omega(2)+m_\mu(3)\,m_\nu(1)+m_\mu(3)\,m_\omega(1)+\\
&\qquad\left.+\;m_\nu(1)\,m_\omega(3)+m_\nu(2)\,m_\omega(2)+m_\nu(3)\,m_\omega(1)\right]+\\
&\quad+(2-4r)\left[m_\mu(1)^2\,m_\nu(1)^2+m_\mu(1)^2\,m_\omega(1)^2+m_\nu(1)^2\,m_\omega(1)^2\right]+\\
&\quad+(1+r)\left[m_\mu(1)^2\,m_\nu(2)+m_\mu(1)^2\,m_\omega(2)+m_\mu(2)\,m_\nu(1)^2+\right.\\
&\qquad\left.+\;m_\mu(2)\,m_\omega(1)^2+m_\nu(1)^2\,m_\omega(2)+m_\nu(2)\,m_\omega(1)^2\right]+\\
&\quad+m_\mu(4)+m_\nu(4)+m_\omega(4).
\end{aligned}$$

Similarly one can get the moment $m_{\mu\boxed{r}(\nu\boxed{r}\omega)}(4)$. The difference between the two is equal to

$$\begin{aligned} m_{(\mu \boxed{r} \nu) \boxed{r} \omega}(4) - m_{\mu \boxed{r} (\nu \boxed{r} \omega)}(4) = \\ = 2\,r\,(1-r)\,m_\nu(1)\left[m_\mu(1)^2\,m_\omega(1) - m_\mu(1)\,m_\omega(1)^2 + \right. \\ \left. - m_\mu(2)\,m_\omega(1) + m_\mu(1)\,m_\omega(2)\right], \end{aligned}$$

and is not zero in general, unless $r = 0$ or $r = 1$. □ □

Remark 11 *The fact that the iterated r–free convolutions are not associative implies that no linearizing cumulants can exist for them.*

The above remark means that it is very difficult to investigate convolutions of long sequences of probability distributions and their limits, as it is required for limit theorems. Indeed, we have been unable to calculate the corresponding central limit theorem measure or the Poisson limit measure.

5.2 Iterated r-free product of states

From Lemma 6 we obtain the following

Theorem 5 *Unless $r = 0, 1$, the r–free product of states is not associative.*

Proof: Taking $a_1, a_2 \in \mathcal{A}_i, b_1, b_2 \in \mathcal{A}_j, c \in \mathcal{A}_k, i \neq j, i \neq k, j \neq k$, we get

$$a_1\,(b_1 c b_2)\,a_2 = (a_1 b_1)\,c\,(b_2 a_2)$$

and therefore, by Lemma 6

$$\begin{aligned} \phi\,(a_1\,(b_1 c b_2)\,a_2) &= (1-r)\,\phi\,(a_1)\,\phi\,(b_1 c b_2)\,\phi\,(a_2) + r\phi\,(b_1 c b_2)\,\phi\,(a_1 a_2) \\ &= (1-r)\,\phi\,(a_1)\left[(1-r)\,\phi\,(b_1)\,\phi\,(c)\,\phi\,(b_2) + \right. \\ &\qquad \left. + r\phi\,(c)\,\phi\,(b_1 b_2)\right]\phi\,(a_2) + \\ &\quad + r\left[(1-r)\,\phi\,(b_1)\,\phi\,(c)\,\phi\,(b_2) + r\phi\,(c)\,\phi\,(b_1 b_2)\right]\phi\,(a_1 a_2) \\ &= (1-r)^2\,\phi\,(a_1)\,\phi\,(a_2)\,\phi\,(b_1)\,\phi\,(b_2)\,\phi\,(c) + \\ &\quad + r\,(1-r)\,\phi\,(a_1)\,\phi\,(a_2)\,\phi\,(b_1 b_2)\,\phi\,(c) + \\ &\quad + r\,(1-r)\,\phi\,(a_1 a_2)\,\phi\,(b_1)\,\phi\,(b_2)\,\phi\,(c) + \\ &\quad + r^2\phi\,(a_1 a_2)\,\phi\,(b_1 b_2)\,\phi\,(c)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} \phi\,((a_1 b_1)\,c\,(b_2 a_2)) &= (1-r)\,\phi\,(a_1 b_1)\,\phi\,(c)\,\phi\,(b_2 a_2) + r\phi\,(c)\,\phi\,(a_1 b_1 b_2 a_2) \\ &= (1-r)\,\phi\,(a_1)\,\phi\,(a_2)\,\phi\,(b_1)\,\phi\,(b_2)\,\phi\,(c) + \\ &\quad + r\,(1-r)\,\phi\,(a_1)\,\phi\,(a_2)\,\phi\,(b_1 b_2)\,\phi\,(c) + \\ &\quad + r^2\phi\,(a_1 a_2)\,\phi\,(b_1 b_2)\,\phi\,(c)\,. \end{aligned}$$

Hence we the following equation must be satisfied

$$
\begin{aligned}
&(1-r)^2\phi(a_1)\phi(a_2)\phi(b_1)\phi(b_2)\phi(c)+r(1-r)\phi(a_1)\phi(a_2)\phi(b_1b_2)\phi(c)\\
&\quad+r(1-r)\phi(a_1a_2)\phi(b_1)\phi(b_2)\phi(c)+r^2\phi(a_1a_2)\phi(b_1b_2)\phi(c)=\\
&=(1-r)\phi(a_1)\phi(a_2)\phi(b_1)\phi(b_2)\phi(c)+r(1-r)\phi(a_1)\phi(a_2)\phi(b_1b_2)\phi(c)\\
&\quad+r^2\phi(a_1a_2)\phi(b_1b_2)\phi(c),
\end{aligned}
$$

and we get a condition

$$
\begin{aligned}
(1-r)^2\phi(a_1)\phi(a_2)\phi(b_1)\phi(b_2)\phi(c)+r(1-r)\phi(a_1a_2)\phi(b_1)\phi(b_2)\phi(c)=\\
=(1-r)\phi(a_1)\phi(a_2)\phi(b_1)\phi(b_2)\phi(c)
\end{aligned}
$$

which gives

$$(1-r)^2=1-r, \qquad r(1-r)=0$$

hence $r=0$, i.e. the product reduces to the boolean one, or $r=1$, i.e. the product reduces to the free one. □ □

6 r^*–convolution

The r^*–convolution $\boxed{r*}$ is defined through the moment–cumulant formula

$$m_{\mu_i}(n)=\sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} r^{uc(\pi)}\prod_{j=1}^{k}R^*_{\mu_i}(|\pi_j|), \tag{7}$$

where $uc(\pi)=\sum_{B\in\pi}uc(B)$ is a suitable function on the set of non–crossing partition. The cumulants of the convolution measure $\mu_1\boxed{r*}\mu_2$ are defined by

$$R^*_{\mu_1\boxed{r*}\mu_2}(n)=R^*_{\mu_1}(n)+R^*_{\mu_2}(n),$$

and its moments are recovered by applying again the relation (7). It clearly is associative.

In [KY1], [Y2] the authors give an explicit formula of the function $uc(\pi)$ Let π be a non-crossing partition. For a block $B=\{b_1,b_2,\dots,b_{|B|}\}$ with $b_1<b_2<\dots<b_{|B|}$, define $uc(B)=\#\{b_i\in B: i>1 \text{ and } b_i\neq b_{i-1}+1\}$. For a non–crossing partition π define $uc(\pi)=\sum_{B\in\pi}uc(B)$.

Lemma 7 *For the function $uc(\pi)$ and appropriate cumulants $R^*(k)$*

$$m_{\mu_i}(n)=\sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} r^{uc(\pi)}\prod_{j=1}^{k}R^*_{\mu_i}(|\pi_j|)=\sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} r^{uc(\pi)}\prod_{j=1}^{k}R^{\boxdot}_{\mu_i,V_r\mu_i}(|\pi_j|).$$

Proof: Let us start by observing the relation between moments of a measure μ_i and its r–deformation $V_r\,\mu_i$:

$$m_{V_r\,\mu}(k) = \begin{cases} r\,m_\mu(k) & k \geq 1, \\ 1 & k = 0. \end{cases}$$

From the equation (5) one has

$$\begin{aligned}
m_{\mu_i}(n) &= \sum_{k=1}^{n} \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ l_1+\ldots+l_k=n-k}} R^{\boxed{c}}_{\mu_i, V_r\,\mu_i}(k) m_{V_r\,\mu_i}(l_1)\ldots m_{V_r\,\mu_i}(l_{k-1}) m_{\mu_i}(l_k) \\
&= \sum_{k=1}^{n} \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ l_1+\ldots+l_k=n-k}} r^{\#\{l_j \neq 0, j=1,\cdots,k-1\}} R^{\boxed{c}}_{\mu_i, V_r\,\mu_i}(k) m_{\mu_i}(l_1)\ldots m_{\mu_i}(l_{k-1}) m_{\mu_i}(l_k) \\
&= \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi=(\pi_1,\ldots,\pi_k)}} r^{uc(\pi)} \prod_{j=1}^{k} R^*_{\mu_i}(|\pi_j|),
\end{aligned}$$

for $R^*_{\mu_i}(k) = R^{\boxed{c}}_{\mu_i, V_r\,\mu_i}(k)$. □ □

Remark 12 *The moment–cumulant formula* (7) *has the following analytic counterpart*

$$\frac{1}{G_\mu(z)} = z - R^*_\mu\left(r\,G_\mu(z) + \frac{1-r}{z}\right).$$

Remark 13 *For the pair $(\mu, V_r\,\mu)$ the above relation is equivalent to the relation between the Cauchy transforms and the conditional cumulant transform*

$$\frac{1}{G_\mu(z)} = z - R^{\boxed{c}}_{(\mu, V_r\,\mu)}\left(G_{V_r\,\mu}(z)\right), \tag{8}$$

$$\frac{1}{G_{V_r\,\mu}(z)} = z - R^{\boxplus}_{V_r\,\mu}\left(G_{V_r\,\mu}(z)\right), \tag{9}$$

but for the pair $(\mu \boxed{r*} \nu, V_r\,\mu \boxplus V_r\,\nu)$ as well as $(\mu \boxed{r} \nu, V_r\,\mu \boxplus V_r\,\nu)$ such connections are no longer valid.

Remark 14 *The fact that the $R^*_\mu(n)$ cumulants do not arise as conditionally free cumulants was first observed by Lehner [Leh].*

We would like to prove now that the r^*–convolution can produce signed measures. A sequence of a real numbers $\{m_n\}$ is the moment sequence of some measure μ if and only if $M_n(\mu) \geq 0$ for all $n \in \mathbb{N}$, where

$$M_n(\mu) = \left| \begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix} \right|,$$

see [C], [S].

Theorem 6 *The r^*–convolution of arbitrary probability measure is positive if and only if $r=0$ or $r=1$.*

Proof: First we will calculate a few moments of the measure $\mu \boxed{\scriptstyle r*} \nu$. We have

$$\begin{aligned}
m_{\mu \boxed{\scriptstyle r*} \nu}(1) &= m_\mu(1) + m_\nu(1)\\
m_{\mu \boxed{\scriptstyle r*} \nu}(2) &= 2m_\mu(1)\, m_\nu(1) + m_\mu(2) + m_\nu(2)\\
m_{\mu \boxed{\scriptstyle r*} \nu}(3) &= (1-r)\, m_\mu(1)\, m_\nu(1)\left[m_\nu(1) + m_\mu(1)\right] +\\
&\quad + (2+r)\left[m_\mu(1)\, m_\nu(2) + m_\mu(2)\, m_\nu(1)\right] +\\
&\quad + m_\mu(3) + m_\nu(3)\\
m_{\mu \boxed{\scriptstyle r*} \nu}(4) &= 2\left(1-r^2\right)\left[m_\mu(1)\, m_\mu(2)\, m_\nu(1) + m_\mu(1)\, m_\nu(1)\, m_\nu(2)\right] +\\
&\quad - 2r(1-r)\left[m_\mu(1)\, m_\nu(1)^3 + m_\mu(1)^3\, m_\nu(1)\right] +\\
&\quad + 2(1+r)\left[m_\mu(1)\, m_\nu(3) + m_\mu(2)\, m_\nu(2) + m_\mu(3)\, m_\nu(1)\right] +\\
&\quad + (2-4r)\, m_\mu(1)^2\, m_\nu(1)^2 +\\
&\quad + (1+r)\left[m_\mu(1)^2\, m_\nu(2) + m_\mu(2)\, m_\nu(1)^2\right] +\\
&\quad + m_\mu(4) + m_\nu(4).
\end{aligned}$$

With the above moments we can produce a counterexample easy to calculate: we take the measures $\mu = \frac{1}{2}\left(\delta_0 + \delta_2\right)$, $\nu = \frac{1}{2}\left(\delta_0 + \delta_{-\frac{1}{2}}\right)$ and we obtain

$$M_2(\mu) = \begin{vmatrix} 1 & \frac{3}{4} & \frac{13}{8} \\ \frac{3}{4} & \frac{13}{8} & 3 - \frac{3r}{16} \\ \frac{13}{8} & 3 - \frac{3r}{16} & \frac{183-35r+17r^2}{32} \end{vmatrix} = \frac{25}{256} - \frac{253r}{512} + \frac{271r^2}{512} < 0$$

for $r \in \left[\frac{253-\sqrt{9809}}{542}, \frac{253-\sqrt{9809}}{542}\right] \subset [0,1]$. The general case follows for instance by taking the sequence of measures $\mu = \frac{1}{2}\left(\delta_0 + \delta_{\frac{2}{r+1}}\right)$, $\nu_n = \frac{1}{2}\left(\delta_0 + \delta_{-\frac{r}{2^n}}\right)$. When $n \to \infty$ we obtain $M_2(\mu \boxed{\scriptstyle r*} \nu_n) < 0$ for all $r \in (0,1)$ hence the result is not a positive measure any longer. □ □

A diagram of $M_2(\mu \boxed{\scriptstyle r*} \nu_5)$ is presented on the following figure.

The central limit theorem for this convolution was proven in [B1] – the central limit measure is measure μ_r, the Cauchy transform of which has the following continued fraction form

$$G_{\mu_r}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{r}{z - \cfrac{1}{z - \cfrac{r}{z - \ddots}}}}}.$$

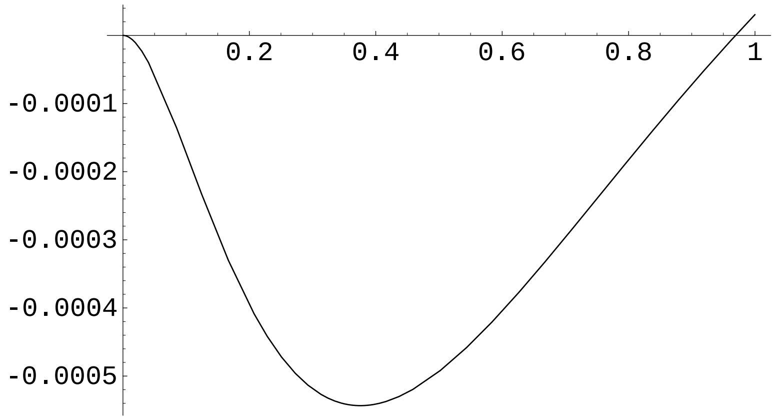

Figure 1: M_2 for the r^*–convolution of μ and ν_5.

Remark 15 *The measure μ_r with periodic continued fraction can be obtained as the central limit measure by considering interacting Fock spaces, see [AB], in the case when even parameters are equal to r and odd equal to 1.*

Remark 16 *For a rational r the measure μ_r can also be obtained by the construction of regular graphs of Obata, see [HO], [Ob].*

The Poisson measure P^r_λ for the $\boxed{r*}$–convolution was given in the paper [KY1]. It has the following form

$$P^r_\lambda(x) \sim c_\lambda(r)\delta_0 + \frac{1}{2\pi r\,|x|}\sqrt{4\lambda r - (x-(\lambda+1))^2}\,\chi_{J_{r,\lambda}}(x)dx,$$

where

$$c_\lambda(r) = 1 - \frac{1+\lambda+\sqrt{(\lambda+1)^2-4\lambda r}}{2r},$$
$$J_{r,\lambda}(x) = [\lambda+1-2\sqrt{\lambda r}, \lambda+1+2\sqrt{\lambda r}].$$

It is probabilistic measure if and only if either $r \leq 1$ or $r > 1$ and

$$\lambda \in [0, 2r-1-\sqrt{r^2-r}] \cup [2r-1+\sqrt{r^2-r}, \infty].$$

7 Δ–convolution

In [B1] Bożejko also introduced a generalization of r–deformation, called the Δ–deformation. We are interested in deformations V_Δ such that the moments of the measure $m_{V_\Delta \mu}(k)$ are of the form $m_{V_\Delta \mu}(0) = 1$ and for $k \geq 1$

$$m_{V_\Delta \mu}(k) = \Delta_k\, m_\mu(k),$$

where $\Delta_k = \int_{\mathbb{R}} x^k d\omega(x)$ are moments of some probability measure. In such a case formula (5) is equivalent to

$$m_{\mu_i}(n) = \sum_{k=1}^{n} \sum_{\substack{l_1,\dots,l_k \geq 1 \\ l_1+\dots+l_k = n-k}} R^{\boxdot}_{\mu_i, V_\Delta \mu_i}(k) m_{\mu_i}(l_1)\dots m_{\mu_i}(l_{k-1}) m_{\mu_i}(l_k) \prod_{j=1}^{k-1} \Delta_{l_j}$$
$$= \sum_{k=1}^{n} \sum_{\substack{l_1,\dots,l_k \geq 1 \\ l_1+\dots+l_k = n-k}} R^{\Delta}_{\mu_i}(k) m_{\mu_i}(l_1)\dots m_{\mu_i}(l_{k-1}) m_{\mu_i}(l_k) \prod_{j=1}^{k-1} \Delta_{l_j}.$$

Definition 4 *We define the Δ–convolution by the formula*

$$\mu_1 \boxslash \mu_2 = (\mu_1, V_\Delta \mu_1) \boxdot (\mu_2, V_\Delta \mu_2).$$

That convolution and related moment-cumulant formulae were considered in [Y2]. A special case when $\Delta_n = s^n$, i.e. when $\omega = \delta_s$ which means that $V_\Delta \mu = D_s \mu$ is a dilation of the measure μ, was also discussed in [Y1]. To be precise, the moment–cumulant formula is in the following form

$$m_{\mu_i}(n) = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \pi = (\pi_1,\dots,\pi_k)}} \mathrm{wt}(\pi) \prod_{j=1}^{k} R^{\Delta}_{\mu_i}(|\pi_j|), \tag{10}$$

where $\mathrm{wt}(\pi) = \prod_{B \in \pi} \mathrm{wt}(B)$ and for the block $B = \{v_1, v_2, \dots, v_{|B|}\}$

$$\mathrm{wt}(B) = \begin{cases} \prod_{i=1}^{|B|-1} \Delta_{v_{i+1}-v_i-1} & \text{if} \quad |B| \geq 2, \\ 1 & \text{if} \quad |B| = 1. \end{cases}$$

However, similarly as in the case of the r–convolution we have the following:

Lemma 8 *If for arbitrary measures μ_1, μ_2*

$$V_\Delta(\mu_1 \boxslash \mu_2) = V_\Delta(\mu_1) \boxplus V_\Delta(\mu_2),$$

then $\Delta = (\Delta_n)$ with $\Delta_n = a^n$ and $a = 0$ or $a = 1$.

Proof: As in the r–case, it is sufficient to show that for some probability measure μ and for some natural number n we have

$$m_{V_\Delta(\mu \boxslash \mu)}(n) \neq m_{V_\Delta(\mu) \boxplus V_\Delta(\mu)}(n).$$

Assume the measure μ has nonzero expectancy. We have

$$m_{V_\Delta(\mu \boxslash \mu)}(2) - m_{V_\Delta(\mu) \boxplus V_\Delta(\mu)}(2) = 2 m_\mu(1)^2 \left(-\Delta_1^2 + \Delta_2\right)$$

and we can see that $m_{V_\Delta(\mu \boxslash \mu)}(2) = m_{V_\Delta(\mu) \boxplus V_\Delta(\mu)}(2)$ if and only if $-\Delta_1^2 + \Delta_2 = 0$. Because the sequence Δ_n is a sequence of moments of some probability measure ω, the condition $-\Delta_1^2 + \Delta_2 = 0$ implies, that $\omega = \delta_a$ for some a and $\Delta_k = a^k$. But in this case

$$m_{V_\Delta(\mu \boxslash \mu)}(3) - m_{V_\Delta(\mu) \boxplus V_\Delta(\mu)}(3) = 2 a^3 (1-a) m_\mu(1) \left(m_\mu(1)^2 - m_\mu(2)\right),$$

which is nonzero unless $a = 0$ or $a = 1$ and the proof is complete. □ □ Moreover,

Theorem 7 *The iterated $\boxed{\Delta}$ convolution is associative only for $\Delta = (\Delta_n)$ with $\Delta_n = a^n$ and $a = 0$ or $a = 1$, i.e. for boolean and free convolution.*

Proof: As in the r–case, it is enough to show, that $m_{(\mu \boxed{\Delta} \nu) \boxed{\Delta} \omega}(4)$ and $m_{\mu \boxed{\Delta} (\nu \boxed{\Delta} \omega)}(4)$ are different. The difference between the two is equal to

$$
\begin{aligned}
m_{(\mu \boxed{\Delta} \nu) \boxed{\Delta} \omega}(4) - m_{\mu \boxed{\Delta} (\nu \boxed{\Delta} \omega)}(4) = \\
2\left(\Delta_2 - \Delta_1^2\right) m_\nu(1) \Big[m_\mu(1)^2\, m_\omega(1) - m_\mu(1)\, m_\omega(1)^2 + \\
+ m_\mu(1)\, m_\omega(2) - m_\mu(2)\, m_\omega(1) \Big],
\end{aligned}
$$

and is nonzero in general, unless $\Delta_2 = \Delta_1^2$. But in that case $\Delta_k = a^k$ for some real a we have

$$
\begin{aligned}
m_{(\mu \boxed{\Delta} \nu) \boxed{\Delta} \omega}(5) - m_{\mu \boxed{\Delta} (\nu \boxed{\Delta} \omega)}(5) = \\
(1-a)\, a^3 \left(m_\nu(1)^2 - m_\nu(2) \right) \Big[m_\omega(1) \left(m_\mu(1)^2 - m_\mu(2) \right) + \\
- m_\mu(1) \left(m_\omega(1)^2 - m_\omega(2) \right) \Big],
\end{aligned}
$$

which is nonzero unless $a = 0$ or $a = 1$. □ □

Problem 3 *We can define another convolution $\mu_1 \boxed{\Delta *} \mu_2$ by the requirement*

$$R^{\Delta}_{\mu_1 \boxed{\Delta *} \mu_2}(n) = R^{\Delta}_{\mu_1}(n) + R^{\Delta}_{\mu_2}(n).$$

The $\boxed{r*}$ is an example of this convolution for a specific choice of the sequence Δ. We have seen that in this special case the convolution produces probability measures only when it reduces to the free or boolean convolution. We would like to know if any other choice of Δ would always give probability measures. However, we have not been able to come to a result and leave it as an open question.

References

[AB] Accardi, L., Bożejko, M.: Interacting Fock spaces and gaussianization of probability measures. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1**(4), 663–670 (1998)

[A] Avitzour, D.: Free product of C*–algebras. Trans. Am. Math. Soc. **271**, 423–435 (1982)

[BGS] Ben Ghorbal, A., Schürmann, M.: Noncommutative notions of stochastic independence. Math. Proc. Camb. Philos. Soc. **133**, 531–561 (2002)

[B1] Bożejko, M.: Deformed Free Probability of Voiculescu. RIMS Kokyuroku **1227**, 96–113 (2001)

[B2] Bożejko, M.: Positive definite functions on free group and noncommutative Riesz product. Boll. Unione Mat. Ital. Serie VI **5 A**(1), 13–21 (1986)

[B3] Bożejko, M.: Uniformly bounded representation of free groups. J. Reine Angew. Math. **377**, 170–186 (1987)

[BB] Bożejko, M., Bryc, W.: On a class of free Lévy laws related to a regression problem. preprint, arXiv:math.PR/0404241 (2004)

[BLS] Bożejko, M., Leinert, M., Speicher, R.: Convolution and limit theorems for conditinally free random variables. Pac. J. Math. **175**(2), 357–388 (1996)

[BS] Bożejko, M., Speicher, R.: Ψ–independent and symmetrized white noises. In: Accardi, L. (ed.) Quantum Probability and Related Topics **VI**, 219–236. World Scientific, Singapore, (1991)

[BW1] Bożejko, M., Wysoczański, J.: New examples of convolutions and non–commutative central limit theorems. Banach Cent. Publ. **43**, 95–103 (1998)

[BW2] Bożejko, M., Wysoczański, J.: Remarks on t-transformations of measures and convolutions. Ann. Inst. Henri Poincare Probab. Stat. **37**(6), 737–761 (2001)

[B] Buchholz, A.: Operator Khinchine inequality in noncommutative probability. Math. Ann. **319**, 1–16 (2001)

[C] Chihara, T. S.: An Introduction to Orthogonal Polynomials. Mathematics and its Applications 13. Gordon and Breach, New York (1978)

[CO] Cuculescu, I., Oprea, A. G.: Noncommutative Probability. Kluwer Academic Publishers, Dordrecht (1994)

[F] Franz, U.: Unification of boolean, monotone, anti-monotone, and tensor independence and Lévy processes. Math. Z. **243**, 779–816 (2003)

[HP] Haagerup, U., Pisier, G.: Bounded linear operator between C^*–algebras. Duke Math. J. **71**, 889–925 (1993)

[HO] Hora, A., Obata, N.: An interacting Fock space with periodic Jacobi parameter obtained from regular graphs in large scale limit. to appear in Quantum Information V

[KW] Krystek, A. D., Wojakowski, Ł. J.: Associative convolutions arising from conditionally free convolution. preprint (2004)

[KY1] Krystek, A. D., Yoshida, H.: The combinatorics of the r–free convolution. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6**(4), 619–628 (2003)

[KY2] Krystek, A. D., Yoshida, H.: Generalized t-transformatons of probability measures and deformed convolution. Probab. Math. Stat. **24**(1), 97–119 (2004)

[Leh] Lehner, F.: Cumulants in noncommutative probability theory I. Math. Z. **248**, 67–100 (2004)

[Len] Lenczewski, R.: Unification of independence in quantum probability. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1**, 384–405 (1998)

[M] Młotkowski, W.: Operator–valued version of conditionally free product. Stud. Math. **153**(1), 13–29 2002

[Ob] Obata, N.: Quantum Probabilistic Approach to Spectral Analysis of Star Graphs. Interdiscip. Inf. Sci. **10**(1), 41–52 2004

[Or] Oravecz, F.: Pure convolutions arising from conditionally free convolution. preprint (2004)

[SY] Saitoh, N., Yoshida, H.: The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory. *Probab. Math. Stat.* **21**(1), 159–170 (2001)

[S] Szegö, G.: Orthogonal Polynomials. AMS Colloquium Publications, **23**, 4th ed., Providence, R. I. (1975)

[VDN] Voiculescu, D., Dykema, K., Nica, A.: Free Random Variables. CRM Monograph Series **1**. AMS, Providence, R. I. (1992)

[W] Wojakowski, Ł. J.: Probability Interpolating between Free and Boolean. Ph.D. Thesis. University of Wrocław (2004)

[Y1] Yoshida, H.: Remarks on the *s*–free convolution. In: Obata, N., Matsui, T., Hora, A. (ed.) Non-commutativity, Infinite Dimensionality and Probability at the Crossroads. QP-PQ Quantum Probability and White Noise Analysis **XVI**, 412–433. World Scientific, Singapore (2001)

[Y2] Yoshida, H.: The weight function on non-crossing partitions for the Δ–convolution. Math. Z. **245**, 105–121 (2003)