Chapter 1

SHIFT INVARIANT SPACES AND WAVELETS

1.1 Introduction

A function $\psi \in L^2(\mathbb{R})$ is called an *orthonormal wavelet* if the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k) \qquad \text{for} \quad j,k \in \mathbb{Z}.$$

The system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is dilation invariant. By this we mean, that the system is invariant under integer powers of the standard "dilation by 2" operator D defined by

$$Df(x) = \sqrt{2}f(2x)$$

for $f \in L^2(\mathbb{R})$ (note, that D is unitary). Thus, the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ can be viewed as the dilates of the family $\mathcal{A} = \{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$. However, this point of view based on dilations is not very popular. Instead, more emphasis is put on translations. The translation (or shift) operator T defined by

$$Tf(x) = f(x-1)$$

for $f \in L^2(\mathbb{R})$ is unitary as well and has an excellent representation via the Fourier transform. The Fourier transform $\widehat{\ }$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx$$

for $f \in L^2(\mathbb{R})$, and the translation of such a function f can be represented as

$$\widehat{(Tf)}(\xi) = e^{-2\pi i x\xi}\hat{f}(\xi).$$

Although the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is not translation invariant the theory of shift invariant spaces (that is, spaces invariant under these translations) can be used for better understanding of wavelets. The reason for this is the following. For every $j \in \mathbb{Z}$ we can define a space

$$W_j = \overline{\operatorname{span}\{\psi_{j,k}\}_{k\in\mathbb{Z}}},$$

which is shift invariant for $j \geq 0$. If ψ is a wavelet then

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j, \tag{1.1.1}$$

therefore, the space

$$V = \bigoplus_{j<0} W_j$$

is also shift invariant (as the orthogonal complement of a shift invariant space $\bigoplus_{j\geq 0} W_j$). Thus, the decomposition (1.1.1) takes the form

$$L^2(\mathbb{R}) = V \oplus \bigoplus_{j\geq 0} W_j, \tag{1.1.2),}$$

that is, the space $L^2(\mathbb{R})$ is an orthogonal sum of shift invariant spaces, which are defined by our wavelet ψ. That is why studying shift invariant spaces is essential for understanding wavelets. In particular, the decomposition (1.1.2) leads to the discrete Calderón reproducing formula, which we shall prove in §1.5.

1.2 Shift invariant spaces

The purpose of this paragraph is to understand the structure of shift invariant subspaces of $L^2(\mathbb{R})$. We begin with their definition:

DEFINITION 1.2.1. *A closed subspace V of $L^2(\mathbb{R})$ is called shift invariant if for every function $f \in V$ we also have $f(\cdot - k) \in V$ when $k \in \mathbb{Z}$.*

For a better understanding of shift invariant spaces we will draw an analogy from Euclidean vector spaces. Every Euclidean space has a set of generators called a basis. The space can be decomposed into a sum of one–dimensional subspaces generated by the elements of the basis. This decomposition is not unique and, in particular, we can find decompositions in which the one–dimensional subspaces are mutually orthogonal. In our case the role of generators will be played by functions in $L^2(\mathbb{R})$ and shift invariant spaces generated by a single function will be the analogs of one–dimensional subspaces.

DEFINITION 1.2.2. *Let φ belong to $L^2(\mathbb{R})$, then the space*

$$V_\varphi = \overline{\operatorname{span}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}}$$

is called a principal shift invariant space.

It is easy to see that any shift invariant space can be decomposed into an orthogonal sum of principal shift invariant spaces:

THEOREM 1.2.3. *If V is a shift invariant space, then there exists a countable set of functions $\{\varphi_n\}_{n=1}^N$, belonging to V such that*

$$V = \bigoplus_{n=1}^{N} V_{\varphi_n}, \tag{1.2.1}$$

where N is a natural number or infinity.

PROOF. The proof is straight forward: select $\varphi_1 \in V$ and, since, V_{φ_1} is shift invariant so is the complement of V_{φ_1} within V. Choose φ_2 in this complement. Zorn's lemma guarantees that, by continuing these choices, we can "exhaust" V and, consequently, obtain equality (1.2.1). ♣

This theorem leads us to investigate the structure of principal shift invariant spaces, since they are the building blocks of all shift invariant spaces. The following result provides us with a better understanding of these building blocks.

THEOREM 1.2.4. *If V_φ is a principal shift invariant space then $f \in V_\varphi$ if and only if*

$$\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi),$$

for some 1–periodic $r \in L^2(\mathbb{T}, \omega)$, where $\omega(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2$.

PROOF. Consider $\mathcal{A}_\varphi = \text{span}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$. That is, $\mathcal{A}_\varphi$ consists of all finite linear combinations of translates of φ and $V_\varphi = \overline{\mathcal{A}_\varphi}$. Let $f \in \mathcal{A}_\varphi$, then

$$f(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(x - k),$$

where only a finite number of elements of $\{a_k\}_{k\in\mathbb{Z}}$ is non–zero. We have

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k\xi} \hat{\varphi}(\xi) = r(\xi)\hat{\varphi}(\xi), \tag{1.2.2}$$

where r is a trigonometric polynomial. Conversly, every trigonometric polynomial will give us a function $f \in \mathcal{A}_\varphi$, via formula (1.2.2). Therefore, we established that $f \in \mathcal{A}_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, where r is a trigonometric polynomial.

Now we introduce an appropriate norm for the space of trigonometric polynomials. Let $\mathcal{P}_\varphi$ be the space of all trigonometric polynomials r with the $L^2(\mathbb{T}, \omega)$ norm, $\|r\|^2_{L^2(\mathbb{T},\omega)} = \int_\mathbb{T} |r(\xi)|^2 \omega(\xi)\, d\xi$. With this norm the operator $U\colon \mathcal{A}_\varphi \to \mathcal{P}_\varphi$ given by $U(f) = r$ (where r is given by (1.2.2)), is an isometry which is onto. In fact:

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \int_\mathbb{T} \sum_{k\in\mathbb{Z}} |\hat{f}(\xi + k)|^2\, d\xi = \int_\mathbb{T} |r(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2\, d\xi \tag{1.2.3}$$

$$= \int_\mathbb{T} |r(\xi)|^2 \omega(\xi)\, d\xi = \|r\|^2_{L^2(\mathbb{T},\omega)}.$$

Therefore, there is a unique isometry $\tilde{U}\colon \overline{\mathcal{A}_\varphi} \to \overline{\mathcal{P}_\varphi}$, which extends U to V_φ and maps it onto the completion $\overline{\mathcal{P}_\varphi}$ of P_φ with respect to the $L^2(\mathbb{T}, \omega)$–norm. To end the proof we observe that $\overline{\mathcal{P}_\varphi} = L^2(\mathbb{T}, \omega)$, which follows from the fact that the trigonometric polynomials are dense in $L^2(\mathbb{T}, \omega)$. ♣

If V_φ is a principal shift invariant space, then, by definition, the function φ is a generator of V_φ. Of course, there are many other functions in V_φ which generate the space. Theorem 1.2.4 provides us with the following immediate characterization of these generators.

Corollary 1.2.5. *If V_φ is a principal shift invariant space and $f \in V_\varphi$, then $V_\varphi = V_f$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, for some 1–periodic $r \in L^2(\mathbb{T}, \omega)$ such that $\operatorname{supp} r = \operatorname{supp} \omega$.*

Remark. We warn the reader that our definition of "support" of ω is, simply, the set of all ξ such that $\omega(\xi) \neq 0$. When we say "$\operatorname{supp} s = \operatorname{supp} \omega$" we mean that these sets differ only by a set of measure zero.

Principal shift invariant spaces appear in the theory of MRA wavelets. The space V_0 generated by the scaling function φ is such a space ($V_0 = V_\varphi$). The nice feature which occurs in this situation is that the shifts $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}}$ form an orthonormal basis for V_φ. It is, therefore, natural to ask if for any given principal shift invariant space we can find a generator whose translates are orthonormal. The answer is negative and can be formulated with the help of the following:

Definition 1.2.6. *If V_φ is a principal shift invariant space and ω is given by $\omega(\xi) = \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + k)|^2$, then the set $\Omega = \operatorname{supp} \omega$ is called the spectrum of V_φ.*

Remark. Note that, by Corollary 1.2.5, the spectrum Ω is well defined (it does not depend on the generator).

It is well known (see [HW]), that for any $\varphi \in L^2(\mathbb{R})$ the system $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}}$ is orthonormal (hence, forms an orthonormal basis for V_φ) if and only if

$$\sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + k)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$. In particular, we can see that in this case the spectrum of V_φ is equal to $\mathbb{R}$. The following example shows existence of principal shift invariant spaces which do not satisfy this property.

EXAMPLE 1.2.7. Let us consider a set $E = [0, \frac{1}{2}]$ and a function φ by given by $\hat{\varphi} = \mathbf{1}_E$. By definition, the spectrum Ω of V_φ is given by $\operatorname{supp}\omega$, where $\omega(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_E(\xi+k)$. Therefore, we can see immediately that Ω is equal to $\bigcup_{k\in\mathbb{Z}}(E+k)$, in particular $\Omega \neq \mathbb{R}$.

The above example indicates that finding a generator whose translates form an orthonormal basis for a given principal shift invariant space is sometimes impossible. However, it does not mean that we can not look for another kind of "nice" generators.

If $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for a Hilbert space $\mathbb{H}$, then for all $f \in \mathbb{H}$ the following Parseval's formula holds:

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |\langle f, e_k\rangle|^2.$$

In general, a system $\{e_k\}_{k\in\mathbb{Z}}$, which satisfies the above formula is called a *tight frame with constant 1*. It is easy to justify (see [HW]), that a system $\{e_k\}_{k\in\mathbb{Z}}$ is a tight frame with constant 1 if and only if for every $f \in \mathbb{H}$ we have the following reproducing formula:

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k,$$

with the convergence in $\mathbb{H}$.

It turns out, that every principle shift invariant space has generators whose translates form a tight frame with constant 1. Before we prove it let us state a simple characterization of functions whose translates form a tight frame with constant 1 (see [BL], [PSWX]).

Theorem 1.2.8. *If $\varphi \in L^2(\mathbb{R})$, then the translates of φ form a tight frame with constant 1 for the space V_φ if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2 = \mathbf{1}_\Omega(\xi),$$

where Ω is the spectrum of V_φ.

Proof. Let us consider V_φ for some $\varphi \in L^2(\mathbb{R})$. By Theorem 1.2.4 for every $f \in V_\varphi$ we have $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, where r is 1–periodic function in $L^2(\mathbb{T}, \omega)$. Let us denote $\varphi(\cdot - k)$ by φ^k. We have

$$\langle f, \varphi^k \rangle = \int_{\mathbb{R}} r(\xi)|\hat{\varphi}(\xi)|^2 e^{2\pi i k\xi}\, d\xi = \int_{\mathbb{T}} r(\xi)\omega(\xi)e^{2\pi i k\xi}\, d\xi,$$

and therefore, by the Plancherel formula,

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi^k\rangle|^2 = \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{T}} r(\xi)\omega(\xi)e^{2\pi i k\xi}\, d\xi \right|^2 = \int_{\mathbb{T}} |r(\xi)|^2|\omega(\xi)|^2\, d\xi. \tag{1.2.4}$$

By definition, the system $\{\varphi^k\}_{k\in\mathbb{Z}}$ is a tight frame with constant 1 for the space V_φ if and only if $\|f\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle f, \varphi^k\rangle|^2$ for every $f \in V_\varphi$. By (1.2.3) and (1.2.4) the above is equivalent to having

$$\int_{\mathbb{T}} |r(\xi)|^2\omega(\xi)\, d\xi = \int_{\mathbb{T}} |r(\xi)|^2|\omega(\xi)|^2\, d\xi$$

or

$$\int_{\mathbb{T}} |r(\xi)|^2\omega(\xi)(\mathbf{1}_\Omega(\xi) - \omega(\xi))\, d\xi = 0 \tag{1.2.5}$$

for all $r \in L^2(\mathbb{T}, \omega)$ (recall that Ω is the spectrum of V_φ, i.e. $\Omega = \operatorname{supp}\omega$). Choosing $r = \mathbf{1}_E$, where E is either $\{\xi \in \Omega\colon 1 > \omega(\xi)\}$ or $\{\xi \in \Omega\colon 1 < \omega(\xi)\}$, we see that (1.2.5) is satisfied if and only if $\omega = \mathbf{1}_\Omega$. ♣

From the above theorem it follows, that if V_φ is a principal shift invariant space and $\omega(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+k)|^2$, then the translates of the function ψ, defined by

$$\hat{\psi}(\xi) = \frac{\hat{\varphi}(\xi)}{\sqrt{\omega(\xi)}} \quad \text{for } \xi \in \Omega \tag{1.2.6}$$

and 0 elsewhere, will generate a tight frame with constant 1 for the space V_φ. Let us present this as the following:

Corollary 1.2.9. *If V_φ is a principal shift invariant space and ψ is defined by (1.2.6), then the system $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a tight frame with constant 1 for the space V_φ.*

Proof. First, let us observe, that ψ belongs to V_φ. In fact, for $\xi \in \Omega$ we have $\hat{\psi}(\xi) = \frac{\hat{\varphi}(\xi)}{\sqrt{\omega(\xi)}}$. Clearly, the function $\frac{1}{\sqrt{\omega}}$ is 1–periodic and belongs to $L^2(\mathbb{T}, \omega)$. Hence, Theorem 1.2.4 implies that $\psi \in V_\varphi$.

To prove that the system $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a tight frame with constant 1 for the space V_φ it is enough to observe that

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+k)|^2 = \sum_{k\in\mathbb{Z}} \frac{|\hat{\varphi}(\xi+k)|^2}{\omega(\xi)} = \mathbf{1}_\Omega(\xi)$$

and apply Theorem 1.2.8. ♣

If V_φ is a principal shift invariant space and the system $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a tight frame with constant 1 for the space V_φ, we will then call the function φ a *tight frame generator* of the space V_φ. This will allow us to avoid repeating the phrase "with constant 1".

Let us gather all of our findings obtained so far in the following:

Theorem 1.2.10. *If V is a shift invariant space, then there exist functions $\{\varphi_n\}_{n=1}^N$, where N is a natural number or infinity, such that*

$$V = \bigoplus_{j=n}^{N} V_{\varphi_n}$$

and φ_n is a tight frame generator for the space V_{φ_n}. Moreover, $f \in V$ if and only if

$$\hat{f}(\xi) = \sum_{n=1}^{N} r_n(\xi)\hat{\varphi}_n(\xi)$$

and $\|f\|_2^2 = \sum_{n=1}^{N} \|r_n\|^2_{L^2(\mathbb{T}\cap\Omega_n)}$, where r_n is a 1–periodic function in $L^2(\mathbb{T}\cap\Omega_n)$ and Ω_n is the spectrum of V_{φ_n} for every natural $n \leq N$.

Remark. Since for every natural $n \leq N$ the function r_n belongs to $L^2(\mathbb{T}\cap\Omega_n)$ we can always assume, that $r_n \in L^2(\mathbb{T})$ and is equal to zero outside of Ω_n. Moreover, if that is the case, then r_n is unique.

Proof of Theorem 1.2.10. The only thing which requires a proof is the characterization of the functions belonging to V. Clearly, if $\hat{f}$ is given by the above formula, then Theorem 1.2.8 implies that f belongs to V To obtain the other implication, let P_n denote the orthogonal projection onto the space V_{φ_n}. For every $f \in V$ we have

$$\hat{f} = \sum_{n=1}^{N} \widehat{(P_n f)}.$$

By Theorem 1.2.4 we obtain $\widehat{(P_n f)}(\xi) = r_n(\xi)\hat{\varphi}_n(\xi)$, where r_n is a 1–periodic function in $L^2(\mathbb{T}\cap\Omega_n)$, for every natural $n \leq N$. Since φ_n is a tight frame generator for V_{φ_n}, from Theorem 1.2.8 and formula (1.2.3) it follows that

$$\|P_n f\|_2^2 = \|r_n\|^2_{L^2(\mathbb{T}\cap\Omega_n)}.$$

To finish the proof we observe that

$$\|f\|_2^2 = \sum_{n=1}^{N} \|P_n f\|_2^2 = \sum_{n=1}^{N} \|r_n\|^2_{L^2(\mathbb{T}\cap\Omega_n)}. \quad \clubsuit$$

Although the above theorem tells us a lot about the general structure of shift invariant spaces it contains some redundancy. The theorem is not optimal, as the following example shows:

EXAMPLE 1.2.11. Let us consider a principal shift invariant space V_φ with the spectrum Ω and two disjoint 1–periodic sets E, F such that $E \cup F = \Omega$. Let $\hat{\varphi}_1 = \hat{\varphi}\mathbf{1}_E$ and $\hat{\varphi}_2 = \hat{\varphi}\mathbf{1}_F$. By Theorem 1.2.4 we have $\varphi_1, \varphi_2 \in V_\varphi$. Theorem 1.2.8 implies that φ_1 and φ_2 are tight frame generators for V_{φ_1} and V_{φ_2}, respectively. Moreover, it is clear that the spectra of V_{φ_1} and V_{φ_2} are equal to E and F, respectively.

From Theorem 1.2.4 it follows that if a function f belongs to a principal shift invariant space V, then the support of $\hat{f}$ must be contained in the spectrum of V. Since the sets E and F are disjoint the above implies that V_{φ_1} is orthogonal to V_{φ_2}. Moreover, the condition $E \cup F = \Omega$ allows us to conclude that $\varphi \in V_{\varphi_1} \oplus V_{\varphi_2}$. Therefore, we obtain the following decomposition of V_φ:

$$V_\varphi = V_{\varphi_1} \oplus V_{\varphi_2}.$$

This shows that, sometimes, we can replace an orthogonal sum of two (and, in general, even more) principal shift invariant spaces by one principal shift invariant space. This leads us to the problem of finding an "optimal" decomposition of a given shift invariant space. One such optimal decomposition was proposed by C. de Boor, R. De Vore and A. Ron in [BDR] (it also appears in the work of M. Papadakis [Pa]). Another, that is closely related, was presented by M. Bownik in [Bo1]. In the next paragraph we will explain Bownik's result and prove its optimality.

To close this paragraph and show how Theorem 1.2.4 works in "reality" let us present the following example related to Example 1.2.7:

EXAMPLE 1.2.12. Let $E \subset \mathbb{R}$ be a set of a finite measure and φ be a function given by $\hat{\varphi} = \mathbf{1}_E$. This situation gives rise to two natural shift invariant spaces. One of them is simply V_φ and the other is given by

$$V = \{f \in L^2(\mathbb{R}) \colon \operatorname{supp} \hat{f} \subset E\}, \tag{1.2.7}$$

that is, $\hat{V} = L^2(E)$. It is natural to ask what is the relationship between these two spaces (note, that their spectra are both equal to $\Omega := \bigcup_{k\in\mathbb{Z}}(E+k)$). Clearly, we always have $V_\varphi \subset V$. Moreover, from Theorem 1.2.4 it follows that these two spaces are equal, provided that the integer translates of the set E are mutually disjoint. It turns out that this disjointness is also a necessary condition for these two spaces to be equal. Indeed, let us assume that there is a $k \in \mathbb{Z}$ such that the measure of $E \cap (E+k)$ is positive and $V_\varphi = V$. By Theorem 1.2.4 every function $f \in V_\varphi$ is of the form $\hat{f} = r\mathbf{1}_E$ for some 1–periodic $r \in L^2(\mathbb{T} \cap \Omega)$. In particular $\hat{f}$ satisfies $\hat{f}(\xi) = \hat{f}(\xi - k)$ for $\xi \in E \cap (E+k)$, which contradicts the assumption that $V_\varphi = V = L^2(E)$. In general, for any measurable subset E of $\mathbb{R}$ we can consider the space V defined by (1.2.7) (E may be of an infinite measure). Then, V is a shift invariant space and from Theorem 1.2.4 it follows that V is principal if and only if the integer translates of the set E are mutually disjoint. If that is the case, then $V = V_\varphi$, where $\hat{\varphi} = \mathbf{1}_E$.

1.3 Range function

In the previous paragraph we proved that every shift invariant space can be decomposed into an orthogonal sum of principal shift invariant spaces. Let us consider a simple situation of an orthogonal sum of two principal shift invariant spaces V_{φ_1} and V_{φ_2}. We shall investigate how the information that the two spaces are orthogonal can be translated into an information about the generators φ_1 and φ_2.

Clearly, V_{φ_1} is orthogonal to V_{φ_2} if and only if the integer translates of φ_1 are orthogonal to the function φ_2, that is,

$$\langle \varphi_1(\cdot - k), \varphi_2 \rangle = 0$$

for all $k \in \mathbb{Z}$. Using the standard "periodization" argument we obtain

$$\langle \varphi_1(\cdot - k), \varphi_2 \rangle = \int_{\mathbb{R}} e^{-2\pi i k\xi} \hat{\varphi}_1(\xi)\overline{\hat{\varphi}_2(\xi)}\, d\xi$$

$$= \int_{\mathbb{T}} e^{-2\pi i k\xi} \sum_{m\in\mathbb{Z}} \hat{\varphi}_1(\xi+m)\overline{\hat{\varphi}_2(\xi+m)}\, d\xi = 0$$

for all $k \in \mathbb{Z}$. This simply means that all Fourier coefficients of the 1–periodic function $\sum_{m\in\mathbb{Z}} \hat{\varphi}_1(\cdot + m)\overline{\hat{\varphi}_2(\cdot + m)}$ are equal to zero. Since the function belongs to $L^2(\mathbb{T})$ (the Schwarz inequality implies that the norm of this function does not exceed $\|\varphi_1\|_2\|\varphi_2\|_2$), we conclude that the above equation is equivalent to having this function equal to zero almost everywhere.

We will use the above result in the future, so let us state it as the following:

PROPOSITION 1.3.1. *The spaces V_{φ_1} and V_{φ_2} are orthogonal if and only if*

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}_1(\xi + k)\overline{\hat{\varphi}_2(\xi + k)} = 0$$

for a.e. $\xi \in \mathbb{R}$.

The above proposition and Theorem 1.2.8 indicate that the vector–valued functions of the form $(\ldots, \hat{f}(\xi-1), \hat{f}(\xi), \hat{f}(\xi+1), \ldots)$ play an important role in analyzing the shift invariant spaces. In fact, both these results can be stated as simple conditions on the norm and the inner product of such vectors in the space $\ell^2(\mathbb{Z})$. The idea of using such vectors was exploited by many researchers. P. Auscher used it to prove that "nice" wavelets have to be associated with a multiresolution analysis (see [Au], [HW]). A. Ron, Z. Shen, P Cassaza and M. Lammers investigate the following *bracket product*

$$[f, g](\xi) = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + k)\overline{\hat{g}(\xi + k)}$$

for any $f, g \in L^2(\mathbb{R})$ and $\xi \in \mathbb{T}$, which is obviously related to the vectors of the form $(\hat{f}(\xi + k))_{k\in\mathbb{Z}}, (\hat{g}(\xi + k))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

The idea is the following. The real line is endowed with the natural definition of the Fourier transform as a function on the dual group of $\mathbb{R}$ that can be identified with $\mathbb{R}$. However, we can also decompose the real line into a disjoint sum of intervals of length equal to 1. Each of these intervals has its own Fourier analysis, which is rather different from the original one (the dual group of $\mathbb{T}$ is $\mathbb{Z}$). The theory of range function we are going to present in this paragraph is based on this situation.

The next proposition we prove can be treated as a precise definition of the family $\{(\hat{f}(\xi+k))_{k\in\mathbb{Z}} \colon \xi \in \mathbb{T}\}$ (note, that the vector $(\hat{f}(\xi+k))_{k\in\mathbb{Z}}$ is "really" defined for a.e. ξ). Beside this, the proposition is nothing more then the standard "periodization" argument we were using before.

Proposition 1.3.2. *The mapping $\mathcal{T} \colon L^2(\mathbb{R}) \to L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ defined by*

$$\mathcal{T}f(\xi) = (\hat{f}(\xi + k))_{k\in\mathbb{Z}}$$

is an isometric isomorphism between $L^2(\mathbb{R})$ and $L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$.

Proof. For every $f \in L^2(\mathbb{R})$ we have

$$\|\mathcal{T}f\|^2_{L^2(\mathbb{T},\ell^2(\mathbb{Z}))} = \int_{\mathbb{T}} \|\mathcal{T}f(\xi)\|^2_{\ell^2(\mathbb{Z})}\, d\xi = \int_{\mathbb{T}} \sum_{k\in\mathbb{Z}} |\hat{f}(\xi+k)|^2\, d\xi$$

$$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}+k} |\hat{f}(\xi)|^2\, d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|_2^2 = \|f\|_2^2.$$

The above calculation establishes that $\mathcal{T}$ is an isometry. Moreover, it justifies our previous claim that $\mathcal{T}f(\xi)$ belongs to $\ell^2(\mathbb{Z})$ for a.e. $\xi \in \mathbb{T}$.

To prove that $\mathcal{T}$ is onto we proceed as follows. Let $\{\mathbf{e_n}\}_{\mathbf{n}\in\mathbb{Z}}$ denote the standard basis of $\ell^2(\mathbb{Z})$. For $L^2(\mathbb{R})$ we choose the Gabor basis $g_{m,n}$, where $m, n \in \mathbb{Z}$ and

$$\widehat{(g_{m,n})}(\xi) = e^{2\pi i m\xi}\mathbf{1}_{[0,1]}(\xi - n).$$

An easy calculation shows that

$$\mathcal{T}(g_{m,n})(\xi) = e^{2\pi i m\xi}\mathbf{e_n}.$$

Let us consider a function $F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$, which is orthogonal to all the functions $\mathcal{T}(g_{m,n})$. We have, for all $m, n \in \mathbb{Z}$, that

$$0 = \langle F, \mathcal{T}(g_{m,n})\rangle_{L^2(\mathbb{T},\ell^2(\mathbb{Z}))} = \int_{\mathbb{T}} \langle F(\xi), \mathcal{T}(g_{m,n})(\xi)\rangle_{\ell^2(\mathbb{Z})}\, d\xi$$

$$= \int_{\mathbb{T}} e^{-2\pi i m\xi} \langle F(\xi), \mathbf{e_n} \rangle_{\ell^2(\mathbb{Z})} \, d\xi,$$

which proves that $\langle F(\xi), \mathbf{e_n} \rangle_{\ell^2(\mathbb{Z})} = 0$ for a.e. $\xi \in \mathbb{T}$ and all $n \in \mathbb{Z}$. This, however, implies that $F(\xi)$ is the null vector in the space $\ell^2(\mathbb{Z})$ for a.e. $\xi \in \mathbb{T}$. Therefore, F must be the null vector in the space $L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$, which proves that $\mathcal{T}$ is onto. ♣

The Hilbert space $L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ is more complicated then $L^2(\mathbb{R})$ (it can be viewed as a family of $\ell^2(\mathbb{Z})$ spaces parametrized by $\xi \in \mathbb{T}$). However, as we shall see in the following example, the images of shift invariant spaces under the transform $\mathcal{T}$ have a very nice form that will be most useful in our further investigations.

Example 1.3.3. Let us consider a principal shift invariant space V_φ with a tight frame generator φ and a spectrum Ω. We are interested in the image of V_φ under the transform $\mathcal{T}$. By Theorem 1.2.4 every function $f \in V_\varphi$ satisfies $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ for some 1–periodic $r \in L^2(\mathbb{T}, \Omega)$. Since r is 1–periodic we obtain that $\mathcal{T}f(\xi) = r(\xi)\mathcal{T}\varphi(\xi)$. Therefore,

$$\mathcal{T}(V_\varphi) = \{F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))\colon F(\xi) = r(\xi)\mathcal{T}\varphi(\xi), \text{ where } r \in L^2(\mathbb{T}, \Omega)\},$$

or equivalently:

$$\mathcal{T}(V_\varphi) = \{F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))\colon F(\xi) \in J(\xi)\}, \tag{1.3.1}$$

where

$$J(\xi) = \operatorname{span}\{\mathcal{T}\varphi(\xi)\} \tag{1.3.2}$$

for a.e. $\xi \in \mathbb{T}$ (note that $J(\xi)$ is a one–dimensional subspace of $\ell^2(\mathbb{Z})$ for a.e. $\xi \in \mathbb{T} \cap \Omega$ and the zero subspace for a.e. $\xi \in \mathbb{T}$ outside of Ω). Since the image of V_φ does not depend on the choice of a generator of V_φ, we conclude, that the space

$J(\xi)$ given in (1.3.2) does not neither. This can be also seen from (1.3.1), which implies that

$$J(\xi) = \{\mathcal{T} f(\xi) \colon f \in V_\varphi\}.$$

The above situation can be extended to the case of an arbitrary shift invariant space V. By Theorem 1.2.3 we know that V is generated by a countable family of functions $\{\varphi_n\}_{n=1}^N$. For a.e. $\xi \in \mathbb{T}$ we have that $\mathcal{T}\varphi_n(\xi) \in \ell^2(\mathbb{Z})$. For each such ξ let

$$J(\xi) = \overline{\operatorname{span}\{\mathcal{T}(\varphi_n)(\xi)\}_{n=1}^N}. \tag{1.3.3}$$

Then, we have, again,

$$\mathcal{T}(V) = \{F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z})) \colon F(\xi) \in J(\xi)\}$$

and

$$J(\xi) = \{\mathcal{T} f(\xi) \colon f \in V\} \tag{1.3.4}$$

for a.e. $\xi \in \mathbb{T}$. Therefore, we see that to every shift invariant space we can associate a function whose values are subspaces of $\ell^2(\mathbb{Z})$. This motivates the following:

Definition 1.3.4. *A mapping*

$$J \colon \mathbb{T} \to \{\textit{closed subspaces of } \ell^2(\mathbb{Z})\}$$

is called a range function.

Moreover, we will always assume that J is *measurable*, which means, that the projections $P(\xi) \colon \ell^2(\mathbb{Z}) \to J(\xi)$ are weakly operator measurable; that is, the function

$$\xi \to \langle P(\xi)v, w\rangle_{\ell^2(\mathbb{Z})}$$

is measurable for each vectors $v, w \in \ell^2(\mathbb{Z})$ (note that the function J defined in (1.3.3) is measurable, simply because the functions φ_n are measurable).

The theory of the range function was developed by H. Helson about 30 years ago (see [He1] and [He2]). It was used by C. de Boor, R.A. DeVore and A. Ron in [BDR] in their study of finitely generated shift invariant spaces. Their results were extended later by M. Bownik to the case of all shift invariant spaces (see [Bo1]). We shall not use the full strength of the range function theory. For our considerations only one thing about the range function is really important: its existence. However, let us present below a theorem, which is a starting point of the range function theory, and which we will use sometimes to clarify some technical difficulties.

We already know that every shift invariant space corresponds to some range function via formula (1.3.3). Of course, every measurable range function allows us to define a subspace of $L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ as follows

$$M_J = \{F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))\colon F(\xi) \in J(\xi) \text{ for a.e. } \xi \in \mathbb{T}\}.$$

Clearly, we can consider the space $V = \mathcal{T}^{-1}(M_J)$, which is a subspace of $L^2(\mathbb{R})$, and ask if it is shift invariant. It turns out that, this is the case. Let us cite the following theorem, which establishes the relation between shift invariant spaces and measurable range functions (see [Bo1]):

THEOREM 1.3.5. *A closed subspace $V \subset L^2(\mathbb{R})$ is shift invariant if and only if*

$$V = \{f \in L^2(\mathbb{R})\colon \mathcal{T}f(\xi) \in J(\xi) \textit{ for a.e. } \xi \in \mathbb{T}\},$$

where J is a measurable range function. The above correspondence is one–to–one under the convention that the range functions are identified if they are equal a.e. Moreover, if $\{\varphi_n\}_{n=1}^N$ is a set of generators of V (N is a natural number or infinity), then

$$J(\xi) = \overline{\operatorname{span}\{\mathcal{T}(\varphi_n)(\xi)\}_{n=1}^N}.$$

The basis for our further considerations of this chapter shall be formula (1.3.3) and the fact that the range function given by (1.3.3) does not depend on the choice of generators (which can be seen from (1.3.4)). We will use the range function to define a functional calculus on shift invariant spaces, which in turn will be applied in solving some problems, which appeared lately in the theory of signal procesing. In many cases our functional calculus approach will provide a new understanding of well known phenomenons, in some other cases it leads to new results.

The first advantage of introducing the range function is a connection it establishes between our study of shift invariant spaces and the treatment of the same topic due to L. Baggett, H. Medina and K. Merrill (see [BMM]). Using metods of abstract harmonic analysis they associated with a given shift invariant space V a multiplicity function m_V (it was done via Stone's Theorem on the unitary representation of the group $\mathbb{Z}$ acting on V by translations). In our approach, as we shall see in the proof of Proposition 1.3.7, we recover the multiplicity function by defining it as follows:

Definition 1.3.6. *If V is a shift invariant space and J is the corresponding range function, then*

$$m_V(\xi) = \dim J(\xi),$$

where $\xi \in \mathbb{T}$, is called the multiplicity function of V.

Remark. Since the multiplicity function is defined on $\mathbb{T}$ we will often treat it as a 1–periodic function defined on $\mathbb{R}$. Note, also, that the multiplicity function can attain infinity as its value.

The multiplicity function in both settings is defined in an abstract way. However, if a shift invariant space is decomposed as in Theorem 1.2.10, then we can easily compute its multiplicity function by using tight frame generators. The following proposition provides us with such an explicit formula.

Proposition 1.3.7. *If V is a shift invariant space with a decomposition*

$$V = \bigoplus_{n=1}^{N} V_{\varphi_n},$$

where φ_n is a tight frame generator for V_{φ_n}, then

$$m_V(\xi) = \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} |\hat{\varphi}_n(\xi + k)|^2.$$

Proof.

Let J_n denote the range function corresponding to V_{φ_n} and J_V be the range function of V. By Proposition 1.3.1 we obtain that the spaces $J_n(\xi)$ are mutually orthogonal for a.e. $\xi \in \mathbb{T}$, and all natural $n \leq N$. Therefore, from formula (1.3.3) it follows, that $J_V(\xi) = \bigoplus_{n=1}^{N} J_n(\xi)$ for a.e. $\xi \in \mathbb{T}$, which in turn implies that

$$m_V(\xi) = \dim J_V(\xi) = \sum_{n=1}^{N} \dim J_n(\xi) = \sum_{n=1}^{N} m_{V_{\varphi_n}}(\xi)$$

for a.e. $\xi \in \mathbb{R}$. From Example 1.3.3 it follows, that the multiplicity function of a principal shift invariant space is equal to the characteristic function of its spectrum. Thus, by Theorem 1.2.8, we obtain that

$$m_{V_{\varphi_n}}(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}_n(\xi + k)|^2$$

for a.e. $\xi \in \mathbb{R}$, which concludes the proof. ♣

Since the same formula for the multiplicity function was obtained in [BM] we conclude that Definition 1.3.6 is equivalent to the definition of the multiplicity function given in [BMM]. The multiplicity function has many other interesting properties and can be succesfuly used in the study of wavelets (see [BMM], [Ba], [We]). We will come back to these issues later. At this point, however, let us present how it works in the optimal decomposition problem, which we mentioned in the previous paragraph. The following theorem is due to M. Bownik (see [Bo1]).

THEOREM 1.3.8. *If V is a shift invariant space and $N = \|m_V\|_\infty$ (N is a natural number or infinity), then there exists a set of functions $\{\varphi_n\}_{n=1}^N$, such that*

$$V = \bigoplus_{n=1}^{N} V_{\varphi_n},$$

where φ_n is a tight frame generator of V_{φ_n} and the spectrum Ω_n of V_{φ_n} satisfies

$$\Omega_n = \{\xi \in \mathbb{R}\colon\, m_V(\xi) \geq n\},$$

for every natural $n \leq N$.

The above theorem gives us a decomposition, which is optimal with respect to N, the number of generators of V. To see it in detail let us consider another decomposition of V:

$$V = \bigoplus_{m=1}^{M} V_{\psi_m},$$

with tight frame generators ψ_m, m natural not exceeding M. We shall prove, that $N \leq M$. Clearly, $N = \|m_V\|_\infty$ and $m_V = \sum_{n=1}^N \mathbf{1}_{\Omega_n}$. Let Ω'_m denote the spectrum of V_{ψ_m}. Proposition 1.3.7 implies that $m_V = \sum_{m=1}^M \mathbf{1}_{\Omega'_m}$. Thus, $M \geq \|m_V\|_\infty = N$.

The generators φ_n, which appear in the optimal decomposition are not unique. Even the spaces V_{φ_n} are not unique. Neverless, the decomposition is unique if we use a criterium proposed by E. Wilson.

COROLLARY 1.3.9. *If V is a shift invariant space, then there exists a unique decreasing sequence of sets $\{\Omega_n\}_{n\in\mathbb{N}}$ such that*

$$V = \bigoplus_{n\in\mathbb{N}} V_n,$$

where V_n is a principal shift invariant space with spectrum Ω_n.

REMARK. Here, we use the convention, that if for some $n \in \mathbb{N}$ the set Ω_n is empty, then V_n is the zero subspace. We will call the sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ the *spectral sequence* of V.

Proof of Corollary 1.3.9. The very existence of such a sequence is guaranteed by Theorem 1.3.8. To see the uniqueness we proceed as follows. By Proposition 1.3.7 we have that $m_V = \sum_{n\in\mathbb{N}} \mathbf{1}_{\Omega_n}$. Since the sequence of the spectra is decreasing, that is, $\Omega_n \supset \Omega_{n+1}$ for all natural n, we obtain that $m_V(\xi) \geq n$ for a.e. $\xi \in \Omega_n$ and $m_V(\xi) \leq n-1$ for a.e. $\xi \in \mathbb{R} \setminus \Omega_n$. Hence, we can see that $\Omega_n = \{\xi \in \mathbb{R}\colon m_V(\xi) \geq n\}$. ♣

It turns out that although the decomposition of Theorem 1.3.8 is not unique there is a nice description of all other optimal decompositions via "unitary" transformations of a given one. We will delay proving this result until paragraph §1.7. Let us, instead, present further applications of the range function in the study of shift invariant spaces. In the next paragraph we will develop our main tool, that is, a functional calculus of shift invariant spaces.

1.4 Functional calculus of shift invariant spaces

The material we present in this paragraph is the main contribution of this dissertation to the study of shift invariant spaces and their connections to signal analysis. We want, however, to point out, that many researchers contributed to this functional calculus. The theory of the range function we use below is due to H. Helson, C. de Boor, R.A. DeVore, A. Ron and M. Bownik. The ideas of using it in signal analysis were developed by A. Ron and Z. Shen. Some related topics were also investigated by J. Benedetto and S. Li.

The nice feature about our functional calculus is that it completely recovers the one introduced by L. Baggett, H. Medina and K. Merrill, that is, the multiplicity function. The authors have shown how to apply their theory in the study of wavelets by generalizing the notion of the multiresolution analysis. Moreover, E. Weber made a connection between the multiplicity function and the dimension function of wavelets, which was introduced by P. Auscher. The spectral function, which we are going to present, containes much more information about shift invariant spaces then the multiplicity function. That is why we hope to recover the main theorem of [BMM], and also, prove some results, which seem inaccesible by using the theory of the multiplicity function.

Our main concern will be an application of the spectral function for proving two completeness results in signal analysis. The first one will be a characterization of the completeness of affine orthonormal systems (that is, wavelet systems). The program of this characterization is due to Y. Meyer and G. Weiss and we will present its history in detail. The second result, known as the Rieffel's theorem, assures that a Gabor system can not be complete if its density ab is greater then 1. Although this result is not new, many researchers attempted to find an elementary proof of it, and that is why we shall show how it can be obtained with a use of the spectral function.

The idea of the functional calculus of shift invariant spaces relies on associating to a given shift invariant space a function which contains essential information about this space. To present the realization of this idea in a transparent manner let us summarize all necessary findings of the previous paragraphs. If V is a shift invariant space, then Zorn's lemma allows us to decompose it into a countable family of mutually orthogonal principal shift invariant spaces (Theorem 1.2.3). A careful study of principal shift invariant spaces reveals the existence of a single tight frame generator for such a space (Theorem 1.2.4 and Corollary 1.2.9). This, in turn, allows us to conclude the existence of a countable family of tight frame generators for a given shift invariant space V. These generators give rise to the range function via formula (1.3.3), which does not depend on the choice of generators because of (1.3.4). The spectral function of the space V is defined with the use of the orthogonal projection of the standard basis of $\ell^2(\mathbb{Z})$ on the range function:

DEFINITION 1.4.1. *Let V be a shift invariant space with the range function J. Let $P_{J(\xi)}$ be the orthogonal projection onto $J(\xi)$ for a.e. $\xi \in \mathbb{T}$ and $\{\mathbf{e_k}\}_{\mathbf{k}\in\mathbb{Z}}$ denote* *the standard basis of* $\ell^2(\mathbb{Z})$. *The function*

$$\sigma_V(\xi + \mathbf{k}) = \|P_{J(\xi)}\mathbf{e}_{\mathbf{k}}\|^2_{\ell^2(\mathbb{Z})}$$

defined for a.e. $\xi \in \mathbb{T}$ *and* $\mathbf{k} \in \mathbb{Z}$ *is called the spectral function of the space* V.

Remark. The range of ξ and k in the above definition allows us to treat the spectral function as a function defined on $\mathbb{R}$. To be precise, if $\xi \in \mathbb{R}$ and $\xi = \xi' + \mathbf{k}$, where $\xi' \in \mathbb{T}$ and $\mathbf{k} \in \mathbb{Z}$, then $\sigma_V(\xi) = \sigma_V(\xi' + \mathbf{k})$.

The above definition is very abstract. In order to make this notion more understandable we offer two other definitions and show that all of them are equivalent.

Definition 1.4.2. *Let* $\mathcal{V}$ *denote the set of all shift invariant subspaces of* $L^2(\mathbb{R})$. *There exists a unique mapping*

$$\sigma : \mathcal{V} \to L^\infty(\mathbb{R}),$$

such that

$$\sigma_{V_\varphi}(\xi) = \frac{|\hat{\varphi}(\xi)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2},$$

for $\xi \in \operatorname{supp}\hat{\varphi}$ *and 0 otherwise and which is additive on orthogonal sums, that is, if* $V = \bigoplus_{n=1}^{\infty} V_n$, *then*

$$\sigma_V = \sum_{n=1}^{\infty} \sigma_{V_n}.$$

Remark. In the above definition V_φ denotes the principal shift invariant space generated by φ, where φ is not necesserily a tight frame generator. However, if that is the case, then the spectral function has a simpler form:

$$\sigma_{V_\varphi}(\xi) = |\hat{\varphi}(\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$ (see Theorem 1.2.8).

We point out that the second definition is related to ideas of abstract harmonic analysis. In fact, the group $\mathbb{Z}$ acts on $L^2(\mathbb{R})$ as a unitary representation obtained by translations:

$$[\tau_z(f)](x) = f(x+z),$$

where $z \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$. The cyclic subspaces of this representation are, what we call, principal shift invariant spaces. If g is a cyclic vector for the subspace V_g, then we can consider

$$p(z) = \langle \tau_z(g), g \rangle,$$

which is a positive definite function on $\mathbb{Z}$. By Bochner's Theorem, there exists a finite Borel measure μ on $\mathbb{T}$ whose Fourier transform is equal to p. In [Ba] it is proved that the density function of this measure is equal to $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2$. Therefore, the spectral function of the cyclic subspace V_g is a quotient of $|\hat{g}|^2$ and this density function (the spectral function is equal to zero outside of the support of $\hat{g}$).

The last definition is very easy to grasp and it is formulated without using the range function theory.

Definition 1.4.3. *Let V be a shift invariant subspace of $L^2(\mathbb{R})$ with a set of generators $\{\varphi_n\}_{n=1}^N$ (N natural or infinity). If the integer translates of the generators form a tight frame with constant 1 for the space V, then the function*

$$\sigma_V(\xi) = \sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2,$$

defined for a.e. $\xi \in \mathbb{R}$, does not depend on the choice of generators.

To prove the equivalence of the three definitions we will need the following lemma (see also [RS1], [Bo1])

Lemma 1.4.4. *Let V be a shift invariant space with a set of generators $\{\varphi_n\}_{n=1}^N$ (N natural or infinity). The integer translates of the generators form a tight* *frame with constant 1 for the space V if and only if for a.e. $\xi \in \mathbb{T}$ the family $\{\mathcal{T}(\varphi_n)(\xi)\}_{n=1}^N$ is a tight frame with constant 1 for the space $J(\xi)$.*

PROOF. The lemma is a simple tautology which relies on the use of the Fourier transform. Let f^m denote the function $f(\cdot - m)$. For any $f, g \in V$ and $m \in \mathbb{Z}$ we have

$$\langle f^m, g \rangle = \langle \mathcal{T} f^m, \mathcal{T} g \rangle = \int_{\mathbb{T}} e^{-2\pi i m \xi} \langle \mathcal{T} f(\xi), \mathcal{T} g(\xi) \rangle \, d\xi \tag{1.4.1}$$

Let φ_n^k denote the function $\varphi_n(\cdot - k)$. We have,

$$\sum_{n=1}^N \sum_{k \in \mathbb{Z}} \langle f^m, \varphi_n^k \rangle \langle \varphi_n^k, g \rangle = \sum_{n=1}^N \sum_{k \in \mathbb{Z}} \langle \mathcal{T}(f^m), \mathcal{T}(\varphi_n^k) \rangle \langle \mathcal{T}(\varphi_n^k), \mathcal{T} g \rangle$$

$$= \sum_{n=1}^N \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \langle \mathcal{T}(f^m)(\xi), \mathcal{T}(\varphi_n)(\xi) \rangle e^{2\pi i k \xi} \, d\xi \int_{\mathbb{T}} \langle \mathcal{T}(\varphi_n)(\xi), \mathcal{T} g(\xi) \rangle e^{-2\pi i k \xi} \, d\xi$$

$$= \sum_{n=1}^N \int_{\mathbb{T}} \langle \mathcal{T}(f^m)(\xi), \mathcal{T}(\varphi_n)(\xi) \rangle \langle \mathcal{T}(\varphi_n)(\xi), \mathcal{T} g(\xi) \rangle \, d\xi$$

$$= \int_{\mathbb{T}} e^{-2\pi i m \xi} \sum_{n=1}^N \langle \mathcal{T} f(\xi), \mathcal{T}(\varphi_n)(\xi) \rangle \langle \mathcal{T}(\varphi_n)(\xi), \mathcal{T} g(\xi) \rangle \, d\xi.$$

That is,

$$\sum_{n=1}^N \sum_{k \in \mathbb{Z}} \langle f^m, \varphi_n^k \rangle \langle \varphi_n^k, g \rangle \tag{1.4.2}$$

$$= \int_{\mathbb{T}} e^{-2\pi i m \xi} \sum_{n=1}^N \langle \mathcal{T} f(\xi), \mathcal{T}(\varphi_n)(\xi) \rangle \langle \mathcal{T}(\varphi_n)(\xi), \mathcal{T} g(\xi) \rangle \, d\xi.$$

From the two formulae above we can establish the equivalency we seek. Indeed, the system $\{\varphi_n^k \colon k \in \mathbb{Z},$ natural $n \le N\}$ is a tight frame with constant 1 for V if and only if for every $f, g \in V$ we have

$$\langle f, g\rangle = \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} \langle f, \varphi_n^k\rangle \langle \varphi_n^k, g\rangle.$$

Since V is shift invariant, the above is equivalent to saying that for every $f, g \in V$ and $m \in \mathbb{Z}$ we have

$$\langle f^m, g\rangle = \sum_{n=1}^{N} \sum_{k\in\mathbb{Z}} \langle f^m, \varphi_n^k\rangle \langle \varphi_n^k, g\rangle.$$

Applying formulae (1.4.1) and (1.4.2) to this equation gives us its equivalent form:

$$\int_{\mathbb{T}} e^{-2\pi i m\xi} \langle \mathcal{T}f(\xi), \mathcal{T}g(\xi)\rangle \, d\xi$$

$$= \int_{\mathbb{T}} e^{-2\pi i m\xi} \sum_{n=1}^{N} \langle \mathcal{T}f(\xi), \mathcal{T}(\varphi_n)(\xi)\rangle \langle \mathcal{T}(\varphi_n)(\xi), \mathcal{T}g(\xi)\rangle \, d\xi$$

for every $f, g \in V$ and $m \in \mathbb{Z}$. This, in turn, can be simply stated as

$$\langle \mathcal{T}f(\xi), \mathcal{T}g(\xi)\rangle = \sum_{n=1}^{N} \langle \mathcal{T}f(\xi), \mathcal{T}(\varphi_n)(\xi)\rangle \langle \mathcal{T}(\varphi_n)(\xi), \mathcal{T}g(\xi)\rangle$$

for every $f, g \in V$ and a.e. $\xi \in \mathbb{T}$. By Theorem 1.3.5, the above is equivalent to

$$\langle F(\xi), G(\xi)\rangle = \sum_{n=1}^{N} \langle F(\xi), \mathcal{T}(\varphi_n)(\xi)\rangle \langle \mathcal{T}(\varphi_n)(\xi), G(\xi)\rangle$$

for every $F(\xi), G(\xi) \in J(\xi)$ and a.e. $\xi \in \mathbb{T}$, which simply means that for a.e. $\xi \in \mathbb{T}$ the system $\{\mathcal{T}(\varphi_n)(\xi)\}_{n=1}^{N}$ is a tight frame with constant 1 for the space $J(\xi)$. ♣

The above lemma allows us to prove that the projection definition of the spectral function is equivalent to the tight frame generators definition.

Proof of the equivalency of Definition 1.4.1 and Definition 1.4.3. Since the projection definition of the spectral function does not depend on generators it is enough to prove that the functions given in both definitions are equal. Let $\xi \in \mathbb{T}$ and $k \in \mathbb{Z}$. By Lemma 1.4.4 we have

$$\sigma_V(\xi + \mathbf{k}) = \|P_{J(\xi)}\mathbf{e_k}\|^2_{\ell^2(\mathbb{Z})} = \sum_{n=1}^{N} |\langle P_{J(\xi)}\mathbf{e_k}, \mathcal{T}(\varphi_n)(\xi)\rangle_{\ell^2(\mathbb{Z})}|^2$$

$$= \sum_{n=1}^{N} |\langle \mathbf{e_k}, \mathcal{T}(\varphi_n)(\xi)\rangle_{\ell^2(\mathbb{Z})}|^2 = \sum_{n=1}^{N} |\hat{\varphi}_n(\xi + \mathbf{k})|^2,$$

hence $\sigma_V(\xi) = \sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2$ for a.e. $\xi \in \mathbb{R}$. ♣

Remark. There is also a less abstract way of proving that the sum $\sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2$ does not depend on the choice of generators if the integer shifts of the system $\{\varphi_n\}_{n=1}^{N}$ form a tight frame with constant 1 for a shift invariant space V. To see it let us consider an orthonormal basis of $L^2(\mathbb{R})$ given by

$$\widehat{(e_{l,m})}(\xi) = e^{2\pi i l\xi}\mathbf{1}_{[0,1]}(\xi - m)$$

for $l, m \in \mathbb{Z}$ and the orthogonal projection P_V onto the space V. A simple calculation relying on the use of the Fourier transform on the interval $[m, m+1]$, $m \in \mathbb{Z}$, shows that for every $l, m \in \mathbb{Z}$ we have

$$\langle P_V e_{l,m}, P_V e_{0,m}\rangle = \sum_{n=1}^{N}\sum_{k\in\mathbb{Z}} \langle \widehat{(e_{l,m})}, \widehat{(\varphi_n^k)}\rangle \langle \widehat{(\varphi_n^k)}, \widehat{(e_{0,m})}\rangle$$

$$= \sum_{n=1}^{N}\sum_{k\in\mathbb{Z}} \int_m^{m+1} e^{2\pi i l\xi}\, \overline{\hat{\varphi}_n(\xi)}\, e^{2\pi i k\xi}\, d\xi \int_m^{m+1} \hat{\varphi}_n(\xi) e^{-2\pi i k\xi}\, d\xi$$

$$= \int_m^{m+1} \sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2 e^{-2\pi i l\xi}\, d\xi;$$

that is,

$$\langle P_V e_{l,m}, P_V e_{0,m}\rangle = \int_m^{m+1} \sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2 e^{-2\pi i l \xi}\, d\xi. \tag{1.4.3}$$

Since the lefthand side of (1.4.3) does not depend on the choice of generators and the righthand side contains the whole information about the function $\sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2$ via its Fourier coefficients on the interval $[m, m+1]$, this justifies our claim.

In our considerations we will often choose the most convenient definition of the spectral function. We shall also do it to prove that Definition 1.4.2 gives us the same spectral function as given in the other definitions.

Proof of the equivalency of Definition 1.4.1 and Definition 1.4.2. By Theorem 1.2.3 it is clear that by the additivity property of the spectral function it is sufficient to determine its value on all shift invariant spaces as long as we know its value on principal shift invariant spaces, thus the uniqueness follows. We only need to justify that the spectral function given by projections satisfies the properties given in Definition 1.4.2. That $\sigma_V \in L^\infty(\mathbb{R})$ is immediate. If V_φ is a principal shift invariant space, then from Corollary 1.2.9 it follows that the function ψ given by

$$\hat{\psi}(\xi) = \frac{\hat{\varphi}(\xi)}{\left(\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+k)|^2\right)^{\frac{1}{2}}}$$

for $\xi \in \operatorname{supp}\hat{\varphi}$ and 0 otherwise is the tight frame generator for the space V_φ. Therefore, Definition 1.4.3 gives us that

$$\sigma_{V_\varphi}(\xi) = |\hat{\psi}(\xi)|^2 = \frac{|\hat{\varphi}(\xi)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+k)|^2}$$

for $\xi \in \operatorname{supp}\hat{\varphi}$ and 0 otherwise. To prove the additivity property let us consider an orthogonal sum of shift invariant spaces $V = \bigoplus_{n=1}^{\infty} V_n$, with corresponding range functions J_V and J_{V_n}, $n \in \mathbb{N}$. As we mentioned before, Proposition 1.3.1 and formula (1.3.3) imply that $J_V(\xi) = \bigoplus_{n=1}^{\infty} J_{V_n}(\xi)$ for a.e. $\xi \in \mathbb{T}$. Therefore, for a.e. $\xi \in \mathbb{T}$ and $\mathbf{k} \in \mathbb{Z}$, we have

$$\sigma_V(\xi + \mathbf{k}) = \|P_{J(\xi)}\mathbf{e_k}\|^2_{\ell^2(\mathbb{Z})} = \sum_{n=1}^{\infty} \|P_{J_n(\xi)}\mathbf{e_k}\|^2_{\ell^2(\mathbb{Z})} = \sum_{n=1}^{\infty} \sigma_{V_n}(\xi + \mathbf{k});$$

hence, $\sigma_V(\xi) = \sum_{n=1}^{\infty} \sigma_{V_n}(\xi)$ for a.e. $\xi \in \mathbb{R}$. ♣

As an immediate consequence of Definition 1.4.2 or Definition 1.4.3 we obtain the following formula for the spectral function of a space V decomposed into principal shift invariant spaces as in Theorem 1.2.10.

Corollary 1.4.5. *If V is a shift invariant space with a decomposition*

$$V = \bigoplus_{n=1}^{N} V_{\varphi_n},$$

where φ_n is a tight frame generator for V_{φ_n} and N is a natural number or infinity, then

$$\sigma_V(\xi) = \sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$.

We shall list properties of the spectral function we will use in the future. In signal analysis, besides of translations, we encounter two other unitary operators: modulations (theory of Gabor systems) and dilations (theory of wavelets). Let $a \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ the modulation by a of f is given by

$$M_a(f)(x) = e^{2\pi i a x} f(x). \tag{1.4.4}$$

For $m \in \mathbb{N}$ the dilation of f by m is given by

$$D_m(f)(x) = \sqrt{m} f(mx). \tag{1.4.5}$$

If V is a shift invariant space then $M_a(V)$ and $D_m(V)$ are also shift invariant and we will be interested in finding their spectral functions.

PROPOSITION 1.4.6. *The spectral function satisfies the following properties:*

(a) $0 \le \sigma_V \le 1$.

(b) If $V = \bigoplus_{n=1}^{\infty} V_n$, *then* $\sigma_V = \sum_{n=1}^{\infty} \sigma_{V_n}$.

(c) If $V \subset W$, *then* $\sigma_V \le \sigma_W$.

(d) If $V \subset W$, *then* $\sigma_V = \sigma_W$ *if and only if* $V = W$.

(e) $\sigma_V = \mathbf{1}_E$ *if and only if* $\hat{V} = L^2(E)$.

(f) $\sigma_{M_a(V)}(\xi) = \sigma_V(\xi - a)$ *for a.e.* $\xi \in \mathbb{R}$ *and* $a \in \mathbb{R}$.

(g) $\sigma_{D_m(V)}(\xi) = \sigma_V\left(\frac{\xi}{m}\right)$ *for a.e.* $\xi \in \mathbb{R}$ *and* $m \in \mathbb{N}$.

(h) $m_V(\xi) = \sum_{k \in \mathbb{Z}} \sigma_V(\xi + k)$ *for a.e.* $\xi \in \mathbb{T}$.

PROOF.

(a) Follows directly from Definition 1.4.1.

(b) This is part of Definition 1.4.2.

(c) If $V \subset W$ and Z is the orthogonal complement of V in W then Z is shift invariant and by property (b) we have $\sigma_V = \sigma_W - \sigma_Z$. Thus, property (a) implies that $\sigma_V \le \sigma_W$.

(d) If $V \subset W$ and $\sigma_V = \sigma_W$, then using the above argument we obtain $\sigma_Z = 0$; therefore, $Z = \{0\}$.

(e) By Definition 1.4.2 it is enough to prove it in the case when V is a principal shift invariant space. If $\sigma_{V_\varphi} = \mathbf{1}_E$, where φ is a tight frame generator, then $|\hat{\varphi}| = \mathbf{1}_E$ (by Definition 1.4.2), hence $\hat{V}_\varphi = L^2(E)$. The other implication is immediate.

(f) Since the operator M_a is unitary it is enough to prove it for principal shift invariant spaces. Clearly, $M_a(V_\varphi) = V_{M_a(\varphi)}$. Furthermore, the Fourier transform of $M_a(\varphi)$ is equal to $\hat{\varphi}(\cdot - a)$. Theorem 1.2.8 implies that if φ is a tight frame generator for V, then $M_a(\varphi)$ is a tight frame generator for $M_a(V_\varphi)$. Therefore, $\sigma_{M_a(V_\varphi)}(\xi) = |\hat{\varphi}(\xi - a)|^2 = \sigma_V(\xi - a)$ for a.e. $\xi \in \mathbb{R}$.

(g) For the same reasons as above we can assume that V is a principal shift invariant space with a tight frame generator φ. We will show that the system $\{D_m(\varphi)(\cdot - i)\colon i = 0, 1, \ldots, m-1\}$ is a tight frame with constant 1 for the space $D_m(V_\varphi)$. Let φ^k denote the function $\varphi(\cdot - k)$ and φ_i denote the function $D_m(\varphi^i)$, $i = 0, 1 \ldots, m-1$. For every $f \in D_m(V_\varphi)$ we have

$$\|f\|_2^2 = \|(D_m)^{-1} f\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle (D_m)^{-1} f, \varphi^k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle f, D_m(\varphi^k)\rangle|^2$$

$$= \sum_{k\in\mathbb{Z}} \sum_{i=1}^{m-1} |\langle f, D_m(\varphi^{mk+i})\rangle|^2 = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{m-1} |\langle f, (\varphi_i)^k\rangle|^2;$$

therefore, the family $\{\varphi_i\colon i = 0, 1, \ldots, m\}$ is a tight frame for $D_m(V_\varphi)$. By Definition 1.4.3 we obtain that for a.e. $\xi \in \mathbb{R}$

$$\sigma_{D_m(V_\varphi)}(\xi) = \sum_{i=0}^{m-1} |\hat{\varphi}_i(\xi)|^2 = \sum_{i=0}^{m-1} m^{-1} \Big|\hat{\varphi}\Big(\frac{\xi}{m}\Big)\Big|^2 = \Big|\hat{\varphi}\Big(\frac{\xi}{m}\Big)\Big|^2 = \sigma_V\Big(\frac{\xi}{m}\Big).$$

(h) This follows immediately from Proposition 1.3.7 and Corollary 1.4.5. However, we can also prove it directly using Definition 1.4.1. For a.e. $\xi \in \mathbb{T}$ we have

$$\sum_{k\in\mathbb{Z}} \sigma_V(\xi + k) = \sum_{k\in\mathbb{Z}} \|P_{J(\xi)} \mathbf{e}_k\|^2 = \dim J(\xi) = m_V(\xi),$$

which concludes the proof. ♣

Let us also state the following characterization of the completness of tight frames of a special form.

Corollary 1.4.7. *Let $\mathcal{A}$ be a countable family of functions in $L^2(\mathbb{R})$. If the integer shifts of the members of $\mathcal{A}$ form a tight frame with constant 1 for a space V, then $V = L^2(\mathbb{R})$ if and only if*

$$\sum_{\varphi\in\mathcal{A}} |\hat{\varphi}(\xi)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$.

PROOF. Clearly, V is a shift invariant subspace of $L^2(\mathbb{R})$ with the spectral function $\sigma_V = \sum_{\varphi\in\mathcal{A}} |\hat{\varphi}|^2$. By property (d) of Proposition 1.4.6 we obtain that $V = L^2(\mathbb{R})$ if and only if $\sigma_V = \sigma_{L^2(\mathbb{R})}$. To finish the proof, we observe, that property (e) of Proposition 1.4.6 gives us that the spectral function of $L^2(\mathbb{R})$ is almost everywhere equal to 1. ♣

In our further investigations we will also encounter shift invariant spaces, which are approximated by a sequence of shift invariant spaces. We shall find it useful to prove the following approximation lemma

LEMMA 1.4.8. *Let V be a shift invariant space and $\{V_j\}_{j\in\mathbb{N}}$ be a sequence of shift invariant spaces. If for every $f \in L^2(\mathbb{R})$ we have that in the $L^2(\mathbb{R})$–norm*

$$\lim_{j\to\infty} P_{V_j} f = P_V f,$$

where P_{V_j} and P_V denote the orthogonal projections, then there exists a subsequence $\{j_n\}_{n\in\mathbb{N}}$ of the sequence $\{j\}_{j\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sigma_{V_{j_n}}(\xi) = \sigma_V(\xi)$$

for a.e. $\xi \in \mathbb{R}$.

To prove the above lemma we shall need the following:

PROPOSITION 1.4.9. *If V is a shift invariant space with the corresponding range function J, then for every $f \in L^2(\mathbb{R})$ and a.e. $\xi \in \mathbb{T}$*

$$\mathcal{T}(P_V f)(\xi) = P_{J(\xi)}(\mathcal{T} f(\xi)), \tag{1.4.6}$$

where $\mathcal{T}$ is the operator of Proposition 1.3.2 and P_V and $P_{J(\xi)}$ are the orthogonal projections onto V and $J(\xi)$, respectively.

PROOF. The argument, which we are going to present here is due to H. Helson. Let V be a shift invariant space and let M denotes the closed subspace of $L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ given by $\mathcal{T}(V)$. Let P_M be the orthogonal projection

$P_M\colon L^2(\mathbb{T}, \ell^2(\mathbb{Z})) \to M$. Since $\mathcal{T}$ is a unitary operator, we have $\mathcal{T}P_V = P_M\mathcal{T}$. Therefore, to prove that (1.4.6) holds, it is enough to show that for every $f \in L^2(\mathbb{R})$ and a.e. $\xi \in \mathbb{T}$ we have

$$P_M(\mathcal{T}f)(\xi) = P_{J(\xi)}(\mathcal{T}f(\xi)).$$

Since T is onto, we can replace Tf by $F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ in the above equation. Thus, if we define an operator P acting on $F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ by

$$PF(\xi) = P_{J(\xi)}(F(\xi)),$$

where $\xi \in \mathbb{T}$; then, (1.4.6) simply means that $P_M = P$.

To prove the equality between P_M and P we proceed as follows. Since $P_{J(\xi)}$ is an orthogonal projection for a.e. $\xi \in \mathbb{T}$, we immediately obtain that P is an orthogonal projection as well. Thus, if M' denotes the range of P, then all we have to prove is that $M' = M$. The inclusion $M' \subset M$ is obvious. Indeed, for every $F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ and and a.e. $\xi \in \mathbb{T}$ we have $PF(\xi) \in J(\xi)$, which simply means that $PF \in M$ (see Theorem 1.3.5). Since M' is closed, the other inclusion will follow if we prove that every $G \in M$, which is in the orthogonal complement of M' must be zero. Let us, therefore, consider such a G. Theorem 1.3.5 asserts that if $G \in M$ then $G(\xi) \in J(\xi)$ for a.e. $\xi \in \mathbb{T}$. Thus, for every $F \in L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ we have the following:

$$0 = \langle PF, G\rangle = \langle F, PG\rangle = \int_{\mathbb{T}} \langle F(\xi), P_{J(\xi)}G(\xi)\rangle\, d\xi = \int_{\mathbb{T}} \langle F(\xi), G(\xi)\rangle\, d\xi = \langle F, G\rangle,$$

which proves that G is zero and ends the proof. ♣

Proof of Lemma 1.4.8. Let J and J_j, where $j \in \mathbb{N}$, be the range functions corresponding to V and V_j respectively. By the above proposition we have that for every $f \in L^2(\mathbb{R})$

$$\|P_V f - P_{V_j} f\|^2 = \|\mathcal{T}(P_V f - P_{V_j} f)\|^2 = \int_{\mathbb{T}} \|\mathcal{T}(P_V f)(\xi) - \mathcal{T}(P_{V_j} f)(\xi)\|^2\, d\xi$$

$$= \int_{\mathbb{T}} \|P_{J(\xi)}(\mathcal{T}f(\xi)) - P_{J_j(\xi)}(\mathcal{T}f(\xi))\|^2 \, d\xi.$$

Since the above equality holds for every $f \in L^2(\mathbb{R})$, by Theorem 1.3.5 we can replace $\mathcal{T}f(\xi)$ by any vector of the space $\ell^2(\mathbb{Z})$. In particular we can consider the standard basis $\{\mathbf{e_k}\}_{\mathbf{k}\in\mathbb{Z}}$ of $\ell^2(\mathbb{Z})$. Therefore, from our assumption that the projections P_{V_j} approach the projection P_V it follows that

$$\lim_{j\to\infty} \int_{\mathbb{T}} \|P_{J(\xi)}\mathbf{e_k} - P_{J_j(\xi)}\mathbf{e_k}\|^2 \, d\xi = 0.$$

To fix our attention let us assume that $\mathbf{k} = 0$. The above equality allows us to conclude that there exists a subsequence $\{j_n\}_{n\in\mathbb{N}}$ of the sequence $\{j\}_{j\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|P_{J(\xi)}\mathbf{e_0} - P_{J_{j_n}(\xi)}\mathbf{e_0}\| = 0$$

for a.e. $\xi \in \mathbb{R}$. Therefore, for such ξ we have

$$\lim_{n\to\infty} \|P_{J_{j_n}(\xi)}\mathbf{e_0}\| = \|P_{J(\xi)}\mathbf{e_0}\|,$$

hence, by Definition 1.4.1, $\lim_{n\to\infty} \sigma_{V_{j_n}}(\xi) = \sigma_V(\xi)$ for a.e. $\xi \in \mathbb{T}$. It is clear, that if we consider an arbitrary vector $\mathbf{e_k}$, then we shall get a subsequence $\{j_n(\mathbf{k})\}_{n\in\mathbb{N}}$ which depends on $\mathbf{k}$, such that $\lim_{n\to\infty} \sigma_{V_{j_n(\mathbf{k})}}(\xi) = \sigma_V(\xi)$ for a.e. $\xi \in \mathbb{T} + \mathbf{k}$. The standard "diagonal subsequence" argument gives us the existence of a subsequence $\{j_n\}_{n\in\mathbb{N}}$, which does not depend on $\mathbf{k}$, such that $\lim_{n\to\infty} \sigma_{V_{j_n}}(\xi) = \sigma_V(\xi)$ for a.e. $\xi \in \mathbb{T} + \mathbf{k}$, and therefore, for a.e. $\xi \in \mathbb{R}$. ♣

As the first application of the approximation lemma we have the following:

Proposition 1.4.10.

(A) Let $\{V_j\}_{j\in\mathbb{N}}$ be a decreasing sequence of shift invariant spaces and V be a shift invariant space contained in V_j for every $j \in \mathbb{N}$. Then

$$\bigcap_{j\in\mathbb{N}} V_j = V \quad \textit{if and only if} \quad \lim_{j\to\infty} \sigma_{V_j}(\xi) = \sigma_V(\xi) \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

(B) Let $\{V_j\}_{j\in\mathbb{N}}$ be an increasing sequence of shift invariant spaces and V be a shift invariant space containing V_j for every $j \in \mathbb{N}$. Then

$$\bigcup_{j\in\mathbb{N}} V_j = V \quad \textit{if and only if} \quad \lim_{j\to\infty} \sigma_{V_j}(\xi) = \sigma_V(\xi) \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

Proof. By considering the orthogonal complements $V_j \ominus V$ in the case (A), and $V \ominus V_j$ in the case (B), where $j \in \mathbb{N}$, and using the additivity property of the spectral function, we can assume, without the loss of generality, that V_j is decreasing and $V = \{0\}$. To prove this simple case we proceed as follows. If $\bigcap_{j\in\mathbb{N}} V_j = \{0\}$ then the orthogonal projections onto V_j satisfy $P_{V_j} f \to 0$ in $L^2(\mathbb{R})$ for every $f \in L^2(\mathbb{R})$. By Lemma 1.4.8 there is a subsequence $\{j_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \sigma_{V_{j_n}}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}$. However, from property (c) of Proposition 1.4.6 it follows that the sequence $\{\sigma_{V_j}(\xi)\}_{j\in\mathbb{N}}$ is decreasing for a.e. $\xi \in \mathbb{R}$, therefore, we obtain that $\lim_{j\to\infty} \sigma_{V_j}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}$. On the other hand, if the limit is equal to zero, then $V' = \bigcap_{j\in\mathbb{N}} V_j$ must be the zero subspace. Indeed, since V' is a shift invariant space contained in V_j for every $j \in \mathbb{N}$, by property (c) of Proposition 1.4.6 we get, that $\sigma_V(\xi) \le \sigma_{V_j}(\xi)$ for every $j \in \mathbb{N}$ and a.e. $\xi \in \mathbb{R}$. Therefore, if we take the limit when $j \to \infty$ we obtain that $\sigma'_V(\xi) = 0$ for a.e. $\xi \in \mathbb{R}$, thus $V' = \{0\}$. ♣

In the following paragraphs we will apply the functional calculus to topics of interest to us. The next paragraph is devoted to a characterization of the completeness of orthonormal wavelet systems. We will present a brief history of this problem and show how the theory of the spectral function can be used to solve it.

1.5 Characterization of wavelets

The theory of wavelets took its origin in the continuous wavelet transform, which is defined for $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, $a > 0$, as

$$\mathcal{W}f(a,b) = \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{a}} \overline{\psi\Big(\frac{x-b}{a}\Big)}\, dx,$$

where ψ is a given function in $L^2(\mathbb{R})$. If for any $\psi \in L^2(\mathbb{R})$ and a, b as above we define

$$\psi^{a,b}(x) = \frac{1}{\sqrt{a}}\psi\Big(\frac{x-b}{a}\Big),$$

then the continuous wavelet transform is just

$$\mathcal{W}f(a,b) = \langle f, \psi^{a,b}\rangle_{L^2(\mathbb{R})}.$$

A basic result in continuous wavelet theory is to find a condition on ψ, which would allow us to reproduce every function belonging to $L^2(\mathbb{R})$ from its continuous wavelet transform. It turns out that the proper condition is

$$\int_{\mathbb{R}+} |\hat{\psi}(a\xi)|^2 \frac{da}{a} = 1 \tag{1.5.1}$$

for a.e. $\xi \in \mathbb{R}$ and, what is sometimes called as the *Calderón reproducing formula*, reads as follows:

$$f = \int_{\mathbb{R}}\int_{\mathbb{R}+} \langle f, \psi^{a,b}\rangle \psi^{a,b}\, \frac{da\, db}{a^2} \tag{1.5.2}$$

holds for $f \in L^2(\mathbb{R})$ if and only if (1.5.1) holds (see [Da]). Condition (1.5.1) is referred to as the *Calderón condition* and by a simple change of variables is equivalent to

$$\int_{\mathbb{R}+} |\hat{\psi}(a)|^2 \frac{da}{|a|} = 1 = \int_{\mathbb{R}-} |\hat{\psi}(a)|^2 \frac{da}{|a|}.$$

The occurence of the measure $\frac{da\,db}{a^2}$ in (1.5.2), as well as the continuous wavelet transform itself, can be understood, in a nice way, from the group representation point of view.

Let G be the affine group associated with $\mathbb{R}$. That is, G consists of all ordered pairs of the form $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ and is equiped with the group operation

$$(a,b)(c,d) = (ac, ad+b).$$

The name of the group is justified by its action on $x \in \mathbb{R}$:

$$(a,b)x = ax + b.$$

This is also the reason why G is also called the *motion group* of $\mathbb{R}$, which in turn explains its appearance in the signal analysis theory. Because of the way the group acts on $\mathbb{R}$ we can consider the following unitary representation of G on $L^2(\mathbb{R})$:

$$T_g\psi(x) = \frac{1}{\sqrt{a}}\psi(g^{-1}x) = \psi^{a,b}(x),$$

where $g = (a,b) \in G$ and $g^{-1}x = \frac{x-a}{b}$ is the action of the inverse of g on x. Then, the Calderón reproducing formula (1.5.2) becomes

$$f = \int_G \langle f, T_g\psi\rangle T_g\psi \, d\lambda(g),$$

where λ is the left Haar measure on G:

$$d\lambda(a,b) = \frac{da\,db}{a^2}.$$

Before we turn to orthonormal wavelets let us dwell a little upon the order of dilations and translations in the definition of the continuous wavelet transform (the discussion we are going to present is due to G. Weiss and M. Paluszyński). We can see, that the functions $\psi^{a,b}$ are defined with respect to a special order of dilations and translations. In fact, we first dilate the function ψ by a and obtain $\frac{1}{\sqrt{a}}\psi(\frac{\cdot}{a})$ and then we translate it by b to get $\psi^{a,b}$. It is naturall to ask what will happen if we reverse the order of this operations and consider a parallel continuous wavelet transform:

$$\mathcal{W}'f(a,b) = \int_{\mathbb{R}} f(x)\frac{1}{\sqrt{a}}\overline{\psi\Big(\frac{x}{a} - b\Big)}\,dx.$$

It turns out that $\mathcal{W}'$ is in some way equivalent to $\mathcal{W}$. In fact, if we consider a group G' consisting of all ordered pairs $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$, with the group operation

$$(a, b)(c, d) = (ac, \frac{b}{c} + d),$$

which acts on $\mathbb{R}$ by $(a, b)x = ax + b$, then the action of the inverse of $(a, b) \in G'$ on $x \in \mathbb{R}$ is given by $\frac{x}{a} - b$, which leads to the following unitary representation of G' on $L^2(\mathbb{R})$

$$T'_g\psi(x) = \frac{1}{\sqrt{a}}\psi(g^{-1}x) = \frac{1}{\sqrt{a}}\psi\Big(\frac{x}{a} - b\Big).$$

Therefore,

$$\mathcal{W}'f(a, b) = \langle f, T'_{(a,b)}\psi\rangle$$

and we can ask for a parallel version of the Calderón reproducing formula. It is shown that the parallel version reads as follows:

$$f = \int_{G'} \langle f, T'_g\psi\rangle T'_g\psi \, d\lambda'(g) \tag{1.5.3}$$

holds for every $f \in L^2(\mathbb{R})$ if and only if (1.5.1) holds, where λ' is the left Haar measure on G' given by

$$d\lambda'(a, b) = \frac{da\, db}{a}.$$

As we can see the Calderón condition in both cases is the same, which allows us to conclude that the order of dilations and translations in $\mathcal{W}$ has no essential importance and can be freely reversed.

Because of the Calderón reproducing formula the system $\{\psi^{a,b}\}_{(a,b)\in\mathbb{R}^+\times\mathbb{R}}$ can be viewed as a "continuous"orthonormal basis of $L^2(\mathbb{R})$. Therefore, we can ask if we can find a natural discrete subset Λ of $\mathbb{R}^+ \times \mathbb{R}$, and a function $\psi \in L^2(\mathbb{R})$ such that the system $\{\psi^{a,b}\}_{(a,b)\in\Lambda}$ will be an orthonormal basis of $L^2(\mathbb{R})$. It turns out that such a natural discrete subset, for which "nice" wavelets ψ exist, is $\{(2^j, k2^j)\}_{j,k\in\mathbb{Z}}$. The sequence gives a rise to the system

$$\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j \cdot -k), \tag{1.5.4}$$

where $j, k \in \mathbb{Z}$, which serves as a base for definition of wavelets. This already indicates that the continuous wavelet transform with the reversed order of dilations and translations, that is $\mathcal{W}'$, is better suited for discretization then the original one. In fact, the analogous discrete subset for $\mathcal{W}'$, which gives us the same system as in (1.5.4), is just $\Lambda' = \{(2^j, k)\}_{j,k\in\mathbb{Z}}$. However, if we really want to make the proces of discretization transparent, we have to replace a in $\mathcal{W}'$ by 2^t, where $t \in \mathbb{R}$ and see what happens.

Formula (1.5.3) reads

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}+}\int_{\mathbb{R}} f(y)\frac{1}{\sqrt{a}}\overline{\psi\left(\frac{y}{a}-b\right)}\,dy\frac{1}{\sqrt{a}}\psi\left(\frac{x}{a}-b\right)\frac{da\,db}{a}.$$

If we replace a by 2^t, where $t \in \mathbb{R}$, then the above becomes

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(y)2^{\frac{t}{2}}\overline{\sqrt{\ln 2}\psi\left(2^t y - b\right)}\,dy\,2^{\frac{t}{2}}\sqrt{\ln 2}\psi\left(2^t x - b\right)\,dt\,db.$$

The same change of variables in (1.5.1) gives us

$$\int_{\mathbb{R}} |\sqrt{\ln 2}\hat{\psi}(2^t\xi)|^2\,dt = 1$$

for a.e. $\xi \in \mathbb{R}$. Therefore, if we drop the redundant term $\sqrt{\ln 2}$ and abuse locally the notation by introducing

$$\psi_{j,k}(x) = \psi(2^j x - k),$$

where j and k are real numbers, then the Calderón reproducing formula takes the following form:

$$f = \int_{\mathbb{R}}\int_{\mathbb{R}}\langle f, \psi_{j,k}\rangle\psi_{j,k}\,dj\,dk \tag{1.5.5}$$

holds for every $f \in L^2(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} |\hat{\psi}(2^j \xi)|^2 \, dj = 1 \tag{1.5.6}$$

holds for a.e. $\xi \in \mathbb{R}$. Within this framework the process of discretization becomes clear. Instead of taking j and k real we assume that they are integer and we are looking for a function $\psi \in L^2(\mathbb{R})$ such that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Two deep problems arise at this point. The first one is the very existence of a ψ such that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal. This fact in general is highly non–obvious (for example, we will show later that another "naturall" discretization of the continuous wavelet transform: $\{2^{\frac{j}{2}}\psi(2^j(x-k)\}_{j,k\in\mathbb{Z}}$ can never give a rise to an orthonormal system); however, the well known example of the Haar basis assures that such a ψ always exists. Thus, we are only left with the second problem, that is: if we find a function ψ such that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a family of orthonormal vectors, how to assure the completeness of this system? To get an insight to this problem let us discretize the Calderón reproducing formula given by (1.5.5) and (1.5.6). The discrete version of (1.5.5) is

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} \tag{1.5.7}$$

for every $f \in L^2(\mathbb{R})$, and this of (1.5.6) is

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$. Since, the former is equivalent to the completeness of the orthonormal system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, the discrete version of the Calderón formula should read:

Conjecture 1.5.1 (Discrete Calderón reproducing formula). *If $\psi \in L^2(\mathbb{R})$ and the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal, then it is complete if and only if* *the following* <u>*discrete Calderón condition*</u> *holds*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \tag{1.5.8}$$

for a.e. $\xi \in \mathbb{R}$.

The above conjecture is due to G. Weiss and its way to become a theorem has an interesting history. In 1985, Y. Meyer applied the Poisson summation formula to (1.5.7) and concluded, that (1.5.7) is equivalent to (1.5.8) and

$$t_q(\xi) := \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+q))} = 0, \tag{1.5.9}$$

which holds for a.e. $\xi \in \mathbb{R}$ and all $q \in 2\mathbb{Z}+1$ (see [Le2]). His original argument, however, was never published, probably because of its technical difficulties. Some of these difficulties were overcome in 1986 by P.G. Lemarié (a former student of Y. Meyer) and he presented the above characterization in [Le1] with mild assumptions on the decay of the Fourier transform of ψ (however, the paper does not include a proof of this result). The first rigorous proof of Meyer's characterization was published in 1993 by A. Bonami, F. Soria and G. Weiss in [BSW], where, for simplicity, band–limited wavelets were considered. Later, this was extended to a general case by Y-H. Ha, H. Kang, J. Lee and J.K. Seo (see [HKLS]).

It may appear, that this result indicates that the discrete Calderón formula alone is too weak to imply the completeness. However, P.G. Lemarié clearly indicated in [Le2], that Y. Meyer was aware of the extreme redundancy of this characterization (both of them have used this redundancy to construct orthonormal wavelets). This was also remarked in [BSW]. To explain this redundancy, let us point out, that (1.5.7) alone is equivalent to saying that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a tight frame with constant 1 for $L^2(\mathbb{R})$. Since a family of unit vectors forms a tight frame with constant 1 for a Hilbert space $\mathbb{H}$ if and only if it forms an orthonormal basis of $\mathbb{H}$, it is completely clear, that (1.5.7) and a simple assumption $\|\psi\|_2 = 1$ already imply that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal. In 1995, G. Grippenberg and X. Wang (a graduate student of G. Weiss) proved independently that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ (not necessarily orthonormal) forms a tight frame with constant 1 for $L^2(\mathbb{R})$ if and only if (1.5.8) and (1.5.9) hold (see [Gr] and [Wa]). Both authors stated explicitly, that (1.5.8) and (1.5.9) are attributed to Y. Meyer, and both of them advertised their results as the following characterization of wavelets:

THEOREM 1.5.2. *If $\psi \in L^2(\mathbb{R})$ then the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ if and only if both (1.5.8) and (1.5.9) hold and $\|\psi\|_2 = 1$.*

The only question, which remained, was if it is possible to avoid this redundancy and find an elegant characterization of the completeness of orthonormal wavelet systems. This question was posed many times by G. Weiss, one of them being at a wavelet seminar at Washington University in 1999. During the seminar it was shown by the author that if $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal system which spans a shift invariant space, then even a slightly weaker condition than (1.5.8), that is

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)| > 0 \quad \text{a.e.},$$

implies the completeness of the system. A question, if (1.5.8) alone implies already the completeness in the general case, was raised by M. Bownik. In the answer to this question G. Weiss pointed out, that since (1.5.8) is a discrete version of the Calderón condition, it should be equivalent to the completeness of the system, provided the system is orthonormal. It turned out to be true, as proved the day after the seminar by M. Bownik and two weeks later by the author. The argument presented by M. Bownik is very clever and boils down to the following observation. If $\tilde{G}$ is an infinite matrix representing an operator bounded on $\ell^2(\mathbb{Z})$, which satisfies $\tilde{G} \leq I$ (I is the identity matrix) and all the terms on the diagonal of $\tilde{G}$ are equal to 1, then $\tilde{G}$ must be equal to the identity matrix. The matrix $\tilde{G}$, which we use in our situation depends on ξ and as easy to predict the terms on the diagonal of $\tilde{G}(\xi)$ are equal to the lefthand side of (1.5.8) and the terms off the diagonal are given by $t_q(\xi)$ as defined in (1.5.9). The existence of such a matrix $\tilde{G}(\xi)$ and the inequality $\tilde{G}(\xi) \leq I$ follows from the orthonormality of the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$. This argument shows in a very transparent manner that the orthogonality together with the discrete Calderón formula imply that (1.5.9) holds, which, in turn, gives us the completeness of the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and proves the conjecture.

At this point we have to give a proper credit to other researchers without whom Conjecture 1.5.1 would still be an unsolved and challenging problem. A. Ron and Z. Shen have developed recently a very powerful theory of *Gramians* and *dual Gramians*, which provides a lot of help in studying shift invariant systems. Their main concern was to understand under which conditions integer shifts of of a given family of functions form a frame, a tight frame or a Bessel family. The results contained in [RS1], [RS2] and [RS3] constitute a whole theory with broad applications. The founders of the theory applied it to the study of Gabor systems and *affine systems*, that is, the systems of the form $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$. Clearly, the Gabor systems are shift invariant, so they fit exactly to the framework of the theory of Gramians and gual Gramians. However, the affine systems are not and this provides an obvious obstacle in applying directly the theory to such systems. This obstacle was overcome in a, let us exaggerate a little, astonishing way. In [RS2] A. Ron and Z. Shen intoduced *quasi affine systems* $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ defined by

$$\tilde{\psi}_{j,k}(x) = \begin{cases} \psi_{j,k}(x) & \text{if} \quad j \geq 0, k \in \mathbb{Z}, \\ 2^j\psi(2^j(x-k)) & \text{if} \quad j < 0, k \in \mathbb{Z} \end{cases}$$

and proved that these systems are equivalent to affine systems, that is, if we are concerned about frames, tight frames or Bessel families, then we can freely raplace affine systems with quasi affine systems, moreover, the frame constants and the Bessel constants are the same for both of the systems. This equivalency was proved originally with mild assumptions on the decay of $\hat{\psi}$. The further investigation of the "quasi affine phenomenon" due to C.K. Chui, X. Shi and J. Stöckler explained it to the end. In [CSS] it was shown that the decay assumption is not essential and the equivalence between $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ holds for every function ψ in $L^2(\mathbb{R})$. Of course, the main advantage and the origin of the quasi affine system is its shift invariance, which allows for applying the theory of Gramians and dual Gramians to such a system. The matrix $\tilde{G}(\xi)$ we were mentioning before is a dual Gramian of a quasi affine system $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$. Theorem 1.5.2 is an immediate consequence of a result which asserts that the system $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ with constants A and B if and only if $AI \leq \tilde{G}(\xi) \leq BI$ for a.e. $\xi \in \mathbb{T}$. A similar result is that $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a Bessel family with a constant B if and only if $\tilde{G}(\xi) \leq BI$ for a.e. $\xi \in \mathbb{T}$. These two results together with the "quasi affine phenomenon" allow us to get the following:

The system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal $\Rightarrow$ $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Bessel family with constant 1 $\Rightarrow$ $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Bessel family with constant 1 $\Rightarrow$ $\tilde{G}(\xi) \leq I$,

therefore, if in addition (1.5.8) holds, then we use the aforementioned matrix argument to get

$\tilde{G}(\xi) \leq I$ and (1.5.8) $\Rightarrow \tilde{G}(\xi) = I \Rightarrow \{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ is a tight frame with constant 1 for $L^2(\mathbb{R})$ $\Rightarrow$ $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a tight frame with constant 1 for $L^2(\mathbb{R})$ $\Rightarrow$ $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is complete.

The argument we just presented is the hard core of [Bo2]. It relies on the "quasi affine phenomenon" as well as the theory of dual Gramians and, as we mentioned before, proved Conjecture 1.5.1. We would like to present a conceptually different proof of the same result. The idea is very simple and uses the functional calculus of shift invariant spaces we developed in the previous paragraph.

Let ψ be a function in $L^2(\mathbb{R})$ such that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal. For every $j \in \mathbb{Z}$ define a space

$$W_j = \overline{\mathrm{span}\{\psi_{j,k}\}_{k\in\mathbb{Z}}}. \tag{1.5.10}$$

Clearly, these spaces are mutually orthogonal and for every $j \in \mathbb{Z}$ we have $D^j(W_0) = W_j$, where D is the standard "dilation by 2" operator defined by

$$Df(x) = \sqrt{2} f(2x)$$

for $f \in L^2(\mathbb{R})$. Since the integer translates of ψ form an orthonormal basis for W_0 we obtain a very simple form for the spectral function of W_0:

$$\sigma_{W_0}(\xi) = |\hat{\psi}(\xi)|^2$$

(see the remark after Definition 1.4.2). Since $D^j(W_0) = W_j$ for $j \in \mathbb{Z}$, by property (g) of Proposition 1.4.6 we obtain

$$\sigma_{W_j}(\xi) = \sigma_{W_0}(2^{-j}\xi) = |\hat{\psi}(2^{-j}\xi)|^2 \tag{1.5.11}$$

Of course, only if $j \geq 0$ we can be sure that the space W_j is shift invariant, however, we shall pretend for a moment that property (g) of Proposition 1.4.6 allows us to extend the definition of the spectral function to the spaces, which are dilates of shift invariant spaces (that is, are invariant under the shifts by elements of $2^j\mathbb{Z}$ for some fixed $j \in \mathbb{Z}$). The completeness of the system $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is equivalent to saying that

$$\bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{R}),$$

which in turn is equivalent to

$$\sum_{j \in \mathbb{Z}} \sigma_{W_j}(\xi) = 1,$$

by properties (b), (d) and (e) of Proposition 1.4.6. Therefore, formula (1.5.11) allows us to conclude, that the completeness of the system $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is equivalent to

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^{-j}\xi)|^2 = 1,$$

which is the discrete Calderón condition.

As we can see, the only obstacle which makes this proof unformal, is the fact, that the spaces W_j are not necessarily shift invariant if $j < 0$. This is exactly the same problem, which originated the introduction of quasi affine systems. However, we shall overcome the obstacle in a different manner. The key is the following result which extends a result due to M. Paluszyński (see Theorem 1.6 §2.1 of [HW]).

PROPOSITION 1.5.3. *If V is a shift invariant space whose spectral function belongs to $L^1(\mathbb{R})$, then*

$$\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}.$$

PROOF. Let us suppose that there exists a non–zero function $f \in \bigcap_{j\in\mathbb{Z}} D^j(V)$. We can, therefore, assume that $\|f\|_2 = 1$. Since $f \in \bigcap_{j\geq 0} D^{-j}(V)$, we have

$$D^j f \in V$$

for $j \geq 0$. Therefore, if we apply Theorem 1.2.10 and use the notation of this theorem we obtain

$$2^{-\frac{j}{2}}\hat{f}(2^{-j}\xi) = \sum_{n=1}^{N} r_n^j(\xi)\hat{\varphi}_n(\xi), \tag{1.5.12}$$

where r_n^j is 1–periodic with $\operatorname{supp} r_n^j \subset \Omega_n$ and

$$\sum_{n=1}^{N} \|r_n^j\|^2_{L^2(\mathbb{T})} = \|D^j f\|_2^2 = 1. \tag{1.5.13}$$

By (1.5.12) we obtain that for $j \geq 0$

$$\hat{f}(\xi) = 2^{\frac{j}{2}} \sum_{n=1}^{N} r_n^j(2^j\xi)\hat{\varphi}_n(2^j\xi).$$

We will show, that $\hat{f}(\xi) = 0$ for a.e. $\xi \in [1,2]$. By Cauchy–Schwartz inequality we have

$$\int_1^2 |\hat{f}(\xi)|\, d\xi \leq 2^{\frac{j}{2}} \int_1^2 \sum_{n=1}^{N} |r_n^j(2^j\xi)\hat{\varphi}_n(2^j\xi)|\, d\xi$$

$$\leq 2^{\frac{j}{2}} \int_1^2 \Big(\sum_{n=1}^{N} |r_n^j(2^j\xi)|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{N} |\hat{\varphi}_n(2^j\xi)|^2\Big)^{\frac{1}{2}}\, d\xi$$

$$\leq 2^{\frac{j}{2}} \Big(\int_1^2 \sum_{n=1}^{N} |r_n^j(2^j\xi)|^2\, d\xi\Big)^{\frac{1}{2}} \Big(\int_1^2 \sum_{n=1}^{N} |\hat{\varphi}_n(2^j\xi)|^2\, d\xi\Big)^{\frac{1}{2}}$$

$$= \Big(2^{-j} \int_{2^j}^{2^{j+1}} \sum_{n=1}^{N} |r_n^j(\xi)|^2\, d\xi\Big)^{\frac{1}{2}} \Big(\int_{2^j}^{2^{j+1}} \sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2\, d\xi\Big)^{\frac{1}{2}}$$

$$= \Big(\int_{2^j}^{2^{j+1}} \sigma_V(\xi)\, d\xi\Big)^{\frac{1}{2}},$$

where, in the last equality, we used (1.5.13) and the 1–periodicity of r_n^j to derive that for every $j \geq 0$

$$\Big(2^{-j} \int_{2^j}^{2^{j+1}} \sum_{n=1}^{N} |r_n^j(\xi)|^2\, d\xi\Big)^{\frac{1}{2}} = 1,$$

as well as Corollary 1.4.5 to replace $\sum_{n=1}^{N} |\hat{\varphi}_n(\xi)|^2$ by $\sigma_V(\xi)$. Since the estimate

$$\int_1^2 |\hat{f}(\xi)|\, d\xi \leq \Big(\int_{2^j}^{2^{j+1}} \sigma_V(\xi)\, d\xi\Big)^{\frac{1}{2}},$$

we just proved, holds for every $j \geq 0$, we can use the Dominated Convergence Theorem and the assumption that $\sigma_V \in L^1(\mathbb{R})$ to conclude that $\hat{f}(\xi) = 0$ for a.e. $\xi \in [1,2]$.

To prove that $\hat{f}(\xi) = 0$ for a.e. $\xi \geq 0$ we argue as follows. For every $i \in \mathbb{Z}$ we have that $D^i f \in \bigcap_{j\in\mathbb{Z}} D^j(V)$, therefore, the above argument shows that $(D^i f)^{\widehat{}}(\xi) = 0$

for a.e. $\xi \in [1,2]$, that is, $\hat{f}(\xi) = 0$ for a.e. $\xi \in [2^{-i}, 2^{-i+1}]$, hence, it follows that $\hat{f}(\xi) = 0$ for a.e. $\xi \geq 0$. Clearly, we can apply the same kind of reasoning for $\xi \leq 0$ to obtain that $\hat{f}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}$, which contradicts our assumption that f is a non–zero function and ends the proof. ♣

We shall use the above proposition to prove Conjecture 1.5.1. However, our approach does not allow us to show that the discrete Calderón condition (1.5.8) is necessary for the system $\{\psi_{j,k}\}$ to be an orthonormal basis. One of the reasons why the necessity is not clear from our point of view, is that the propostition we just proved provides only a sufficient but not a necessary condition under which the intersection $\bigcap_{j\in\mathbb{Z}} D^j(V)$ is equal to $\{0\}$. The example: $\hat{V} = L^2([1,\infty))$ shows that this is indeed the case. Of course, the necessity of the condition follows from Theorem 1.5.2, so we can concentrate on proving the sufficiency of this condition.

THEOREM 1.5.4. *The discrete Calderón reproduction formula holds.*

PROOF. Let ψ be a function in $L^2(\mathbb{R})$ such that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal. Let W_j, $j \in \mathbb{Z}$, be the space defined in (1.5.10). As we pointed out before the spaces W_j are mutually orthogonal for all $j \in \mathbb{Z}$ and shift invariant for $j \geq 0$. Let Z denote the orthogonal complement of $\bigoplus_{j\in\mathbb{Z}} W_j$ in $L^2(\mathbb{R})$. The system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is complete if and only if the space Z is equal to $\{0\}$. Therefore, we shall concentrate on proving that the discrete Calderón condition:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$ implies that $Z = \{0\}$.

Let us consider

$$V = \Big(\bigoplus_{j\geq 0} W_j\Big)^{\perp}.$$

Since W_j is shift invariant for $j \geq 0$ we obtain that V is shift invariant as well.

Moreover, we have

$$D^i(V) = \Big(\bigoplus_{j\geq i} W_j\Big)^{\perp}$$

for $i \in \mathbb{Z}$, where, as we recall, D denotes the "dilation by 2" operator. Thus, we have

$$Z \subset \Big(\bigoplus_{j\geq i} W_j\Big)^{\perp} = D^i(V)$$

for every $i \in \mathbb{Z}$ and therefore

$$Z \subset \bigcap_{i\in\mathbb{Z}} D^i(V),$$

which motivates us to use Proposition 1.5.3, to prove that $Z = \{0\}$. In order to do this we have to show that the spectral function of the space V is integrable. Since

$$L^2(\mathbb{R}) = V \oplus \bigoplus_{j\geq 0} W_j,$$

by properties (b) and (e) of Proposition 1.4.6 we obtain that

$$1 = \sigma_V(\xi) + \sum_{j=0}^{\infty} \sigma_{W_j}(\xi). \tag{1.5.14}$$

for a.e. $\xi \in \mathbb{R}$. Let us also recall, that in (1.5.11) we already computed the spectral functions of the spaces W_j, $j \geq 0$, with the use of property (g) of Proposition 1.4.6 and they are given by

$$\sigma_{W_j}(\xi) = |\hat{\psi}(2^{-j}\xi)|^2.$$

Therefore, from (1.5.14) we obtain

$$\sigma_V(\xi) = 1 - \sum_{j=0}^{\infty} |\hat{\psi}(2^{-j}\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$, so if we use the discrete Calderón condition, then the above gives us

$$\sigma_V(\xi) = \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 - \sum_{j=0}^{\infty} |\hat{\psi}(2^{-j}\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2.$$

This already implies that σ_V is integrable. In fact,

$$\int_{\mathbb{R}} \sigma_V(\xi)\, d\xi = \int_{\mathbb{R}} \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2\, d\xi = \sum_{j=1}^{\infty} 2^{-j} \int_{\mathbb{R}} |\hat{\psi}(\xi)|^2\, d\xi = \|\hat{\psi}\|_2^2 = 1,$$

hence, from Proposition 1.5.3 if follows that $Z = \{0\}$, that is, the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is complete. ♣

Let us conclude this paragraph by presenting a very interesting observation due to D. Speegle.

Corollary 1.5.5. *If $\psi \in L^2(\mathbb{R})$ and the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal, then it is complete if and only if*

$$\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} = 2\ln 2. \tag{1.5.15}$$

Proof. Let us denote the set $[-2,-1]\cup[1,2]$ by S. We shall use the following standard argument

$$\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} = \sum_{j\in\mathbb{Z}} \int_{2^jS} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} = \int_S \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 \frac{d\xi}{|\xi|}. \tag{1.5.16}$$

From the above we can see that (1.5.15) follows immediately from the discrete Calderón condition (1.5.8). The other implication relies on the fact that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is orthonormal.

Let $W = \bigoplus_{j\geq 0} W_j$, where W_j defined as in (1.5.10). By (1.5.11) and property (b) of Proposition 1.4.6 we have

$$\sigma_W(\xi) = \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$. Since $\sigma_W \le 1$ a.e. (see property (a) of Proposition 1.4.6), for every $N \in \mathbb{N}$ we have

$$\sum_{j=-\infty}^{N} |\hat{\psi}(2^j\xi)|^2 = \sigma_W(2^N\xi) \le 1$$

for a.e. $\xi \in \mathbb{R}$, which implies that

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 \le 1,$$

for a.e. $\xi \in \mathbb{R}$. Therefore, if we assume that (1.5.15) holds, then (1.5.16) and the above give us

$$2\ln 2 = \int_S \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 \frac{d\xi}{|\xi|} \le \int_S 1 \frac{d\xi}{|\xi|} = 2\ln 2,$$

which proves that (1.5.8) holds. ♣

1.6 Generalized Multiresolution Analysis

In the previous paragraph we have shown how the continuous wavelet transform gave rise to orthonormal wavelets. It is not clear if the theory of the continuous wavelet transform does imply by itself, that there exist functions $\psi \in L^2(\mathbb{R})$ such that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{R})$. However, the well known Haar basis and Shannon basis indicate that in fact it is possible to find such ψ's. Because ψ corresponding to the Haar basis is a sum of two characteristic functions and the Fourier transform of ψ corresponding to the Shannon basis is a characteristic function, this displays an anology to the Balian-Low theorem for Gabor systems (see Theorem 1.8.1). An equivalent version of this theorem for wavelets might assert that, essentially, every wavelet ψ has to exhibite bad behaviour either in the time domain or on the Fourier transform side. When he attempted to establish this fact Y. Mayer discovered that it is not true and, instead, wavelets, that are in the Schwartz class do exists (see [Le1] and [Le2] for this historical account).

Having done this he then tried to extend his construction to a multi–dimensional setting. In the case of the Haar basis this can be easily done because if ψ is the Haar wavelet then there is a complementary function φ such that three functions: $\varphi \otimes \psi$, $\psi \otimes \varphi$ and $\psi \otimes \psi$, give rise to an orthonormal wavelet basis of $L^2(\mathbb{R}^2)$ (an analogous construction works for $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$). Y. Meyer, instead of carrying over his calculations in R^n, simply conjectured that his wavelet ψ also has a complementary function φ which allows one to obtain multi–dimensional wavelets via tensor products. The conjecture turned out to be true as shown by P.G. Lemarié who constructed this complementary function. A "complementary" function φ of a wavelet ψ is what we call today a scaling function associated to ψ. This was the origin of the idea of the multiresolution analysis, which was developed in 1986 by Y. Meyer and S. Mallat. The idea was fully presented in the famous paper *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$ by S. Mallat. The original definition given in [Ma] says that a multiresolution approximation (analysis) of $L^2(\mathbb{R})$ is an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ invariant under the shifts $2^{-j}k$, where $k \in \mathbb{Z}$, with the sum $\bigcup_{j\in\mathbb{Z}} V_j$ being dense in $L^2(\mathbb{R})$, the intersection $\bigcap_{j\in\mathbb{Z}} V_j$ equal to the zero subspace $\{0\}$; and satysfying $D(V_j) = V_{j+1}$ (D is the standard "dilation by 2" operator). Moreover, there exists an isomorphism I from V_0 onto $\ell^2(\mathbb{Z})$ which commutes with the action of $\mathbb{Z}$ (the action over V_0 is the translation of functions by integers and the action over $\ell^2(\mathbb{Z})$ is the usual translation). In the paper, S. Mallat has shown that the last condition is equivalent to saying that there exists a function $\varphi \in V_0$ whose integer shifts form an orthonormal basis of V_0. This and other improvements have led to the following (see [Da], [HW]):

Definition 1.6.1. *A multiresolution analysis (MRA) of* $L^2(\mathbb{R})$ *is a sequence* $\{V_j\}_{j\in\mathbb{Z}}$ *of closed subspaces of* $L^2(\mathbb{R})$ *such that the following hold:*

(M1) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$,

(M2) $D(V_j) = V_{j+1}$ *for all* $j \in \mathbb{Z}$,

(M3) $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$,

(M4) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(M5) V_0 *is shift invariant,*

(M6) there exists a function $\varphi \in V_0$ *such that* $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis of* V_0.

REMARK. The function φ, which appears in condition (M6) is called a *scaling function.*

Seemingly, the above definition has nothing in common with wavelets. S. Mallat has shown, however, that if $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis, then we can always find a function ψ belonging to $V_1 \ominus V_0$, which is an orthonormal wavelet. Although this method of finding wavelets proved to be very succesful it was still not clear if that was the only way of doing so. The situation became clear when Journé found an example of a wavelet, which can not be obtained from an MRA. Neverless, P.G. Lemarié and P. Auscher have shown that all "nice" wavelets can be obtained from a multiresolution analysis (see [Le3], [Le4], [Au]). Since all the above results are basics to wavelet theory let us translate them into the framework of shift invariant spaces.

We will need the following easy application of Theorem 1.3.8.

PROPOSITION 1.6.2. *Let* V *be a shift invariant space. There exists a function* $\varphi \in V$ *such that* $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis of* V *if and only if the multiplicity function of* V *is a.e. equal to 1.*

PROOF. We have mentioned already that the translates of a $\varphi \in L^2(\mathbb{R})$ are orthonormal if and only if

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2 = 1 \tag{1.6.1}$$

holds for a.e. $\xi \in \mathbb{R}$. Therefore, Proposition 1.3.7 already implies that if the integer shifts of φ form an orthonormal basis of V, then $m_V(\xi) = 1$ for a.e. $\xi \in \mathbb{R}$. The other implication makes use of Theorem 1.3.8. If $m_V(\xi) = 1$ a.e., then from this theorem it follows that

$$V = V_\varphi, \tag{1.6.2}$$

where φ is a tight frame generator for V_φ and the spectrum of V_φ is equal to $\mathbb{R}$. Therefore, by Theorem 1.2.8 we obtain, that

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2 = \mathbf{1}_{\mathbb{R}}(\xi),$$

hence, (1.6.1) shows that the system $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is orthonormal. To finish the proof we observe, that, by (1.6.2), the system is complete in V. ♣

The second ingredient will be the following formula for the multiplicity function, which is also proved in [BMM]:

Proposition 1.6.3. *If V is shift invariant and D is the "dilation by 1" operator, then*

$$m_{D(V)}(\xi) = m_V\left(\frac{\xi}{2}\right) + m_V\left(\frac{\xi}{2} + \frac{1}{2}\right).$$

Proof. By property (g) of Proposition 1.4.6 we see that the spectral function of $D(V)$ satisfies

$$\sigma_{D(V)}(\xi) = \sigma_V\left(\frac{\xi}{2}\right)$$

for a.e. $\xi \in \mathbb{R}$. Therefore, by property (h) of the same proposition we obtain that

$$m_{D(V)}(\xi) = \sum_{k\in\mathbb{Z}} \sigma_{D(V)}(\xi + k) = \sum_{k\in\mathbb{Z}} \sigma_V\left(\frac{\xi}{2} + \frac{k}{2}\right)$$

$$= \sum_{k\in\mathbb{Z}} \sigma_V\left(\frac{\xi}{2} + k\right) + \sum_{k\in\mathbb{Z}} \sigma_V\left(\frac{\xi}{2} + \frac{1}{2} + k\right) = m_V\left(\frac{\xi}{2}\right) + m_V\left(\frac{\xi}{2} + \frac{1}{2}\right). \quad ♣$$

As an immediate consequence of properties (b) and (h) of Proposition 1.4.6 we also have the following:

$$m_{V\oplus W} = m_V + m_W \tag{1.6.3}$$

for any shift invariant spaces V and W.

The above results allow us to see why any multiresolution analysis gives rise to a wavelet. If $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA, then we use property (M1) of Definition 1.6.1 to define spaces

$$W_j = V_{j+1} \ominus V_j,$$

which are mutually orthogonal for $j \in \mathbb{Z}$ and shift invariant for $j \geq 0$. By property (M2) we have that $D(W_j) = W_{j+1}$. From property (M4) it follows that

$$V_0 = \bigoplus_{j<0} W_j.$$

Therefore, property (M3) implies that

$$\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}).$$

Thus, any function $\psi \in W_0$, whose integer shifts form an orthonormal basis of W_0 is automatically an orthonotmal wavelet. To see that such a function exists we can use Proposition 1.6.2; that is, we can simply check that the multiplicity function of the space W_0 is almost everywhere equal to 1. By property (M6) and Proposition 1.6.2 we have that $m_{V_0} = 1$ a.e. Since $V_1 = D(V_0)$, Proposition 1.6.3 implies that $m_{V_1} = 2$ a.e. On the other hand $V_1 = V_0 \oplus W_0$, therefore, by (1.6.3), it is clear that $m_{W_0} = 1$ a.e.

As we mentioned before, the definition of multiresolution analysis does not explain by itself why it should be considered as a tool in the investigation of wavelets.

However, there is a standard way of defining an MRA if an orthonormal wavelet ψ is given (see [Au], [HW], [BMM]). It is done by considering the usual spaces

$$(1.6.4) \qquad W_j = \overline{\operatorname{span}\{\psi_{j,k}\}_{k\in\mathbb{Z}}},$$

defining the space V_0 by

$$(1.6.5). \qquad V_0 = \bigoplus_{j<0} W_j$$

and the spaces V_j by $V_j = D^j(V_0)$ for $j \in \mathbb{Z}$. Then, properties (M1)–(M5) simply follow from the fact that ψ is a wavelet (note that (M5) is true because $V_0^{\perp}$ is shift invariant). The only property which is not satisfied in general is (M6). Wavelets for which this property is satisfied are called *MRA wavelets*. Let us recall, that (M6) simply says, that we can find a single function φ belonging to V_0 (defined in (1.6.5)) such that the integer shifts of φ form an orthonormal basis of V_0. this, however, is equivalent to the fact, that the multiplicity function of V_0 is almost everywhere equal to 1, as proved in Proposition 1.6.2. Therefore, if we want to find out, which wavelets come from an MRA it is neccesary to compute the multiplicity function of V_0 if the wavelet ψ is given. Toward this end we shall prove the following formula for the spectral function of V_0:

PROPOSITION 1.6.4. *If $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet and V_0 is defined by (1.6.5), then*

$$\sigma_{V_0}(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$.

PROOF. Clearly,

$$V_0 \oplus \bigoplus_{j\geq 0} W_j = L^2(\mathbb{R}).$$

Therefore, by Proposition 1.4.6 we obtain

$$\sigma_{V_0} + \sum_{j\geq 0} \sigma_{W_j} = 1. \tag{1.6.6}$$

Using the proposition again or simply (1.5.10) we establish that $\sigma_{W_j}(\xi) = |\hat{\psi}(2^{-j}\xi)|^2$ for a.e. $\xi \in \mathbb{R}$ and $j \geq 0$. Therefore, the discrete Calderón condition (1.5.7) together with (1.6.6) gives us that

$$\sigma_{V_0}(\xi) = 1 - \sum_{j\geq 0} \sigma_{W_j}(\xi) = \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 - \sum_{j\geq 0} |\hat{\psi}(2^{-j}\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$. ♣

As an immediate consequence of the above proposition and property (h) of Proposition 1.4.6 we obtain the following formula for the multiplicity function of the space V_0 due to E. Weber (see [We]).

Corollary 1.6.5. *If $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet and V_0 is defined by (1.6.5), then*

$$m_{V_0}(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + k))|^2$$

for a.e. $\xi \in \mathbb{R}$.

The function

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + k))|^2, \tag{1.6.7}$$

appear originally in the work of P.G. Lemarié. It was named as the *dimension function* of a wavelet (see [HW]) and we shall fully investigate it in Chapter 2. The dimension function was used to determine, which wavelets can be associated to a multiresolution analysis. The problem of characterizing MRA wavelets was solved independently by G. Grippenberg and X. Wang (see [Gr] and [Wa]). In both solutions basic ideas developed by P.G. Lemarié and P. Auscher were used. The result is the following:

THEOREM 1.6.6. *If $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet, then ψ is an MRA wavelet if and only if the dimension function of ψ is almost everywhere equal to 1.*

PROOF. If ψ is a wavelet and V_0 is the space defined in (1.6.5), then using Proposition 1.6.2 and Corollary 1.6.5 we obtain the following equivalency:

ψ is an MRA wavelet $\iff$ *condition (M6) for V_0 is satisfied* $\iff$ $m_{V_0} = 1$ *a.e.* $\iff$ $D_\psi = 1$ *a.e.* ♣

To see why all "nice" wavelets can be associated to an MRA we repeat an argument due to P. Auscher. If ψ is an orthonormal wavelet, then its dimension function D_ψ is integer valued (we can see that it follows immediately from Corollary 1.6.5. Moreover, by definition (1.6.7) we obtain that

$$\int_0^1 D_\psi(\xi)\,d\xi = \int_0^1 \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+k))|^2\,d\xi = \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\hat{\psi}(2^j\xi)|^2\,d\xi = \|\hat{\psi}\|_2^2 = 1.$$

This allowed P. Auscher to conclude the following:

COROLLARY 1.6.7. *If $D_\psi > 0$ a.e., then $D_\psi = 1$ a.e.*

COROLLARY 1.6.8. *If D_ψ is continuous in the open interval $(0,1)$, then $D_\psi = 1$ a.e.*

The first of the above corollaries takes care of bounded wavelets with an exponential decay. In fact, if ψ is a wavelet, such that $|\psi(x)| \le Ce^{-a|x|}$ for some $C, a > 0$, then by Paley–Wiener's theorem (see §7.1 of [Ka]) the Fourier transform $\hat{\psi}$ is the restriction to $\mathbb{R}$ of a function holomorphic in a strip containing $\mathbb{R}$. In particular, $|\hat{\psi}| > 0$ a.e. and we can apply Corollary 1.6.7 to conclude that ψ is an MRA wavelet. This shows, that, for example, all compactly supported wavelets, like the Haar wavelet and the Daubechies wavelets, are MRA wavelets.

The second corollary implies that every wavelet ψ such that $\frac{d\psi}{dx} \in L^1(\mathbb{R})$ must be an MRA wavelet. Indeed, in this case we have $\widehat{(\frac{d\psi}{dx})} \in C_0(\mathbb{R})$ and

$(\widehat{\frac{d\psi}{dx}})(\xi) = 2\pi i\xi\hat{\psi}(\xi)$. Therefore, $\hat{\psi}$ is continuous on $\mathbb{R} \setminus \{0\}$ and $|\hat{\psi}(\xi)| \leq \frac{C}{|\xi|}$ for some $C > 0$, this, however, implies that D_ψ is continuous in the interval $(0, 1)$, so by Corollary 1.6.8 the wavelet ψ is associated to a multiresolution analysis. In particular, all wavelets in the Schwartz class, like the Meyer wavelet, are MRA wavelets.

So to speak, if we care about either a good decay or a minimal smoothness in wavelets, then we can restrict our attention to MRA wavelets. This had a great impact on the wavelet community, which was interested in finding out: "what are wavelets and how to get them?". The beauty of the above result simply killed the topic. The result was deciphered as the following message: "Let's don't worry about wavelets, the construction of S. Mallat and Y. Meyer produces the only legitimate members of this family – MRA wavelets". Of course, this point of view is completely correct from a standpoint of a signal analyst. The power and the origin of wavelets is their applications in signal processing, where the ugly non–MRA wavelets are not welcome. However, there was still a chance for these wavelets to survive. For sentimental reasons (the Shannon sampling theorem) the Shannon wavelet is well accepted in this community, and the result of P. Auscher can not be applied to prove that it is an MRA wavelet. Unfortunately, this was already taken care of by Y. Meyer and S. Mallat who proved "on the beginning" that the Shannon wavelet comes from an MRA (the scaling function φ is given by $\hat{\varphi} = \mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}$). That was the last nail in the coffin containing non–MRA wavelets. Their fate was sealed and the "ugly" wavelets were buried deep underground.

Well, David Hilbert is in a grave himself, and as we can see, some of his ideas as well. The whole concept of the Hilbert space $L^2(\mathbb{R})$ is that we free our mind of thinking that only continuous functions have a right to exist. In the theory of the space $L^2(\mathbb{R})$ all functions f with a finite energy (that is, such that $\int |f(x)|^2\,dx < \infty$) are "equal". That is why we will treat wavelets as a whole. The phenomenon that there exist functions $\psi \in L^2(\mathbb{R})$ such that the system $2^{\frac{j}{2}}\psi(2^j \cdot -k)$, $j, k \in \mathbb{Z}$, forms an orthonormal basis of $L^2(\mathbb{R})$ is amazing and it is not necessary to settle for less in understanding it.

To see that the above system is nothing ordinary let us reverse the order of translations and dilations, exactly like we did for the continuous wavelet transform. If $\psi \in L^2(\mathbb{R})$ and we consider the system $2^{\frac{j}{2}}\psi(2^j(\cdot - k))$, $j, k \in \mathbb{Z}$, then we will never get an orthonormal basis. Even more is true, the system can not be a tight frame with constant 1 for the shift invariant space V, which it spans (unless ψ is the zero function). In particular, **the reversed system can never be orthonormal**. To see this in detail, let us consider a family $\mathcal{A} = \{\psi_j\}_{j\in\mathbb{Z}}$, where

$$\psi_j(\xi) = 2^{\frac{j}{2}}\psi(2^j x)$$

for $j \in \mathbb{Z}$. Clearly, the set of all integer shifts of the members of $\mathcal{A}$ is exactly equal to $\{2^{\frac{j}{2}}\psi(2^j(\cdot - k))\}_{j,k\in\mathbb{Z}}$, which is the reversed system we are interested in. Let us denote by V the shift invariant space spanned by the reversed system. If the system would form a tight frame with constant 1 for the space V, then by Definition 1.4.3 of the spectral function we would obtain that

$$\sigma_V(\xi) = \sum_{j\in\mathbb{Z}} 2^j |\hat{\psi}(2^j\xi)|^2$$

for a.e. $\xi \in \mathbb{R}$. In particular, we could conclude that $\sigma_V(2^{-n}\xi) = 2^n\sigma_V(\xi)$ for a.e. $\xi \in \mathbb{R}$ and every natural n. This, however, contradicts with the fact that $\sigma_V \leq 1$, unless $V = \{0\}$.

We already excavated the non–MRA wavalets by proving the discrete Calderón reproducing formula in Theorem 1.5.4. The formula has nothing to do with MRA wavelets although its proof relies on the idea of the multiresolution analysis. No wonder, the approach of Y. Meyer and S. Mallat to understandig wavelets presented in Definition 1.6.1 is one of the basic features of the wavelet theory and it would be a pity to discard it simply because the condition (M6) of this definition does not hold for all wavelets. This was noticed by L. Baggett, H. Medina and K. Merrill who in [BMM] proposed a unified approach to wavelets based on the following:

Definition 1.6.9. *A sequence* $\{V_j\}_{j\in\mathbb{Z}}$ *of closed subspaces of* $L^2(\mathbb{R})$ *is called a generalized multiresolution analysis (GMRA) if it satisfies conditions (M1)–(M5) of Definition 1.6.1.*

Remark. The space V_0 belonging to a GMRA is called the *core subspace* of a GMRA.

As we mentioned before, there is a standard way of getting a GMRA from a given wavelet ψ. The core subspace V_0 of the corresponding GMRA is defined by (1.6.5) and this is all we need to determine the rest of the sequence $\{V_j\}_{j\in\mathbb{Z}}$ because of condition (M2). What we can see immediately, is that in this case the multiplicity function of V_0 satisfies

$$\int_0^1 m_{V_0}(\xi)\, d\xi = 1. \tag{1.6.8}$$

Indeed, since $m_{V_0} = D_\psi$ we can use the calculation which appears directly above Corollary 1.6.7 to conclude that (1.6.8) holds. What is more interesting, is that m_{V_0} has to satisfy the following *consistency equation*:

$$m_{V_0}(\xi) + 1 = m_{V_0}\left(\frac{\xi}{2}\right) + m_{V_0}\left(\frac{\xi}{2} + \frac{1}{2}\right) \tag{1.6.9}$$

for a.e. $\xi \in \mathbb{R}$. To see the reason for this, let us recall, that $V_0 \oplus W_0 = V_1 = D(V_0)$, where W_0 is the space generated by integer shifts of the wavelet ψ. Therefore, by (1.6.3) and Proposition 1.6.3 we obtain that

$$m_{V_0}(\xi) + m_{W_0}(\xi) = m_{V_0}\left(\frac{\xi}{2}\right) + m_{V_0}\left(\frac{\xi}{2} + \frac{1}{2}\right) \tag{1.6.10}$$

for a.e. $\xi \in \mathbb{R}$. Since Proposition 1.6.2 implies that $m_{W_0}(\xi) = 1$ a.e., it follows that the consistency equation (1.6.9) holds.

Conversely, let us assume that $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA and the multiplicity function of the core subspace V_0 satisfies (1.6.8) and (1.6.9). If we denote $V_1 \ominus V_0$ by W_0, then we again have that $V_0 \oplus W_0 = V_1 = D(V_0)$, which leads to (1.6.10). Therefore, we obtain the following equality:

$$m_{W_0}(\xi) = m_{V_0}\left(\frac{\xi}{2}\right) + m_{V_0}\left(\frac{\xi}{2} + \frac{1}{2}\right) - m_{V_0}(\xi)$$

(here, we have silently used (1.6.8) to conclude that m_{V_0} is finite a.e.). Thus, by the consistency equation (1.6.9), we obtain that $m_{W_0}(\xi) = 1$ a.e., so from Proposition 1.6.2 it follows, that we can find a function $\psi \in W_0$, whose integer shifts form an orthonormal basis of W_0. Since the definition of GMRA guaranties that $\bigoplus_{j\in\mathbb{Z}} D^j(W_0) = L^2(\mathbb{R})$, we conclude, that ψ is an orthonormal wavelet. We also point out, that a simple calculation shows that (1.6.8) follows from the consistencty equation (1.6.9) if we assume that m_{V_0} is integrable (on the interval $[0,1]$). In this way we have proved the main theorem of [BMM]:

THEOREM 1.6.10. *If $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA such that the multiplicity function of its core subspace V_0 is integrable and satisfies the consistency equation (1.6.9), then there exists an orthonormal wavelet $\psi \in V_1 \ominus V_0$. Conversely, if ψ is an orthonormal wavelet and V_0 is defined by (1.6.5) then $\{D^j(V_0)\}_{j\in\mathbb{Z}}$ is a GMRA, moreover, the multiplicity function of V_0 is integrable and satisfies the consistency equation.*

The above theorem shows that in some sense every wavelet is an "MRA" wavelet. All we have to do is to replace the condition (M6) in Definition 1.6.1 by

(M6') the multiplicity function of V_0 is integrable and satisfies the consistency equation. Even more is true, in the next paragraph we will show that for every wavelet we can also find appropriate "scaling functions", which generate the core subspace V_0 (a similar approach was taken by M. Papadakis in [Pa]). Before we do so, let us show how the spectral function of §1.4 can be used for a better understanding of GMRA's.

If ψ is an orthonormal wavelet and W_j is the space defined in (1.6.4) for $j \in \mathbb{Z}$, then the shift invariant space

$$V = \bigoplus_{j<0} W_j \tag{1.6.11}$$

shall be reffered to as the *core space* of ψ. This will allow us to avoid the subindex 0, which is not essential for our further considerations.

We will also say that a shift invariant space V is *refinable* if

$$V \subset D(V),$$

where D is the "dilation by 2" operator.

In connection with condition (M3), which appears in the definition of a GMRA, we have the following:

PROPOSITION 1.6.11. *Let V be a refinable shift invariant space. The sum $\bigcup_{j\in\mathbb{Z}} D^j(V)$ is dense in $L^2(\mathbb{R})$ if and only if*

$$\lim_{j\to\infty} \sigma_V(2^{-j}\xi) = 1 \tag{1.6.12}$$

for a.e. $\xi \in \mathbb{R}$.

PROOF. By Proposition 1.4.10 the fact that $\bigcup_{j\in\mathbb{Z}} D^j(V)$ is dense in $L^2(\mathbb{R})$ is equivalent to saying that $\lim_{j\to\infty} \sigma_{D^j(V)}(\xi) = 1$ a.e. (note that $D^j(V)$ is shift invariant for $j \geq 0$). From property (g) of Proposition 1.4.6 it follows that for $j \geq 0$ we have $\sigma_{D^j(V)}(\xi) = \sigma_V(2^{-j}\xi)$ a.e., which ends the proof. ♣

Condition (M4) seems to be less accessible, in the sense that it is not clear how to characterize it in therms of the spectral function of the space V_0. However, in Proposition 1.5.3 we proved a sufficient condition on σ_{V_0}, which guaranties that (M4) holds. Therefore, as an immediate consequence of this proposition and Proposition 1.6.11 we obtain the following:

Corollary 1.6.12. *If V is a refinable shift invariant space such that its spectral function is integrable and satisfies (1.6.12), then the sequence $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA.*

In spite of the aforementioned obstacle we can still characterize shift invariant spaces, which are core spaces of wavelets.

Theorem 1.6.13. *A shift invariant space V is a core space of some orthonormal wavelet if and only if the following conditions hold:*

(C1) V is refinable,

(C2) m_V is integrable on $[0,1]$ (and the integral is equal to 1),

(C3) m_V satisfies the consistency equation (1.6.9),

(C4) $\lim_{j\to\infty}\sigma_V(2^{-j}\xi)=1$ for a.e. $\xi\in\mathbb{R}$.

Proof. If ψ is an orthonormal wavelet and V is its core space, then the sequence $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA and, by Theorem 1.6.10, conditions (C1)–(C3) follow. By Proposition 1.6.11 we obtain that (C4) holds as well.

Conversely, let us assume that V is a shift invariant space, such that (C1)–(C4) hold. Because of (C2) and property (h) of Proposition 1.4.6 we obtain that σ_V is integrable. Therefore, from Corollary 1.6.12 it follows, that $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA, hence, Theorem 1.6.10 guaranties the existence of an orthonormal wavelet $\psi\in D(V)\ominus V$. Let $V^\sharp$ denote the core space of ψ. We will show that $V^\sharp=V$. Since $\psi\in D(V)$ and V is refinable we have that $D^j\psi\in V$ for $j<0$, which in turn implies that $V^\sharp\subset V$. Thus, by property (c) of Proposition 1.4.6, we have that $\sigma_{V^\sharp}\le\sigma_V$ a.e. Moreover, since $V^\sharp$ is the core space of ψ, its multiplicity function $m_{V^\sharp}$ satisfies (C2), which, together, with property (h) of Proposition 1.4.6 allows us to conclude that $\int_{\mathbb{R}}\sigma_{V^\sharp}=1$. Since we assume that m_V also satisfies (C2), the integral of σ_V is equal to 1 as well. Therefore, the inequality between the spectral functions turns into equality and we can use property (d) of Proposition 1.4.6 to obtain the equality between $V^\sharp$ and V. ♣

REMARK. The way we proved that $V^\sharp = V$ allows us to conclude that a refinable shift invariant space V is the core space of an orthonormal wavelet ψ if and only if $\psi \in D(V)$ and $\int_{\mathbb{R}} \sigma_V = 1$. In particular, we obtain, that an orthonormal wavelet ψ is an MRA wavelet if and only if there is an MRA $\{V_j\}_{j\in\mathbb{Z}}$, such that $\psi \in V_1$ (which was already noticed by C.K. Chui and W. He in [CH]).

Proposition 1.6.11 can be also used to answer a question which is of interest in light of [BM]. The authors of [BM] were concerned about determining what are the most general properties of the multiplicity function of the core subspace of a GMRA. They have found two conditions which are always satisfied by the multiplicity function. They also proved that if in addition a third condition is satisfied, then there exists a GMRA corresponding to this multiplicity function (see Theorem 1.5 of [BM]). We can use Proposition 1.6.11 to prove that the third condition is necessary as well. To formulate this condition let us consider the multiplicity function m_V of the core subspace V of a GMRA and a set $S = \{\xi \in \mathbb{R} \colon m_V(\xi) \geq 1\}$, which is the support of m_V. Define $\Delta = \bigcap_{n=0}^{\infty} 2^n S$. The condition we are interested in is the following:

$$\bigcup_{p\in\mathbb{Z}} 2^p \Delta = \mathbb{R} \tag{1.6.13}$$

By Proposition 1.6.11 the spectral function of V satisfies (1.6.12). We will show that this already implies that (1.6.13) must hold. Indeed, since $m_V(\xi) = \sum_{k\in\mathbb{Z}} \sigma_V(\xi+k)$, we can see that (1.6.12) gives us the following:

$$\liminf_{j\to\infty} m_V(2^{-j}\xi) \geq 1$$

for a.e. $\xi \in \mathbb{R}$. This means, that for a.e. $\xi \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that $2^{-n}\xi \in S$ for all natural $n \geq N$, or equivalently $\xi \in \bigcap_{n=N}^{\infty} 2^n S$. Therefore, we have that $2^{-N}\xi \in \bigcap_{n=0}^{\infty} 2^n S = \Delta$, or simply $\xi \in 2^N \Delta$, which proves (1.6.13).

1.7 Scaling vectors

In the previous paragraph we have shown that every wavelet can be viewed as an "MRA" wavelet. Of course, we do not opt for changing Definition 1.6.1 by replacing condition (M6) by (M6'). What we wish to do, however, is to show that the whole theory of multiresolution analysis is a part of a very general scheme, which governs all wavelets. The fact, that condition (M6) follows immediately from (M6') is literary as obvious as "one plus one equals two" (if V_0 satisfies (M6), then $m_{V_0} = 1$ a.e. and the consistency equation (1.6.10) takes form 1+1=1+1). Therefore, we can see already, that (M6) describes the simplest possible case, in which (M6') holds. The good point about (M6) is that it automatically gives us a scaling function φ contained in V_0. The scaling function is the starting point for development of the theory of multiresolution analysis. We shall cary on a similar program for all wavelets. Let us point out, however, that the scheme we are going to present has no applications in signal analysis and its only purpose is to justify our claim that all wavelets are "MRA" wavelets.

We begin by indicating crucial points in the theory of multiresolution analysis. Let us consider an MRA $\{V_j\}$ with a scaling function φ. From Proposition 1.6.2 it follows that φ has to satisfy (1.6.1). Since the space V_0 is refinable, we have $\varphi \in V_0 \subset D(V_0)$, or equivalently $D^{-1}\varphi \in V_0$. therefore, by Theorem 1.2.4, we obtain the following *scaling equation*:

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi), \tag{1.7.1}$$

where m is a 1–periodic function in $L^2(\mathbb{T})$ (note, that by (1.6.1) m is unique). Sometimes, a function φ, which satisfies the scaling equation is called *refinable.* It is easy to see that a function φ is refinable if and only if the principal shift invariant space V_φ is refinable. The scaling function φ also satisfies:

$$\lim_{j\to\infty} |\hat{\varphi}(2^{-j}\xi)| = 1 \tag{1.7.2}$$

for a.e. $\xi \in \mathbb{R}$ (this can be easily seen if we use condition (M3) of Definition 1.6.1, Proposition 1.6.11 and the fact, that $\sigma_{V_0} = |\hat{\varphi}|^2$). It turns out, that (1.6.1), (1.7.1) and (1.7.2) characterize all scaling functions. This result was proven in [HW] and let us state it for a future referrence.

THEOREM 1.7.1. *A function $\varphi \in L^2(\mathbb{R})$ is a scaling function if and only if φ satisfies (1.6.1), (1.7.1) and (1.7.2).*

REMARK. To see that conditions (1.6.1), (1.7.1) and (1.7.2) are indeed sufficient to conclude that φ is a scaling function we can simply use Proposition 1.6.1, Proposition 1.6.11 and Theorem 1.6.13. Indeed, (1.7.1) implies that condition (C1) of Theorem 1.6.13 is satisfied for $V = V_\varphi$. Proposition 1.6.1 together with (1.6.1) gives us that $m_{V_\varphi} = 1$ a.e., which in turn implies that (C2) and (C3) hold. Finally, by Proposition 1.6.11, we have that (C4) is satisfied as well. Thus, the sequence $\{D^j(V_\varphi)\}$ is an MRA (condition (M6) holds because $m_{V_\varphi} = 1$ a.e.), so φ is a scaling function.

The function m, which is present in (1.7.1) is called the *low–pass filter* of a scaling function φ. As a consequence of (1.6.1) and (1.7.1) the following *Smith-Barnwell* equation:

$$|m(\xi)|^2 + \left|m\left(\xi + \frac{1}{2}\right)\right|^2 = 1 \tag{1.7.3}$$

is satisfied by the low–pass filter for a.e. $\xi \in \mathbb{R}$. Moreover, the low–pass filter can be used to obtain an orthonormal wavelet ψ associated to the MRA with the scaling function φ via

$$\hat{\psi}(2\xi) = l(\xi)\hat{\varphi}(\xi), \tag{1.7.4a}$$

where l is a 1–periodic function in $L^2(\mathbb{T})$ such that the matrix

$$U(\xi) = \begin{bmatrix} l(\xi) & l(\xi + \frac{1}{2}) \\ m(\xi) & m(\xi + \frac{1}{2}) \end{bmatrix} \tag{1.7.4b}$$

is unitary. A simple explicit formula for an l that satisfies (1.7.4b) is

$$l(\xi) = e^{2\pi i\xi}\overline{m\Big(\xi + \frac{1}{2}\Big)}s(2\xi), \tag{1.7.4c}$$

where s is any 1–periodic unimodular function.

To see how the above ideas should be extended to include all wavelets, let us consider the well–known non–MRA Journé wavelet ψ given by $\hat{\psi} = \mathbf{1}_W$, where

$$W = \Big[-\frac{16}{7}, -2\Big] \cup \Big[-\frac{1}{2}, -\frac{2}{7}\Big] \cup \Big[\frac{2}{7}, \frac{1}{2}\Big] \cup \Big[2, \frac{16}{7}\Big].$$

The space $W_0 = V_\psi$, which was considered in (1.5.9), is simply given by $\hat{W}_0 = L^2(W)$. The core space of ψ defined in (1.6.11) is, therefore, given by $\hat{V} = L^2(S)$, where

$$S = \bigcup_{j>0} 2^{-j}W = \Big[-\frac{8}{7}, -1\Big] \cup \Big[-\frac{4}{7}, -\frac{1}{2}\Big] \cup \Big[-\frac{2}{7}, \frac{2}{7}\Big] \cup \Big[\frac{1}{2}, \frac{4}{7}\Big] \cup \Big[1, \frac{8}{7}\Big].$$

The dimension function D_ψ of this wavelet can by computed directly from (1.6.7) and is given by

$$D_\psi(\xi) = \begin{cases} 2 & \text{for } x \in [-\frac{1}{7}, \frac{1}{7}] \\ 1 & \text{for } x \in [-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, -\frac{1}{7}] \cup [\frac{1}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}] \\ 0 & \text{for } x \in [-\frac{2}{7}, -\frac{3}{7}] \cup [\frac{2}{7}, \frac{3}{7}] \end{cases}$$

for $\xi \in \mathbb{T}$ (note, that $D_\psi(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + k)$). This already indicates, that ψ can not be associated to a multiresolution analysis. By Corollary 1.6.5, the multiplicity function m_V of the core space V of ψ is equal to the dimension function D_ψ of ψ. Therefore, from Theorem 1.3.8 it follows that we can decompose V into an orthogonal sum of two principal shift invariant spaces:

$$V = V_{\varphi_1} \oplus V_{\varphi_2}, \tag{1.7.5}$$

where φ_1 and φ_2 is a tight frame generator of V_{φ_1} and V_{φ_2}, respectively. Moreover, the spectrum Ω_1 of V_{φ_1} is given by

$$\Omega_1 \cap \mathbb{T} = \{\xi \in \mathbb{T} : m_V(\xi) \geq 1\} = \Big[-\frac{1}{2}, -\frac{3}{7}\Big] \cup \Big[-\frac{2}{7}, \frac{2}{7}\Big] \cup \Big[\frac{3}{7}, \frac{1}{2}\Big]$$

and the spectrum Ω_2 of V_{φ_2} is given by

$$\Omega_2 \cap \mathbb{T} = \{\xi \in \mathbb{T} : m_V(\xi) \geq 2\} = \Big[-\frac{1}{7}, \frac{1}{7}\Big]$$

(both spectra are 1–periodic sets, so it is enough to determine their restriction to $\mathbb{T}$). Since the decomposition (1.7.5) is not unique, we have some freedom in choosing the "scaling functions" φ_1 and φ_2. Our choice will be $\hat{\varphi}_1 = \mathbf{1}_{S_1}$ and $\hat{\varphi}_2 = \mathbf{1}_{S_2}$, where

$$S_1 = \Big[-\frac{4}{7}, -\frac{1}{2}\Big] \cup \Big[-\frac{2}{7}, \frac{2}{7}\Big] \cup \Big[\frac{1}{2}, \frac{4}{7}\Big] \quad \text{and} \quad S_2 = \Big[-\frac{8}{7}, -1\Big] \cup \Big[1, \frac{8}{7}\Big]$$

Note, that $S_1 \cup S_2 = S$ and the integer translates of S_i are mutually disjoint for $i = 1, 2$. In general, every two sets with these properties lead to the decomposition (1.7.5), for example, we also could have chosen sets $S_1' = S \backslash [-\frac{1}{7}, \frac{1}{7}]$ and $S_2' = [-\frac{1}{7}, \frac{1}{7}]$ to define φ_1 and φ_2. Let us consider the following:

$$\Phi(\xi) = \begin{bmatrix} \varphi_1(\xi) \\ \varphi_2(\xi) \end{bmatrix} \quad \text{and} \quad \hat{\Phi}(\xi) = \begin{bmatrix} \hat{\varphi}_1(\xi) \\ \hat{\varphi}_2(\xi) \end{bmatrix},$$

which are column vectors for $\xi \in \mathbb{R}$. We shall refer to Φ as the scaling vector and to $\hat{\Phi}$ as its Fourier transform. The scaling vector will be an analog of the scaling function in our scheme. To see what should correspond to formula (1.6.1), let us define the following:

$$\tilde{\Omega}(\xi) := \begin{bmatrix} \mathbf{1}_{\Omega_1}(\xi) & 0 \\ 0 & \mathbf{1}_{\Omega_2}(\xi) \end{bmatrix},$$

which is a 1–periodic matrix defined for $\xi \in \mathbb{R}$. It is easy to check that the Fourier transform of our scaling vector $\hat{\Phi}$ and its adjoint $\hat{\Phi}^*$ (which is a row vector) satisfy

$$\sum_{k \in \mathbb{Z}} \hat{\Phi}(\xi + k)\hat{\Phi}^*(\xi + k) = \tilde{\Omega}(\xi),$$

which is a generalization of (1.6.1). We have also, that $\|\hat{\Phi}(\xi)\|^2 = |\hat{\varphi}_1(\xi)|^2 + |\hat{\varphi}_2(\xi)|^2 = \mathbf{1}_S(\xi)$, which allows us to obtain

$$\lim_{j \to \infty} \|\hat{\Phi}(2^{-j}\xi)\| = 1$$

for $\xi \in \mathbb{R}$, which corresponds to (1.7.2). Since φ_1 and φ_2 belong to V, which is a refinable shift invariant space, we can conclude, that $D^{-1}\varphi_1, D^{-1}\varphi_1 \in V$. Therefore, Theorem 1.2.10 implies that there exist unique 1–periodic functions $m_{ij} \in L^2(\mathbb{T})$ with $\operatorname{supp} m_{i,j} \subset \Omega_j$, where $i, j = 1, 2$, such that

$$\hat{\varphi}_1(2\xi) = m_{11}(\xi)\hat{\varphi}_1(\xi) + m_{12}(\xi)\hat{\varphi}_2(\xi)$$

and

$$\hat{\varphi}_2(2\xi) = m_{21}(\xi)\hat{\varphi}_1(\xi) + m_{22}(\xi)\hat{\varphi}_2(\xi).$$

Therefore, if we consider the following analog of the low–pass filter:

$$M(\xi) = \begin{bmatrix} m_{11}(\xi) & m_{12}(\xi) \\ m_{21}(\xi) & m_{22}(\xi) \end{bmatrix},$$

which is a 1–periodic matrix defined for $\xi \in \mathbb{R}$, then we obtain

$$\hat{\Phi}(2\xi) = M(\xi)\hat{\Phi}(\xi),$$

that is, a generalization of (1.7.1). In our case, we have that $m_{11} = \mathbf{1}_{F_1}$, $m_{21} = \mathbf{1}_{F_2}$ and $m_{12} = m_{22} = 0$, where F_1 is the 1–periodization of $\frac{S_1}{2}$ and F_2 is the 1–periodization of $\frac{S_2}{2}$. As a consequence, the following generalized version of the Smith-Barnwell equation (1.7.3):

$$M(\xi)M^*(\xi) + M\Big(\xi + \frac{1}{2}\Big)M^*\Big(\xi + \frac{1}{2}\Big) = \tilde{\Omega}(2\xi)$$

holds for $\xi \in \mathbb{R}$. We also have $W = 2S \setminus S$, that is, $\hat{\psi}(2\xi) = \hat{\varphi}_1(\xi) + \hat{\varphi}_2(\xi) - \hat{\varphi}_1(2\xi) - \hat{\varphi}_2(2\xi) = (1 - m_{11}(\xi) - m_{21}(\xi))\hat{\varphi}_1(\xi) + \hat{\varphi}_2(\xi)$. Therefore, we can see that ψ is given by

$$\hat{\psi}(2\xi) = L(\xi)\hat{\Phi}(\xi), \quad \text{where} \quad L(\xi) = [l_1(\xi), l_2(\xi)]$$

and $l_1 = (1 - m_{11} - m_{21})\mathbf{1}_{\Omega_1}$, $l_2 = \mathbf{1}_{\Omega_2}$. The above equation corresponds to (1.7.4a) and it is easy to check that the following generalization of (1.7.4b) holds: the matrix

$$U(\xi) = \begin{bmatrix} l_1(\xi) & l_2(\xi) & l_1(\xi + \frac{1}{2}) & l_2(\xi + \frac{1}{2}) \\ m_{11}(\xi) & m_{12}(\xi) & m_{11}(\xi + \frac{1}{2}) & m_{12}(\xi + \frac{1}{2}) \\ m_{21}(\xi) & m_{22}(\xi) & m_{21}(\xi + \frac{1}{2}) & m_{22}(\xi + \frac{1}{2}) \end{bmatrix}$$

has orthogonal rows and the norm of the first row is equal to 1 for a.e. $\xi \in \mathbb{R}$. Unfortunately, we can not use our low–pass filter matrix to find an explicit formula for L similar to (1.7.4c). The reason for this is that in our case the matrix U is more complicated and it is not clear how to determine L knowing that this matrix has the above properties.

The approach to the Journé wavelet, which we presented above, is motivated by the multiresolution analysis theory of multiwavelets. This theory deals with the case, where instead of condition (M6) of Definition 1.6.1 we assume that the multiplicity function of V_0 is equal a.e. to a fixed natural n. Before we extend the above approach to the case of an arbitrary orthonormal wavelet, we shall develop necessary tools and introduce the appropriate notation. Moreover, since the "scaling functions" for a given wavelet are not going to be unique, we will try to understand this non-uniqueness as well.

Let V be a shift invariant space with the multiplicity function m_V. In Corollary 1.3.9 we proved that to a given shift invariant space V we can associate the spectral sequence, that is, a unique decreasing sequence of sets $\{\Omega_n\}_{n\in\mathbb{N}}$ such that

$$V = \bigoplus_{n\in\mathbb{N}} V_n, \tag{1.7.6}$$

where V_n is a principal shift invariant space with spectrum Ω_n. Moreover, in the proof of the corollary we found out that Ω_n is given by

$$\Omega_n = \{\xi \in \mathbb{R}\colon m_V(\xi) \geq n\} \tag{1.7.7}$$

(here, again, we use the convention that V_n is the zero subspace if $\Omega_n = \emptyset$). Since every space V_n in the above decomposition is principal we can consider its tight frame generator φ_n for each natural n. The decomposition (1.7.6) allows us to define the following *genarating vector* of V:

$$\Phi(\xi) = (\varphi_n(\xi))_{n\in\mathbb{N}} \tag{1.7.8}$$

and its Fourier transform

$$\hat{\Phi}(\xi) = (\hat{\varphi}_n(\xi))_{n\in\mathbb{N}}$$

for $\xi \in \mathbb{R}$. From Corollary 1.4.5 it follows that

$$\|\hat{\Phi}(\xi)\|^2_{\ell^2(\mathbb{N})} = \sigma_V(\xi), \tag{1.7.9}$$

so will treat $\hat{\Phi}(\xi)$ as a column vector, which belongs to $\ell^2(\mathbb{N})$ for a.e. $\xi \in \mathbb{R}$. We can also consider the following diagonal matrix

$$\tilde{\Omega}(\xi) := \begin{bmatrix} \mathbf{1}_{\Omega_1}(\xi) & 0 & 0 & \cdots \\ 0 & \mathbf{1}_{\Omega_2}(\xi) & 0 & \cdots \\ 0 & 0 & \mathbf{1}_{\Omega_3}(\xi) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.7.10}$$

where Ω_n are given by (1.7.7) and $\xi \in \mathbb{R}$. The 1–periodic matrix $\tilde{\Omega}$ shall be called the *spectral matrix* of V.

We introduced the generating vector and the spectral matrix for notational purposes. This notation will allow us to obtain compact formulae and carry on the program of finding "scaling functions" for a given wavelet ψ. To see how this notation works, let us assume the decomposition (1.7.6) of a shift invariant space V. Recall, that by Theorem 1.2.8, we have

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}_n(\xi+k)|^2 = \mathbf{1}_{\Omega_n}(\xi)$$

for every natural n and a.e. $\xi \in \mathbb{R}$. Moreover, by Proposition 1.3.1 we have also

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}_n(\xi+k)\overline{\hat{\varphi}_m(\xi+k)} = 0$$

for every natural $n \neq m$ and a.e. $\xi \in \mathbb{R}$. Both these statements can be written as the following formula involving the Fourier transform of the generating vector given by (1.7.8) and the spectral matrix of V:

$$\sum_{k\in\mathbb{Z}} \hat{\Phi}(\xi+k)\hat{\Phi}^*(\xi+k) = \tilde{\Omega}(\xi) \tag{1.7.11}$$

for a.e. $\xi \in \mathbb{R}$. In the above formula the vector $\hat{\Phi}^*(\xi)$ is the row vector, which is the adjoint of $\hat{\Phi}(\xi)$.

Formulae (1.7.9) and (1.7.11) can be used to give a simple characterization of generating vectors of a given shift invariants space V. If V is a shift invariant space and $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a set of functions belonging to V, then we can define $\hat{\Phi}$ via (1.7.8) and ask when Φ is a generating vector of V. If $\hat{\Phi}$ satisfies (1.7.11), then it is clear that $V \supset \bigoplus_{n=1}^{\infty} V_{\varphi_n}$. Since the trace of the spectral matrix of V is equal to the multiplicity function of V, we can conclude, that $m_V = m_W$, where W denotes $\bigoplus_{n=1}^{\infty} V_{\varphi_n}$. Therefore, from Theorem 1.3.5 it follows that $V = W$, in the case, when the multiplicity function of V is finite a.e. However, in some cases (1.7.11) alone is not sufficient for concluding that $\hat{\Phi}$ is a generating vector of V. For example, the multiplicity function of the space $L^2(\mathbb{R})$ is equal to the multiplicity function of $V = \{f \in L^2(\mathbb{R})\colon \operatorname{supp} \hat{f} \subset \mathbb{R}^+\}$ although these two spaces are obviously different. Therefore, if we want to make sure that $V = W$ we have to use (1.7.9), which together with property (d) of Proposition 1.4.6 proves that the equality holds. This gives us the following:

PROPOSITION 1.7.2. *Let V be a shift invariant space and $\{\varphi_n\}_{n\in\mathbb{Z}}$ be a set of functions belonging to V. Then Φ defined in (1.7.8) is a generating vector of V if and only if it satisfies (1.7.9) and (1.7.11).*

We shall use the introduced notation to describe all optimal decompositions (1.7.6) of a given shift invariant space V. This was mentioned at the closure of §1.3. Let us recall, that the decomposition (1.7.6) is not unique because the spaces V_n of (1.7.6) are not unique. Let us, therefore, allow to consider two alternative optimal decompositions

$$V = \bigoplus_{n\in\mathbb{N}} V_n \quad \text{and} \quad V = \bigoplus_{n\in\mathbb{N}} V_n' \tag{1.7.12}$$

If we choose tight frame generators for V_n and V_n', then (1.7.12) can be written as

$$V = \bigoplus_{n\in\mathbb{N}} V_{\varphi_n} \quad \text{and} \quad V = \bigoplus_{n\in\mathbb{N}} V_{\psi_n}.$$

We are interested in characterizing tight frame generators $\{\psi_n\}_{n\in\mathbb{N}}$ in terms of a fixed set of optimal tight frame generators $\{\varphi_n\}_{n\in\mathbb{N}}$. With the use of the notation, we just introduced, this problem can be stated in the following way: "Given a generating vector Φ of a shift invariant space V what are all other generating vectors Ψ of V?". To avoid technical difficulties we answer the question in the case when the multiplicity function of V is finite a.e.

THEOREM 1.7.3. *If V is a shift invariant space with a generating vector Φ and the multiplicity function of V is finite a.e., then Ψ is a generating vector of V if and only if*

$$\hat{\Psi}(\xi) = R(\xi)\hat{\Phi}(\xi) \tag{1.7.13}$$

for a.e. $\xi \in \mathbb{R}$, where R is a 1–periodic matrix with entries $r_{mn} \in L^2(\mathbb{T})$, $m, n \in \mathbb{N}$, $\operatorname{supp} r_{mn} \subset \Omega_n$, which satisfies

$$R(\xi)R^*(\xi) = R^*(\xi)R(\xi) = \tilde{\Omega}(\xi) \tag{1.7.14}$$

for a.e. $\xi \in \mathbb{T}$. Moreover, if Ψ is a generating vector of V then the corresponding R is unique.

Proof. Let us assume that (1.7.13) and (1.7.14) hold. Thus, we have $\hat{\Psi}(\xi) = R(\xi)\hat{\Phi}(\xi)$, where Φ is a generating vector of V and $\hat{\Psi}(\xi) = (\hat{\psi}_n(\xi))_{n\in\mathbb{N}}$. We shall prove that Ψ is a generating vector of V, provided R is 1–periodic with the entries described above. From (1.7.13) it follows that

$$\hat{\psi}_n(\xi) = \sum_{m\in\mathbb{N}} r_{mn}(\xi)\hat{\varphi}_n(\xi)$$

for a.e. $\xi \in \mathbb{R}$. Therefore, Theorem 1.2.10 implies that $\psi_n \in V$ for $n \in \mathbb{N}$. This allows us to apply Proposition 1.7.2 as long as we prove that (1.7.9) and (1.7.11) are satisfied by $\hat{\Psi}$. To check (1.7.9) we use (1.7.14) and the fact that (1.7.9) is satisfied by $\hat{\Phi}$:

$$\|\hat{\Psi}(\xi)\|^2_{\ell^2(\mathbb{N})} = \langle R(\xi)\hat{\Phi}(\xi), R(\xi)\hat{\Phi}(\xi)\rangle = \langle R^*(\xi)R(\xi)\hat{\Phi}(\xi), \hat{\Phi}(\xi)\rangle$$

$$= \langle \hat{\Phi}(\xi), \hat{\Phi}(\xi)\rangle = \sigma_V(\xi)$$

(here, we used $\tilde{\Omega}(\xi)\hat{\Phi}(\xi) = \hat{\Phi}(\xi)$); which proves that $\hat{\Psi}$ satisfies (1.7.9). Moreover, by (1.7.14) and (1.7.11) for $\hat{\Phi}$, we have

$$\sum_{k\in\mathbb{Z}} \hat{\Psi}(\xi+k)\hat{\Psi}^*(\xi+k) = \sum_{k\in\mathbb{Z}} R(\xi)\hat{\Phi}(\xi+k)\hat{\Phi}^*(\xi+k)R^*(\xi) = R(\xi)\tilde{\Omega}(\xi)R^*(\xi) = \tilde{\Omega}(\xi)$$

(here we used $R(\xi)\tilde{\Omega}(\xi) = R(\xi)$); that is, (1.7.11) holds for $\hat{\Psi}$ as well). Therefore, by Proposition 1.7.2, Ψ is a generating vector of V.

Conversly, let us assume that Φ and Ψ are generating vectors of V. By Theorem 1.2.10 we conclude that there exists a unique 1–periodic matrix R with entries $r_{mn} \in L^2(\mathbb{T})$, $m, n \in \mathbb{N}$, $\operatorname{supp} r_{mn} \subset \Omega_n$, such that $\hat{\Psi}(\xi) = R(\xi)\hat{\Phi}(\xi)$. Therefore, to finish the proof, we need to show that R satisfies (1.7.14). The fact, that (1.7.11) holds for both $\hat{\Phi}$ and $\hat{\Psi}$ allows us to conclude that

$$\tilde{\Omega}(\xi) = \sum_{k\in\mathbb{Z}} \hat{\Psi}(\xi+k)\hat{\Psi}^*(\xi+k) = \sum_{k\in\mathbb{Z}} R(\xi)\hat{\Phi}(\xi+k)\hat{\Phi}^*(\xi+k)R^*(\xi) = R(\xi)R^*(\xi),$$

which is a part of (1.7.14). To see why R^*R is equal to $\tilde{\Omega}$ as well we will use the assumption, that m_V is finite a.e. Towards this end let us fix an $N \in \mathbb{N}$ and consider a set $A_N = \Omega_N \setminus \Omega_{N+1}$. Since $A_N \cap \Omega_n = \emptyset$ for $n > N$ it is clear that if $\xi \in A_N$, then the entries $d_n(\xi)$ on the diagonal of the spectral matrix $\tilde{\Omega}(\xi)$ are equal to 1 for $n \le N$ and are equal to zero otherwise. Moreover, since $\operatorname{supp} r_{mn} \subset \Omega_n$ we obtain that $r_{mn}(\xi) = 0$ for $\xi \in A_N$ and $n > N$. This, however, together with $RR^* = \tilde{\Omega}$ implies that $r_{mn}(\xi) = 0$ for $\xi \in A_N$ and $m > N$. Therefore, for $\xi \in A_N$ we can discard the zero entries of the matrix $R(\xi)$ and consider the truncated $N \times N$ matrix $R_N(\xi)$. It is also clear, that the spectral matrix $\tilde{\Omega}(\xi)$ can be truncated to an $N \times N$ identity matrix if $\xi \in A_N$. Therefore, the equation $RR^* = \tilde{\Omega}$ has the following form:

$$R_N(\xi)R_N^*(\xi) = Id_N$$

for $\xi \in A_N$. This means, that $R_N(\xi)$ is a unitary matrix for $\xi \in A_N$, which allows us to conclude that $R_N^*(\xi)R_N(\xi) = Id_N$ as well, provided $\xi \in A_N$. Thus, we obtain that $R^*(\xi)R(\xi) = \tilde{\Omega}(\xi)$ for $\xi \in A_N$. Since m_V is finite a.e., we have that $\bigcup_{N\in\mathbb{N}} A_N = \mathbb{R}$, so $R^*(\xi)R(\xi) = \tilde{\Omega}(\xi)$ for a.e. $\xi \in \mathbb{T}$, which concludes the proof. ♣

To close this general discussion concerning generating vectors let us prove the following corollary of Theorem 1.2.10:

Corollary 1.7.4. *Let V be a shift invariant space with a generating vector Φ and the spectral sequence $\{\Omega_n\}_{n\in\mathbb{N}}$. The space V is refinable if and only if*

$$\hat{\Phi}(2\xi) = M(\xi)\hat{\Phi}(\xi), \tag{1.7.15}$$

where M is a 1–periodic matrix with entries $m_{ln} \in L^2(\mathbb{T})$, $l, n \in \mathbb{Z}$, such that $\operatorname{supp} m_{ln} \subset \Omega_n$. *Moreover, if such an M exists, then it is unique.*

Proof. By definition, V is refinable if $V \subset D(V)$, where D is the "dilation by 2" operator. If Φ is a generating vector of V then $\hat{\Phi} = (\hat{\varphi}_n)_{n\in\mathbb{N}}$ and

$$V = \bigoplus_{n\in\mathbb{N}} V_{\varphi_n}.$$

Thus, the condition $V \subset D(V)$ is equivalent to saying that $\varphi_l \in D(V)$ for every $l \in \mathbb{N}$. This, in turn, can be formulated as $\hat{\varphi}_l(2\cdot) \in V$ for $l \in \mathbb{N}$, which, by Theorem 1.2.10, is equivalent to (1.7.15). ♣

At this point we are ready to present a general scheme based on the idea of the multiresolution analysis, which applies to all wavelets.

The basic notion is the following:

Definition 1.7.5. *If $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet, then a generating vector Φ (see (1.7.8)) of its core space V (see (1.6.11)) is called a scaling vector of ψ.*

The notion of the scaling vector generalizes the notion of the scaling function. We shall, therefore, concentrate on proving properties of scaling vectors which are analogous to those of scaling functions.

Let $\psi \in L^2(\mathbb{R})$ be an orthonormal wavelet. Consider its core space V defined by (1.6.11) with the multiplicity function m_V, the spectral sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ given by (1.7.7) and the spectral matrix (1.7.10). If Φ is a generating vector of V its Fourier transform $\hat{\Phi}$ satisfies (1.7.9) and (1.7.11). The latter corresponds to (1.6.1) and the former, together with Proposition 1.6.11, gives us

$$\lim_{j\to\infty} \|\hat{\Phi}(2^{-j}\xi)\| = 1, \tag{1.7.16}$$

which corresponds to (1.7.2). Since V is refinable, formula (1.7.15) holds for $\hat{\Phi}$, which generalizes the scaling equation (1.7.1). We shall show that the low–pass filter matrix M given in (1.7.15) satisfies a generalized version of the Smith-Barnwell equation (1.7.3). The condition on the support of M implies that $M\tilde{\Omega} = M$, thus, by (1.7.11) and (1.7.15), we obtain

$$\begin{aligned}
\tilde{\Omega}(2\xi) = \sum_{k\in\mathbb{Z}} \hat{\Phi}(2\xi+k)\hat{\Phi}^*(2\xi+k) &= \sum_{k\in\mathbb{Z}} M\Big(\xi+\frac{k}{2}\Big)\hat{\Phi}\Big(\xi+\frac{k}{2}\Big)\hat{\Phi}^*\Big(\xi+\frac{k}{2}\Big)M\Big(\xi+\frac{k}{2}\Big)^* \\
&= M(\xi)\tilde{\Omega}(\xi)M(\xi)^* + M\Big(\xi+\frac{k}{2}\Big)\tilde{\Omega}\Big(\xi+\frac{k}{2}\Big)M\Big(\xi+\frac{k}{2}\Big)^* \\
&= M(\xi)M(\xi)^* + M\Big(\xi+\frac{k}{2}\Big)M\Big(\xi+\frac{k}{2}\Big)^*
\end{aligned}$$

for a.e. $\xi \in \mathbb{R}$. That is, (1.7.3) corresponds to

$$M(\xi)M(\xi)^* + M\Big(\xi+\frac{k}{2}\Big)M\Big(\xi+\frac{k}{2}\Big)^* = \tilde{\Omega}(2\xi)$$

for a.e. $\xi \in \mathbb{R}$. As we mentioned before the problem of determining the wavelet ψ from its scaling vector and its low–pass filter matrix is more complicated than the corresponding problem in the case of usual MRA wavelets. Formula (1.7.4a) can be easily generalized. Indeed, since $\psi \in D(V)$, by Theorem 1.2.10 we have

$$\hat{\psi}(2\xi) = L(\xi)\hat{\Phi}(\xi),$$

where $L(\xi)$ is a row vector of the form $L(\xi) = (l_n(\xi))_{n\in\mathbb{Z}}$ with 1–periodic entries in $L^2(\mathbb{T})$ such that $\operatorname{supp} l_n \subset \Omega_n$ for $n \in \mathbb{N}$. The corresponding version of (1.7.4b) can be formulated as follows: the rows of the matrix

$$U(\xi) = \begin{bmatrix} L(\xi) & L(\xi+\frac{1}{2}) \\ M(\xi) & M(\xi+\frac{1}{2}) \end{bmatrix}$$

are orthogonal for a.e. $\xi \in \mathbb{R}$ and the norm of the first row is equal to 1 a.e. However, formula (1.7.4c) can not be easily generalized to fit our scheme. In spite of this obstacle we can readily find the absolute value of $\hat{\psi}$. This can be done by using Proposition 1.6.4, which implies that

$$|\hat{\psi}(2\xi)|^2 = \sigma_V(\xi) - \sigma_V(2\xi);$$

that is,

$$|\hat{\psi}(2\xi)|^2 = \|\hat{\Phi}(\xi)\|^2_{\ell^2(\mathbb{N})} - \|\hat{\Phi}(2\xi)\|^2_{\ell^2(\mathbb{N})}$$

(here, we have used (1.7.9)). The above equation corresponds to

$$|\hat{\psi}(2\xi)|^2 = |\hat{\varphi}(\xi)|^2 - |\hat{\varphi}(2\xi)|^2,$$

which is satisfied by every MRA wavelet ψ and its scaling function φ.

In light of Theorem 1.7.1 it is natural to ask if we can find a characterization of scaling vectors based on (1.7.11), (1.7.15) and (1.7.16). It turns out, that Theorem 1.6.13 provides us with such a characterization. However, because of condition (1.7.11) the characterization is more complicated then the one given in Theorem 1.7.1. To see the reason for this, let us point out, that condition (1.7.11) is stated in terms of the spectral matrix of a shift invariant space V (the space which is supposed to be the core space of a wavelet). The spectral matrix is defined by the spectral sequence of V. Furthermore, this sequence determines the multiplicity function m_V of V, which has to satisfy the consistency equation (1.6.9) in order to give rise to a wavelet. In practice it is rather hard to find a 1–periodic function attaining only natural number values which satisfies the consistency equation. Therefore, it is better to assume that an appropriate multiplicity function is already given, and we shall take this approach. This motivates a way of understanding wavelets that we present below.

The set of all orthonormal wavelets in $L^2(\mathbb{R})$ can be partitioned into equivalence classes via:

$$\psi \equiv \eta \quad \Leftrightarrow \quad D_\psi = D_\eta \quad \text{a.e.},$$

where ψ, η are wavelets and D_ψ, D_η are their dimension functions defined in (1.6.7). Let $\mathcal{D}$ denote the set of all dimension functions of wavelets. The set $\mathcal{D}$ is **uncountable** and connected in the $L^1(\mathbb{T})$–norm. The elements of $\mathcal{D}$ are characterized by six conditions: $d \in \mathcal{D}$ if and only if

- d is 1–periodic,
- d attains only natural number values,
- d satisfies the consistency equation,
- $\|d\|_{L^1(\mathbb{T})} = 1$,
- $\liminf_{n\to\infty} d(2^{-n}\xi) \geq 1$,
- $\sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + k) \geq d(\xi)$,

where $\Delta = \{\xi \in \mathbb{R}\colon d(2^{-j}\xi) \geq 1 \text{ for } j \in \mathbb{N} \cup \{0\}\}$. Every element of $\mathcal{D}$ is equal to the dimension function of a wavelet whose Fourier transform is a characteristic function of some subset of $\mathbb{R}$. For every $n \in \mathbb{N}$ we can find an element of $\mathcal{D}$ whose $L^\infty(\mathbb{T})$–norm is equal to n. Moreover, $\mathcal{D}$ contains elements which are not bounded in this norm[1]. This gives us a "bird's-eye" view on the set of all wavelets. $\mathcal{D}$ is a "humongous" set with an interesting structure. Every element d of this set represents a whole family of wavelets whose dimension functions are equal to d. MRA wavelets are just a tiny point on this map corresponding to the element of $\mathcal{D}$ which is a constant function equal to 1. However, the idea of the multiresolution analysis which is the essence of wavelets given by this simple dimension function can be extended to all wavelets. Moreover, this idea can be used to "construct" all wavelets corresponding to a given element d of $\mathcal{D}$. Indeed, let $d \in \mathcal{D}$. Define

$$\Omega_n = \{\xi \in \mathbb{R}\colon d(\xi) \geq n\} \tag{1.7.17}$$

for every $n \in \mathbb{N}$ and a 1–periodic matrix $\tilde{\Omega}$ given by the above sets via (1.7.10). Consider a family of functions $\{\varphi_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ and form a corresponding "vector"

[1] All these statements about $\mathcal{D}$ shall be proven in Chapter 2.

Φ using (1.7.3). We shall prove that Φ is a scaling of an orthonormal wavelet, provided (1.7.11), (1.7.15) and (1.7.16) are satisfied by its Fourier transform $\hat{\Phi}$. Let V be the shift invariant space generated by the integer translates of φ_n's, $n \in \mathbb{N}$. Since (1.7.11) holds for $\hat{\Phi}$ we obtain the following optimal decomposition of V:

$$V = \bigoplus_{n \in \mathbb{N}} V_{\varphi_n}.$$

Moreover, the spectral sequence of V is equal to the sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ given by (1.7.17). Thus, $\tilde{\Omega}$ is the spectral matrix of V and the multiplicity function m_V of V is equal to d. By definition, Φ is a generating vector of V. Since $\hat{\Phi}$ satisfies condition (1.7.15), from Corollary 1.7.4 it follows that V is refinable; that is, condition (C1) of Theorem 1.6.13 is satisfied. The next condition of the theorem follows from the fact that $m_V = d$ and $\|d\|_{L^1(\mathbb{T})} = 1$. We also have that $m_V = d \in \mathcal{D}$, which implies that m_V satisfies the consistency equation; that is, (C3) holds as well. Finally, from formula (1.7.9) it follows that condition (1.7.16) is equivalent to (C4). Therefore, by Theorem 1.6.13 we obtain that V is the core space of an orthonormal wavelet ψ. Since Φ is a generating vector of V, by Definition 1.7.5, it is a scaling vector of ψ.

The above "constructive" procedure allows us to obtain all orthonormal wavelets. However, in reality, condition (1.7.15) is very hard to impose. This program will be caried out in Chapter 3 in the simple case of $d \in \mathcal{D}$ being a constant function equal to 1 (that is, for MRA wavelets). This case is easy to consider, mainly because every scaling vector of an MRA wavelet has only one non–trivial component. In general, Theorem 1.3.8 and Corollary 1.6.5 give us immediately the following:

Corollary 1.7.6. *If ψ is an orthonormal wavelet then every scaling vector of ψ has exactly N non–trivial components if and only if $\|D_\psi\|_{L^\infty(\mathbb{T})} = N$.*

In the final paragraph of this chapter we will use the spectral function to prove Rieffel's incompleteness theorem for Gabor systems. We then shall return to wavelets and cover the topics mentioned above.

1.8 Gabor systems

In 1946, Denis Gabor published a paper, *Theory of Communication* [Ga], in which he considered the following system:

$$\{g_{l,m}\}_{l,m\in\mathbb{Z}}, \qquad \text{where} \qquad g_{l,m}(x) = e^{2\pi i l x} g(x-m) \tag{1.8.1}$$

and g is a Gaussian, that is, $g(x) = e^{-cx^2}$, where $c > 0$. (The same system appeared in the theory of quantum mechanics and was named as *Weyl–Heisenberg system.*) Gabor's idea was to use this system for expanding functions (signals) $f \in L^2(\mathbb{R})$ via

$$f = \sum_{l,m\in\mathbb{Z}} c_{l,m}\, g_{l,m},$$

where $\{c_{l,m}\}_{l,m\in\mathbb{Z}}$ is a set of complex coefficients depending on f. It turned out, however, that in "real life" one would like to reconstruct a given function $f \in L^2(\mathbb{R})$ from the set of coefficients of the form $\{\langle f, g_{l,m}\rangle\}_{l,m\in\mathbb{Z}}$. Only later it was shown by Bacry, Grossmann and Zak [BGZ] that if g is a Gaussian then

$$\inf_{\|f\|_2=1} \sum_{l,m\in\mathbb{Z}} |\langle f, g_{l,m}\rangle|^2 = 0,$$

which leads to numerical instabilities in algorithms recovering f from the coefficients $\{\langle f, g_{l,m}\rangle\}_{l,m\in\mathbb{Z}}$. Moreover, in early 80's, Balian and Low proved independently that any other choice of a "nice" function $g \in L^2(\mathbb{R})$ does not work either[2]. The most general statement of this result is the following (see [HW], [BHW]):

THEOREM 1.8.1. *If $g \in L^2(\mathbb{R})$ and $\{g_{l,m}\}_{l,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ then either*

$$\int_{\mathbb{R}} x^2 |g(x)|^2\, dx = \infty \qquad \text{or} \qquad \int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2\, d\xi = \infty.$$

The Balian–Low obstruction was overcome with a help of the following:

[2] Note, that if we choose $g = \mathbf{1}_{[0,1]}$ then the system given in (1.8.1) is an orthonormal basis of $L^2(\mathbb{R})$ but the function $\mathbf{1}_{[0,1]}$ is not "nice".

DEFINITION 1.8.2. *If $g \in L^2(\mathbb{R})$ and $a, b > 0$, then the system*

$$\{g_{bl,am}\}_{l,m\in\mathbb{Z}}, \qquad \textit{where} \qquad g_{bl,am}(x) = e^{2\pi i b l x} g(x - am)$$

is called a Gabor system.

In 1986, I. Daubechies, A. Grossmann and Y. Meyer constructed a Gabor system forming a tight frame for $L^2(\mathbb{R})$ with g being a compactly supported function in the Schwartz class for every positve a and b, such that $ab < 1$ (see [DGM]). Lyubarskii [Ly] and independently Seip and Wallstén [SW] proved that the original choice of Gabor (that is, g being a Gaussian) gives rise to a frame of $L^2(\mathbb{R})$ if $ab < 1$ (this was conjectured by I. Daubechies and Grossmann in 1988).

As we can see, both of these results utilize the situation $ab < 1$. This is because if $ab > 1$, then the system $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ **can never be complete** in $L^2(\mathbb{R})$ for any choice of $g \in L^2(\mathbb{R})$. The first person who communicated this simple fact was I. Daubechies in [Da]. While doing research on this topic she was informed by her colleagues R. Howe and T. Steger that this incompleteness follows from the work of M.A. Rieffel [Ri] on von Neumann algebras. The argument of R. Howe and T. Steger is very simple but based completely on the paper [Ri], which by itself is not easy to understand without a reasonable knowlege of the theory of C^*–algebras. That is how a quest for finding an "elementary" proof of the incompleteness result was started.

I. Daubechies was able to find a very elegant argument in the case when $ab > 1$ is rational. She constructed a function orthogonal to every element of $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ and the problem of constructing such a function in the general case $ab > 1$ is still open (see [Da]). In 1993, Landau proved that $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ can not be a frame for $L^2(\mathbb{R})$ if $ab > 1$. His result was obtained under a certain assumptions on the decay of g and $\hat{g}$ but applied to irregular Gabor systems as well[3] (see [Lau]).

[3] In irregular Gabor systems the sampling set $\{bl, am\}_{l,m\in\mathbb{Z}}$ is replaced by an arbitrary discrete subset of $\mathbb{R}^2$. Landau has proved his result for irregular sampling sets with uniform Beurling density.

One year later, Janssen showed that for every $g \in L^2(\mathbb{R})$ the system $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ can not be a frame for $L^2(\mathbb{R})$ if $ab > 1$ (see [Ja]). The quest was finalized in a paper *Incompleteness of Sparse Coherent States* by J. Ramanathan and T. Steger which was communicated by G. Weiss and appeared in ACHA [RSt]. The authors were able to prove Landau's result for irregular Gabor systems without any decay assumptions. Their methods allowed them also to recover Rieffel's incompletness theorem[4].

In this paragraph we shall present how the incompletness theorem follows from our study of shift invariant spaces. The purpose of this presentation is to exhibit the strength of the spectral function theory. The basic idea which we are going to utilize can be seen in our proof of the following well known result:

PROPOSITION 1.8.3. *If $g \in L^2(\mathbb{R})$ and the system $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ is orthogonal then $ab \geq 1$ (unless g is the zero function).*

PROOF. Since $\|g_{bl,am}\|_2 = \|g\|_2$ for $l, m \in \mathbb{Z}$, we can assume without loss of generality that the system $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ is orthonormal. Another standard argument allows us to assume that $a = 1$. Indeed, since the "dilation by $\frac{1}{a}$" operator $D_{\frac{1}{a}}$ defined via (1.4.5) is unitary it makes no difference if, instead of $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$, we consider $\{g'_{abl,m}\}_{l,m\in\mathbb{Z}}$, where $g' = D_{\frac{1}{a}}(g)$. Therefore, we assume that the system $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ is orthonormal and we shall prove that $b \geq 1$.

Let M_b be the modulation operator defined in (1.4.4). For every $l \in \mathbb{Z}$ define

$$G_l = \overline{\operatorname{span}\{g_{bl,m}\}_{m\in\mathbb{Z}}}. \tag{1.8.2}$$

The space G_l is a principal shift invariant space generated by $M_{lb}(g)$. Therefore,

$$G_l = M_b^l(G_0) \tag{1.8.3}$$

[4]They also conjectured that the incompletness result can be extended to the case of irregular sampling sets with a uniform density smaller then 1. This conjecture was later disproved in [BHW].

(note that this resembles the multiresolution analysis situation, where $V_0 = D^j(V_0)$). Since the system $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ is orthonormal we can consider the following shift invariant space

$$S = \bigoplus_{l\in\mathbb{Z}} G_l.$$

We shall use Definition 1.4.2 to compute the spectral function of S. We have

$$\sigma_S = \sum_{l\in\mathbb{Z}} \sigma_{G_l}.$$

By (1.8.3) and property (f) of Proposition 1.4.6 we obtain

$$\sigma_{G_l}(\xi) = \sigma_{G_0}(\xi - bl)$$

for a.e. $\xi \in \mathbb{R}$ and $l \in \mathbb{Z}$. The spectral function of G_0 is easy to compute because the integer translates of g form an orthonormal basis of G_0. Indeed, the remark below Definition 1.4.2 allows us to conclude that $\sigma_{G_0} = |\hat{g}|^2$. All of the above shows that

$$\sigma_S(\xi) = \sum_{l\in\mathbb{Z}} |\hat{g}(\xi + bl)|^2 \tag{1.8.4}$$

for a.e. $\xi \in \mathbb{R}$. Property (a) of Proposition 1.4.6 assures that $1 \geq \sigma_S$. Integrating this inequality over the interval $[0, b]$ and utilizing (1.8.4) leads to the following:

$$b \geq \int_0^b \sigma_S(\xi)\, d\xi = \int_0^b \sum_{l\in\mathbb{Z}} |\hat{g}(\xi + bl)|^2\, d\xi = \sum_{l\in\mathbb{Z}} \int_{bl}^{b(l+1)} |\hat{g}(\xi)|^2\, d\xi \tag{1.8.5}$$

$$= \int_{\mathbb{R}} |\hat{g}(\xi)|^2\, d\xi = \|g\|_2^2 = 1.$$

That is, $b \geq 1$, which is exactly what we wanted to prove. ♣

To prove the incompleteness result we shall use the "multiresolution" scheme (1.8.3) to show that the spectral function of $S = \overline{\operatorname{span}\{g_{bl,m}\}_{l,m\in\mathbb{Z}}}$ is given roughly by (1.8.4). Then we will use the assumption $S = L^2(\mathbb{R})$ (that is, $\sigma_S = 1$ a.e.) to conclude that $b \leq 1$ (this final step will be done via an "integral" argument analogous to (1.8.5)). The main ingrediens in achieving all of this will be Corollary 1.2.9 and the following:

PROPOSITION 1.8.4. *If V is a shift invariant space and W is a principal shift invariant space, then $(V+W)\ominus V$ is principal as well.*

PROOF. This is a simple consequence of the following general fact from linear algebra: If A is a closed linear subspace (of $\ell^2(\mathbb{Z})$) and B is a 1–dimensional subspace (of $\ell^2(\mathbb{Z})$), then $dim((A+B)\ominus A) \leq 1$. To be precise, let us denote the multiplicity function of the space $(V+W)\ominus V$ by $m_{(V+W)\ominus V}$. The above gives us immediately

$$m_{(V+W)\ominus V}(\xi) \leq 1$$

for a.e. $\xi \in \mathbb{T}$, which proves that the space $(V+W)\ominus V$ is principal. ♣

At this point we are ready to prove the incompleteness result for Gabor systems.

THEOREM 1.8.5. *If $g \in L^2(\mathbb{R})$ and $ab > 1$ then the system $\{g_{bl,am}\}_{l,m\in\mathbb{Z}}$ is incomplete in $L^2(\mathbb{R})$.*

PROOF. The standard argument given in the proof of Proposition 1.8.3 allows us to assume that $a = 1$. We present a proof by contradiction; that is, we shall assume that the system $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ is complete and we will prove that $b \leq 1$ follows.

We start by constructing a "multiresolution" scheme for Gabor systems. Let $g \in L^2(\mathbb{R})$ and $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ be the corresponding Gabor system. For every $l \in \mathbb{Z}$ define G_l via (1.8.2). G_l is a principal shift invariant space generated by $M_{bl}(g)$. Let

$$V_l = \overline{\operatorname{span}\{g_{bk,m} \colon k < l, m \in \mathbb{Z}\}} = \overline{\biguplus_{k<l} G_k} \tag{1.8.6}$$

and S denote the closed subspace of $L^2(\mathbb{R})$ spanned by $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ (the discussion is general and we do not assume yet that the system is complete). We have the following

(G1) $V_l \subset V_{l+1}$ for all $l \in \mathbb{Z}$,

(G2) $M_b(V_l) = V_{l+1}$ for all $l \in \mathbb{Z}$,

(G3) $\bigcup_{l\in\mathbb{Z}} V_l$ is dense in S,

(G4) $\bigcap_{l\in\mathbb{Z}} V_l = \{0\}$,

(G5) V_0 is shift invariant.

Indeed, all the above conditions, except (G4), follow immediately from (1.8.6). In general, condition (G4) is hard to prove and we shall not do it. Instead, we will, at first, assume that g is band–limited. This assumption easily implies that (G4) holds. In fact, if $\operatorname{supp}\hat{g} \subset [-L, L]$ for some positive L then $\hat{V}_0 \subset L^2([-L,\infty))$. Thus,

$$\bigcap_{l\in\mathbb{Z}} \hat{V}_l = \bigcap_{l\in\mathbb{Z}} \widehat{\left(M_{bl}(V_0)\right)} \subset \bigcap_{l\in\mathbb{Z}} L^2([-L+bl,\infty)) = \{0\},$$

hence, (G4) follows.

We shall use (G1)–(G5) in exactly the same way we have used (M1)–(M5) of Definition 1.6.1 to construct wavelets. In our case for every $l \in \mathbb{Z}$ we define a space

$$W_l = V_{l+1} \ominus V_l. \tag{1.8.7}$$

By (G2) we have

$$M_b(W_l) = W_{l+1}. \tag{1.8.8}$$

From (G4) it follows that for every $l \in \mathbb{Z}$

$$V_l = \bigoplus_{k<l} W_k. \tag{1.8.9}$$

Indeed, the inclusion $\bigoplus_{k<l} W_k \subset V_l$ is clear. To see the other inclusion, observe that if $f \in V_l$ and $f \perp W_k$ for $k < l$, then by (1.8.7) we have $f \in V_k$ for $k < l$. Thus, $f \in \bigcap_{k<l} V_k = \bigcap_{k\in\mathbb{Z}} V_k = \{0\}$, hence f is the zero function, which proves (1.8.9). Finally, from (G3) and (1.8.8) it follows immediately that

$$S = \bigoplus_{l\in\mathbb{Z}} W_l. \tag{1.8.10}$$

The above decomposition allows us to see that the spectral function of S is given by a formula analogous to (1.8.4). In fact, we have

$$W_0 = V_1 \ominus V_0 = (V_0 + G_0) \ominus V_0,$$

so that, by Proposition 1.8.4, the space W_0 is principal. From Corollary 1.2.9 it follows that W_0 has a tight frame generator φ. Therefore, the spectral function of W_0 is given by $\sigma_{W_0} = |\hat{\varphi}|^2$. Thus, (1.8.8) and (1.8.10) give us

$$\sigma_S(\xi) = \sum_{l\in\mathbb{Z}} |\hat{\varphi}(\xi + bl)|^2 \tag{1.8.11}$$

for a.e. $\xi \in \mathbb{R}$.

The system $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ is complete in $L^2(\mathbb{R})$ if and only if $S = L^2(\mathbb{R})$, which by property (e) of Proposition 1.4.6 is equivalent to having $\sigma_S = 1$ a.e. If we integrate the equality over the interval $[0, b]$ then, by (1.8.11), we obtain

$$b = \int_0^b \sum_{l\in\mathbb{Z}} |\hat{\varphi}(\xi + bl)|^2 \, d\xi = \|\varphi\|_2^2.$$

Since φ is a tight frame generator of W_0 we must have $\|\varphi\|_2 \leq 1$ (if we use Theorem 1.2.8 we can even see that $\|\varphi\|_2 = |\mathbb{T} \cap \Omega|$, where Ω is the spectrum of W_0), which proves that $b \leq 1$.

To drop the assumption that g is band–limited we approximate g with a sequence of band–limited functions $\{g^j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R})$ in the $L^2(\mathbb{R})$–norm. For example, we can define such a sequence by $g^j = (\mathbf{1}_{[-j,j]}\hat{g})\check{}$ for $j \in \mathbb{N}$, and then

$$\lim_{j\to\infty} \|g - g^j\|_2 = 0. \tag{1.8.12}$$

Let S^j denote the closure of the space spanned by the Gabor system $\{g^j_{bl,m}\}_{l,m\in\mathbb{Z}}$ and P_{S^j} be the orthogonal projection onto S^j for $j \in \mathbb{N}$. For every $l, m \in \mathbb{Z}$ we have

$$\|g_{bl,m} - g^j_{bl,m}\|_2 = \|g - g^j\|_2.$$

Thus, the completeness of the system $\{g_{bl,m}\}_{l,m\in\mathbb{Z}}$ and (1.8.12) imply that for every $f \in L^2(\mathbb{R})$

$$\lim_{j\to\infty} \|P_{S^j} f - f\|_2 = 0.$$

This allows us to apply Lemma 1.4.8. Since $\sigma_{L^2(\mathbb{R})} = 1$ a.e., from the lemma it follows that

$$\lim_{n\to\infty} \sigma_{S^{j_n}}(\xi) = 1 \text{ a.e.},$$

where $\{j_n\}_{n\in\mathbb{N}}$ is a subsequence of $\{j\}_{j\in\mathbb{N}}$. For every $n \in \mathbb{N}$ the function g^{j_n} is band–limited, therefore, by (1.8.11) we have

$$\sigma_{S^{j_n}}(\xi) = \sum_{l\in\mathbb{Z}} |\hat{\varphi}_n(\xi + bl)|^2,$$

where $\varphi_n \in L^2(\mathbb{R})$ and $\|\varphi_n\|_2 \le 1$ for $n \in \mathbb{N}$. The above formula and (1.8.13) give us

$$\lim_{n\to\infty} \sum_{l\in\mathbb{Z}} |\hat{\varphi}_n(\xi + bl)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$. Integrating this limit equality over the interval $[0, b]$ proves that $b \le 1$. Indeed, by Fatou's lemma we have

$$b = \int_0^b \lim_{n\to\infty} \sum_{l\in\mathbb{Z}} |\hat{\varphi}_n(\xi+bl)|^2 \, d\xi \le \liminf_{n\to\infty} \int_0^b \sum_{l\in\mathbb{Z}} |\hat{\varphi}_n(\xi+bl)|^2 \, d\xi = \liminf_{n\to\infty} \|\varphi_n\|_2^2 \le 1,$$

which concludes the proof. ♣

Chapter 2

DIMENSION FUNCTION OF ORTHONORMAL WAVELETS

2.1 Background

Due to historicall reasons, in Chapter 2 and Chapter 3 we shall use the Fourier transform $\widehat{\ }$ defined by

$$f(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$$

for $f \in L^2(\mathbb{R})$.

The *dimension function* of an orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is defined as

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2.$$

The importance of the dimension function was discovered by P.G. Lemarié, who used it to prove that certain wavelets are associated with a multiresolution analysis (see [Le3], [Le4]). After that G. Grippenberg in [Gr] and Wang in [Wa] independently characterized all wavelets associated with an MRA. The well known characterization establishes that a wavelet ψ is an MRA wavelet if and only if $D_\psi(\xi) = 1$ for almost every $\xi \in \mathbb{R}$. However P. Auscher [Au] proved something more. His deep theorem shows that the dimension function of a wavelet describes dimensions of certain subspaces of $\ell^2(\mathbb{Z})$; in particular, it is integer valued. The proper attention to the dimension function was brought by G. Weiss and E. Hernández in [HW], where the name "dimension function" appears for the first time. The dimension function plays an important role in understanding wavelets which cannot be associated with a multiresolution analysis. We are, therefore, interested in the study of the dimension functions of orthonormal wavelets; in particular in a characterization of dimension functions of orthonormal wavelets. Such a characterization can be obtain with a use of generalized scaling sets, which will be discussed in the next section.

Another tool we are going to use in the characterization is the *translation projection* τ, which is defined on $\mathbb{R}$ by $\tau(\xi) = \xi'$, where $\xi' \in [-\pi, \pi)$ and $\xi' - \xi = 2k\pi$ for some $k \in \mathbb{Z}$. For a given set $E \subset \mathbb{R}$ we shall denote its 2π–periodization $\bigcup_{k\in\mathbb{Z}}(E+k)$ by E^P. A basic property of the translation projection we shall need is the following:

LEMMA 2.1.1. *Let $\tilde{E}$ be a measurable subset of $\mathbb{R}$. Then there exists a measurable set $E \subset \tilde{E}$, such that $\tau(E) = \tau(\tilde{E})$ and $\tau|_E$ is injective.*

PROOF. Let us define $I_k = [2k\pi, 2(k+1)\pi)$ for $k \in \mathbb{Z}$. Fix any bijection $\alpha\colon \mathbb{N} \to \mathbb{Z}$ and define $E'_{\alpha(n)} = \tilde{E} \cap I_{\alpha(n)}$ for $n \in \mathbb{N}$. Let $E_{\alpha(1)} = E'_{\alpha(1)}$ and $E_{\alpha(n)} = E'_{\alpha(n)} \setminus \bigcup_{i=1}^{n-1} E^P_{\alpha(i)}$ for $n \geq 2$. We claim that $E = \bigcup_{n=1}^{\infty} E_{\alpha(n)}$ is the appropriate subset of $\tilde{E}$.

To check that $\tau(E) = \tau(\tilde{E})$, it is enough to prove that $\tilde{E} \subset E^P$. Let $\xi \in \tilde{E}$; it is clear that $\xi \in E'_{\alpha(n_0)}$ for some $n_0 \in \mathbb{N}$. If $n_0 = 1$, then $\xi \in E'_{\alpha(1)} = E_{\alpha(1)} \subset E^P$. For $n_0 \geq 2$, we have either $\xi \in E'_{\alpha(n_0)} \setminus \bigcup_{i=1}^{n_0-1} E^P_{\alpha(i)} = E_{\alpha(n_0)} \subset E^P$, or $\xi \in \bigcup_{i=1}^{n_0-1} E^P_{\alpha(i)} \subset E^P$, either of which implies that $\xi \in E^P$.

To prove the second condition, let us consider $\xi_1, \xi_2 \in E$ such that $\tau(\xi_1) = \tau(\xi_2)$. Without loss of generality we can assume that $\xi_1 \in E_{\alpha(n)}$ and $\xi_2 \in E_{\alpha(m)}$, where $n \leq m$. If $n < m$, then $\xi_2 \notin E^P_{\alpha(n)}$, which contradicts $\tau(\xi_1) = \tau(\xi_2)$; therefore, $n = m$. This means that $\xi_1, \xi_2 \in E_{\alpha(n)} \subset I_{\alpha(n)}$, but since $\tau|_{I_{\alpha(n)}}$ is injective, we obtain $\xi_1 = \xi_2$. ♣

It is also of interest to point out that for every function $f \in L^2(\mathbb{R})$ we can define

$$D_f(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{f}(2^j(\xi + 2k\pi))|^2. \tag{2.1.1}$$

The following observation allows to conclude that the dimension function of orthonormal wavelets is finite almost everywhere

Proposition 2.1.2. *If* $f \in L^2(\mathbb{R})$, *then* $\|D_f\|_{L^1(\mathbb{T})} = 2\pi\|f\|^2_{L^2(\mathbb{R})}$.

Proof. We have

$$\int_{-\pi}^{\pi} D_f(\xi)d\xi = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi}|\hat{f}(2^j(\xi+2k\pi))|^2 d\xi =$$

$$\sum_{j=1}^{\infty}\int_{\mathbb{R}}|\hat{f}(2^j\xi)|^2 d\xi = \|\hat{f}\|_2^2\sum_{j=1}^{\infty}2^{-j} = \|\hat{f}\|_2^2 = 2\pi\|f\|^2_{L^2(\mathbb{R})}. \quad \clubsuit$$

In our considerations we will also use the discrete Calderon reproducing formula

$$\sum_{j\in\mathbb{Z}}|\hat{\psi}(2^j\xi)|^2 = 1 \quad \text{a.e.} \tag{2.1.2}$$

and

$$\sum_{k\in\mathbb{Z}}|\hat{\psi}(\xi+2k\pi)|^2 = 1 \quad \text{a.e.} \tag{2.1.3}$$

Both of the above formulae hold for every wavelet ψ (see Theorem 1.5.2 and Theorem 1.2.8).

2.2 Generalized scaling sets

An *MSF wavelet* is an orthonormal wavelet ψ such that $|\hat{\psi}| = \mathbf{1}_W$ for some measurable set $W \subset \mathbb{R}$, where $\mathbf{1}_W$ denotes the characteristic function of a set W. In this case the set W is called a *wavelet set.* The following theorem characterizes all wavelet sets (Theorem 2.3 §7.2 in [HW])

THEOREM 2.2.1. *A measurable set $W \subset \mathbb{R}$ is a wavelet set if and only if*

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_W(\xi + 2k\pi) = 1 \quad a.e. \tag{2.2.1}$$

and

$$\sum_{j\in\mathbb{Z}} \mathbf{1}_W(2^j\xi) = 1 \quad a.e. \tag{2.2.2}$$

If ψ is an MSF wavelet which comes from a multiresolution analysis, then its scaling function φ satisfies $|\hat{\varphi}| = \mathbf{1}_S$ for some measurable set $S \in \mathbb{R}$. Conversely, if φ is a scaling function of some MRA, such that $|\hat{\varphi}| = \mathbf{1}_S$, then every orthonormal wavelet generated by the MRA is an MSF wavelet. Such a set S is called a *scaling set,* and the following characterization of scaling sets can be find in [PSW].

THEOREM 2.2.2. *A measurable set $S \subset \mathbb{R}$ is a scaling set if and only if $S \subset 2S$, $\lim_{n\to\infty} \mathbf{1}_S(2^{-n}\xi) = 1$ for almost all $\xi \in \mathbb{R}$ and $\sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi) = 1$, a.e..*

In [PSW], it is also shown that a wavelet set W and the associated scaling set S satisfy $W = 2S \setminus S$, $|S| = 2\pi$ and $S = \bigcup_{j=1}^{\infty} 2^{-j}W$. These equations allow us to define a "scaling set" for every MSF wavelet, not only for these coming from a multiresolution analysis.

DEFINITION 2.2.3. *A set $S \subset \mathbb{R}$ is called a generalized scaling set if $|S| = 2\pi$ and $2S \setminus S$ is a wavelet set.*

An equivalent definition can be formulated as follows

PROPOSITION 2.2.4. *A set $S \subset \mathbb{R}$ is a generalized scaling set if and only if $S = \bigcup_{j=1}^{\infty} 2^{-j}W$ for some wavelet set W.*

PROOF. Let us assume that $S = \bigcup_{j=1}^{\infty} 2^{-j}W$ for some wavelet set W. From (2.2.2) it follows that the union is disjoint; therefore, $2S \setminus S = W$. Moreover, (2.2.1) implies that $|W| = 2\pi$; hence, $|S| = \sum_{j=1}^{\infty} 2^{-j}|W| = 2\pi$.

To prove the other implication, denote $2S \setminus S$ by W and observe that since $|W| = 2\pi$ we must have $S \subset 2S$. From this easily follows that $\bigcup_{j=1}^{\infty} 2^{-j} W \subset S$, but since both these sets have the same measure they must be equal. ♣

An example of a generalized scaling set that is not a scaling set is the following:

$$S = \Big[-\frac{16\pi}{7}, -2\pi \Big] \cup \Big[-\frac{8\pi}{7}, -\pi \Big] \cup \Big[-\frac{4\pi}{7}, \frac{4\pi}{7} \Big] \cup \Big[\pi, \frac{8\pi}{7} \Big] \cup \Big[2\pi, \frac{16\pi}{7} \Big].$$

The set S given above is the generalized scaling set associated to the Journé wavelet set.

A theorem similar to Theorem 2.2.2 which characterizes all generalized scaling sets can be found in [BMM]. The proof presented there uses methods of abstract harmonic analysis (in particular, the multiplicity function of projection valued measures). Let us present the theorem with an elementary proof.

THEOREM 2.2.5. *A measurable set $S \subset \mathbb{R}$ is a generalized scaling set if and only if*

(i) $|S| = 2\pi$,

(ii) $S \subset 2S$,

(iii) $\lim_{n\to\infty} \mathbf{1}_S(2^{-n}\xi) = 1$ *for almost all* $\xi \in \mathbb{R}$,

(iv) $D(\xi) + D(\xi + \pi) = D(2\xi) + 1$ *a.e.*,

where $D(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi)$.

PROOF. Let us prove that conditions (i)-(iv) are necessary. The first one is guaranteed by definition, the second one follows easily from Proposition 2.2.4. Moreover, since $S = \bigcup_{j=1}^{\infty} 2^{-j} W$ for some wavelet set W and the union is disjoint, we can write $\mathbf{1}_S(2^{-n}\xi) = \sum_{j=-n+1}^{\infty} \mathbf{1}_W(2^j\xi)$ and use (2.2.2) to obtain that the limit as $n \to \infty$ is 1, which establishes (iii). A simple calculation shows that

$$(2.2.3)\quad D(\xi)+D(\xi+\pi) = \sum_{k\in\mathbb{Z}}\sum_{j=1}^{\infty} \mathbf{1}_W(2^j(\xi+2k\pi)) + \sum_{k\in\mathbb{Z}}\sum_{j=1}^{\infty} \mathbf{1}_W(2^j(\xi+\pi+2k\pi)) =$$

$$\sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\mathbf{1}_W(2^j(2\xi+2k\pi)) = D(2\xi) + \sum_{k\in\mathbb{Z}}\mathbf{1}_W(2\xi+2k\pi),$$

therefore (iv) follows from (2.2.1).

To prove sufficiency define W as $2S\setminus S$. It is easy to see that W and 2^jW are disjoint for every $j\in\mathbb{N}$. In fact, $2^jW = 2^{j+1}S\setminus 2^jS$, and by (ii) $W\subset 2^jS$, which together imply $W\cap 2^jW=\emptyset$. In this way we obtain that $2^jW\cap 2^kW=\emptyset$ for $j,k\in\mathbb{Z}$, $j\neq k$. This allows us to check that $S=\bigcup_{j=1}^{\infty}2^{-j}W$. Indeed, by (ii) we have $\bigcup_{j=1}^{\infty}2^{-j}W\subset S$, but both these sets have the same measure; therefore, they are equal. This shows that $\mathbf{1}_S(\xi)=\sum_{j=1}^{\infty}\mathbf{1}_W(2^j\xi)$, so (iii) implies that W satisfies (2.2.2). Using (iv) together with (2.2.3) we obtain that (2.2.1) is fulfilled as well; therefore, W is a wavelet set. ♣

Remark. Suppose that a measurable set S satisfies conditions (i)-(iv) of Theorem 2.2.5. Then, as we have shown, $W=2S\setminus S$ is a wavelet set and

$$\sum_{k\in\mathbb{Z}}\mathbf{1}_S(\xi+2k\pi) = D_\psi(\xi)\quad a.e. \tag{2.2.4}$$

where $\hat{\psi}=\mathbf{1}_W$.

2.3 Characterization of dimension functions of wavelets

As we mentioned before the dimension function can attain only integer values. Although this fact follows immediately form Corollary 1.6.5 let us present briefly the original argument of P. Auscher. For an orthonormal wavelet ψ, we can define the vector

$$\Psi_j(\xi)=\{\hat{\psi}(2^j(\xi+2k\pi))\colon k\in\mathbb{Z}\},\quad j\geq 1,$$

which belongs to $\ell^2(\mathbb{Z})$ for almost every ξ. It turns out that the vectors Ψ_j satisfy the following reproduction formula

$$\Psi_n(\xi)=\sum_{j=1}^{\infty}\langle\Psi_n(\xi),\Psi_j(\xi)\rangle_{\ell^2}\Psi_j(\xi).$$

Surprisingly, this implies that $\dim \overline{\operatorname{span}\{\Psi_j(\xi) \colon j \geq 1\}} = \sum_{j=1}^{\infty} \|\Psi_j(\xi)\|_{\ell^2}^2$, and a simple calculation shows that the last sum is equal to the dimension function of ψ (for references see §7.3 in [HW]).

Recently, L. Baggett, H. Medina and K. Merrill observed that the dimension function has one more property, that is, it satisfies the following *consistency equation*

$$D_\psi(\xi) + D_\psi(\xi + \pi) = D_\psi(2\xi) + 1 \quad a.e.$$

Another key contribution to the study of the dimension function of a wavelet is due to D. Speegle, who observed that is has to satisfy the following condition

$$\sum_{k \in \mathbb{Z}} \mathbf{1}_\Delta(\xi + 2k\pi) \geq D_\psi(\xi) \quad a.e. \tag{2.3.1}$$

where $\Delta = \{\xi \in \mathbb{R} \colon D_\psi(2^{-j}\xi) \geq 1 \text{ for } j \in \mathbb{N} \cup \{0\}\}$. Before we proceed further let us prove a technical lemma which converts condition (2.3.1) into a statement more convenient for the proof of the characterization.

LEMMA 2.3.1. *Let $D : \mathbb{R} \to \mathbb{N} \cup \{0\}$ be a measurable 2π-periodic function which satisfies (2.3.1). Let $A_j = \{\xi \in [-\pi, \pi] \colon D(\xi) \geq j\}$ for $j \in \mathbb{N}$ and let $\{S_i\}_{i=1}^n$, where $n \in \mathbb{N}$ is fixed, be a collection of measurable sets such that $\tau(S_i) = A_i$ and $\tau|_{S_i}$ is injective for $i = 1, \ldots, n$. Then*

(a) there exists a measurable set G, such that $\tau(G) = A_1$ and $D(2^{-j}\xi) \geq 1$ for $\xi \in G$ and $j \geq 0$,

(b) there exists a measurable set H disjoint from $\bigcup_{i=1}^n S_i$, such that $\tau(H) = A_{n+1}$ and $D(2^{-j}\xi) \geq 1$ for $\xi \in H$ and $j \geq 0$,

PROOF. (a) Recall that $\Delta = \{\xi \in \mathbb{R} : D(2^{-j}\xi) \geq 1 \text{ for all } j \geq 0\}$. Define $G = \Delta$. All we have to prove is that $\tau(G) = A_1$. Since the inclusion $\tau(G) \subset A_1$ is obvious it is enough to check that the opposite inclusion holds. By condition (2.3.1) we have $\sum_{k \in \mathbb{Z}} \mathbf{1}_G(\xi + 2k\pi) \geq D(\xi) \geq \mathbf{1}_{A_1}(\xi)$, which implies that $A_1 \subset G^P$; i.e. $A_1 \subset \tau(G)$.

(b) Define $H = (\Delta \setminus \bigcup_{i=1}^n S_i) \cap A_{n+1}^P$. Since $H \cap \bigcup_{i=1}^n S_i = \emptyset$ and $\tau(H) \subset A_{n+1}$ we can see that all we have to do is prove that $A_{n+1} \subset H^P$. By condition (2.3.1) and our assumption about $\{S_i\}_{i=1}^n$ we have

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + 2k\pi) \geq D(\xi) \geq \sum_{k\in\mathbb{Z}} \sum_{i=1}^{n+1} \mathbf{1}_{A_i}(\xi + 2k\pi) = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{n} \mathbf{1}_{S_i}(\xi + 2k\pi) +$$

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{A_{n+1}}(\xi + 2k\pi) \geq \sum_{k\in\mathbb{Z}} \mathbf{1}_{\cup_{i=1}^n S_i}(\xi + 2k\pi) + \mathbf{1}_{A_{n+1}}(\xi).$$

In this way we obtain

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{\Delta\setminus\bigcup_{i=1}^n S_i}(\xi + 2k\pi) \geq \sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + 2k\pi) - \sum_{k\in\mathbb{Z}} \mathbf{1}_{\cup_{i=1}^n S_i}(\xi + 2k\pi) \geq \mathbf{1}_{A_{n+1}}(\xi),$$

which implies that $A_{n+1} \subset (\Delta \setminus \bigcup_{i=1}^n S_i)^P$, i.e. $A_{n+1} \subset H^P$. ♣

The following theorem, which was proven in collaboration with D. Speegle gives a full characterization of dimension functions of wavelets

THEOREM 2.3.2. *Let $D : \mathbb{R} \to \mathbb{N} \cup \{0\}$ be a measurable 2π-periodic function. Then, D is a dimension function of some wavelet if and only if the following conditions are satisfied.*

(i) $\int_{-\pi}^{\pi} D(\xi)\, d\xi = 2\pi$,

(ii) $\liminf_{n\to\infty} D(2^{-n}\xi) \geq 1$ *a.e.*,

(iii) $D(\xi) + D(\xi + \pi) = D(2\xi) + 1$ *a.e.*,

(iv) $\sum_{k\in\mathbb{Z}} \mathbf{1}_\Delta(\xi + 2k\pi) \geq D(\xi)$ *a.e.*,

where $\Delta = \{\xi \in \mathbb{R}\colon D(2^{-j}\xi) \geq 1$ *for* $j \in \mathbb{N} \cup \{0\}\}$.

PROOF. Let us begin by proving that the conditions (i)-(iv) are necessary. To do so we assume that D is a dimension function of some wavelet ψ, i.e. $D(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2$. Therefore, by Proposition 2.1.2 we obtain

$\int_{-\pi}^{\pi} D(\xi)\, d\xi = 2\pi\|\psi\|^2 = 2\pi$, so (i) is proven. To see that (ii) is true let us observe that $D(\xi) \geq \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$. Thus, (ii) follows from (2.1.2). An easy computation similar to (2.2.3) shows that

$$D(\xi) + D(\xi + \pi) = D(2\xi) + \sum_{k\in\mathbb{Z}} |\hat{\psi}(2\xi + 2k\pi)|^2;$$

hence, after applying (2.1.3) we obtain (iii). To prove the last condition let us denote $s(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$. By (2.1.2) it is clear that $s(\xi) \leq 1$; moreover, we have $s(\xi) \geq s(2\xi)$ and $D(\xi) = \sum_{k\in\mathbb{Z}} s(\xi + 2k\pi)$. Therefore all we have to prove is $s(\xi) \leq \mathbf{1}_\Delta(\xi)$ a.e. It is true for $\xi \in \Delta$. On the other hand if $\xi \notin \Delta$ then $D(2^{-j}\xi) = 0$ for some $j \geq 0$, therefore $s(2^{-j}\xi) = 0$, but $s(2^{-j}\xi) \geq s(\xi)$, so we obtain that $s(\xi) = 0$, i.e. $s(\xi) \leq \mathbf{1}_\Delta$ holds almost everywhere.

We now turn to proving the sufficiency of conditions (i)-(iv). By Theorem 2.2.5 and the remark following it, it suffices to find a measurable set S such that

(a) $|S| = 2\pi$,

(b) $S \subset 2S$,

(c) $\lim_{n\to\infty} \mathbf{1}_S(2^{-n}\xi) = 1$ for almost all $\xi \in \mathbb{R}$, and

(d) $D(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi)$.

Denote $A_k = \{\xi \in [-\pi, \pi] : D(\xi) \geq k\}$, and fix any measurable set $Q \subset \mathbb{R}$, such that $Q \subset 2Q$, $\tau|_Q$ is injective, $\lim_{n\to\infty} \mathbf{1}_Q(2^{-n}\xi) = 1$ a.e. and $D(\xi) \geq 1$ for $\xi \in Q$. Condition (ii) implies that $Q = [-\pi, \pi) \cap \Delta$ is an example of a set which satisfies these properties.

Claim 1. *There is a measurable set $S_1 \subset \mathbb{R}$ such that*

(1) $Q \subset S_1$,

(2) $S_1 \subset 2S_1$,

(3) $\tau|_{S_1}$ is injective, and

(4) $\tau(S_1) = A_1$.

Proof of claim 1. Let us denote $E_1 = Q$. For $n \in \mathbb{N}$ define

$$\tilde{E}_{n+1} = (2E_n \setminus \bigcup_{i=1}^{n} E_i^P) \cap A_1^P,$$

and let $E_{n+1} \subset \tilde{E}_{n+1}$ be the set guaranteed to exist by Lemma 2.1.1.

Define $S_1 = \bigcup_{i=1}^{\infty} E_i$. We submit that S_1 satisfies conditions (1) through (4) above. Indeed, (1) is obvious from the definition. To see (2), note that by the construction, $E_1 \subset 2E_1$ and $E_n \subset 2E_{n-1}$ for all $n \geq 2$. Therefore, $2S_1 = \bigcup_{i=1}^{\infty} 2E_i \supset \bigcup_{i=1}^{\infty} E_i = S_1$.

To see (3), if $\xi_1, \xi_2 \in S_1$ and $\tau(\xi_1) = \tau(\xi_2)$, then there exists $j, k \in \mathbb{N}$ such that $\xi_1 \in E_j$ and $\xi_2 \in E_k$. Without loss of generality, $j \leq k$. If $j < k$, then $\xi_2 \notin E_j^P$, which contradicts $\tau(\xi_1) = \tau(\xi_2)$. Therefore, $j = k$; however, $\tau|_{E_j}$ is injective, so $\xi_1 = \xi_2$.

We now turn to proving (4). By definition of S_1 we have $\tau(S_1) \subset A_1$. It remains to show that $A_1 \subset \tau(S_1)$. By condition (iv) and Lemma 2.3.1, there is a set $G \subset \mathbb{R}$ such that $\tau(G) = A_1$, and $D(2^{-j}\xi) \geq 1$ for every $j \geq 0$ and every $\xi \in G$.

Subclaim. *For $j \geq 1$ we have $G \cap 2^j S_1^P \subset 2^{j-1} S_1^P$.*

Proof of subclaim. Let $\xi \in G \cap 2^j S_1^P$. Since $2^{-j}\xi + 2l\pi \in S_1$ for some $l \in \mathbb{Z}$, it follows that $2^{-j}\xi + 2l\pi \in E_m$ for some $m \in \mathbb{N}$. Then, $\xi' := 2^{-j+1}\xi + 4l\pi \in 2E_m$, and by definition of G, $D(\xi') \geq 1$, so $\xi' \in 2E_m \cap A_1^P$. Recall that $\tilde{E}_{m+1} = (2E_m \setminus \bigcup_{i=1}^{m} E_i^P) \cap A_1^P$. Therefore, if $\xi' \notin \bigcup_{i=1}^{m} E_i^P$, then $\xi' \in \tilde{E}_{m+1} \subset S_1^P$, i.e. $\xi \in 2^{j-1} S_1^P$. On the other hand, if $\xi' \in E_i^P$ for some $i = 1, \ldots, m$, then, again, $\xi' \in S_1^P$, so $\xi \in 2^{j-1} S_1^P$ and the proof of the subclaim is completed.

To finish the proof of claim 1, notice that from the assumptions we made about Q it follows that $\bigcup_{j=1}^{\infty} 2^j Q = \mathbb{R}$. Therefore, $G = \bigcup_{j=1}^{\infty} (G \cap 2^j Q) \subset \bigcup_{j=1}^{\infty} (G \cap 2^j S_1^P)$. By iterating the subclaim, we obtain that $G \cap 2^j S_1^P \subset S_1^P$ for $j \geq 1$. So, $G \subset S_1^P$, i.e. $\tau(G) \subset \tau(S_1)$, and since $\tau(G) = A_1$ the proof of claim 1 is finished.

We continue defining the set S. Suppose that there exist sets $S_1, \dots, S_n$ such that if for $i \le n$ we define $P_i = \bigcup_{j=1}^{i} S_j$, then the collections $\{S_i\}_{i=1}^n$ and $\{P_i\}_{i=1}^n$ satisfy

(1) $Q \subset P_1$,

(2) $P_i \subset 2P_i$ for $i = 1, \dots, n$,

(3) $\tau|_{S_i}$ is injective for $i = 1, \dots, n$,

(4) $S_i \cap S_j = \emptyset$ for all $i, j = 1, \dots, n$, where $i \neq j$ and

(5) $\tau(S_i) = A_i$ for $i = 1, \dots, n$.

Then, we will construct S_{n+1} such that $S_1, \dots, S_{n+1}$ and $P_1, \dots, P_{n+1}$ satisfy (1)-(5). Recall that $A_{n+1} = \{\xi \in [-\pi, \pi] : D(\xi) \ge n+1\}$.

Define $\tilde{F}_1 = (2P_n \setminus P_n) \cap A_{n+1}^P$. By Lemma 2.1.1, there is a measurable subset $F_1 \subset \tilde{F}_1$ such that $\tau|_{F_1}$ is injective and $\tau(F_1) = \tau(\tilde{F}_1)$.

For $m \in \mathbb{N}$ define

$$\tilde{F}_{m+1} = (2F_m \setminus \bigcup_{i=1}^{m} F_i^P) \cap A_{n+1}^P,$$

and let F_{m+1} be the subset of $\tilde{F}_{m+1}$ guaranteed to exist by Lemma 2.1.1.

Let $S_{n+1} = \bigcup_{m=1}^{\infty} F_m$. We claim that the collections $\{S_i\}_{i=1}^{n+1}$ and $\{P_i\}_{i=1}^{n+1}$ satisfy the five conditions above. The first one is obvious. For condition (2), it suffices to show that $S_{n+1} \subset 2P_{n+1}$. This follows from the fact that $F_1 \subset 2P_n$, while $F_m \subset 2F_{m-1} \subset 2P_{n+1}$ for $m \ge 2$.

For condition (3), we must show that $\tau|_{S_{n+1}}$ is injective. Suppose that $\xi_1, \xi_2 \in S_{n+1}$ and $\tau(\xi_1) = \tau(\xi_2)$; then, without loss of generality, for some $j \le k$, $\xi_1 \in F_j$ and $\xi_2 \in F_k$. If $j < k$, then $\xi_2 \notin F_j^P$, which contradicts $\tau(\xi_1) = \tau(\xi_2)$. Therefore, $j = k$; however, $\tau|_{F_k}$ is injective, so $\xi_1 = \xi_2$.

To see (4) it is enough to prove that $S_{n+1} \cap P_n = \emptyset$. As we noticed before, for $m \ge 2$ we have $F_m \subset 2F_{m-1}$. But since $F_1 \subset 2P_n \setminus P_n$, by induction we obtain $F_m \subset 2^m P_n \setminus 2^{m-1} P_n$ for $m \ge 1$. From (2) with $i = n$, it follows that $P_n \subset 2^m P_n$ for $m \ge 1$. Therefore, for such m we obtain $F_m \cap P_n = \emptyset$, i.e. $S_{n+1} \cap P_n = \emptyset$.

The proof of (5) is more difficult. First note that since $\tau(F_m) \subset A_{n+1}$, we have $\tau(S_{n+1}) \subset A_{n+1}$. For the reverse inclusion, we will find it useful to prove the following

Claim 2. *We have* $A^P_{n+1} \cap 2S^P_{n+1} \subset S^P_{n+1}$.

Proof of claim 2. If $\xi \in A^P_{n+1} \cap 2S^P_{n+1}$, then $2^{-1}\xi + 2k\pi \in F_m$ for some $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. Therefore, $\xi' := \xi + 4k\pi \in 2F_m$, moreover, since $\xi \in A^P_{n+1}$, we obtain $\xi' \in 2F_m \cap A^P_{n+1}$. Now, if $\xi' \notin \bigcup_{i=1}^m F^P_i$, then $\xi' \in \tilde{F}_{m+1} \subset S^P_{n+1}$, i.e. $\xi \in S^P_{n+1}$. On the contrary, if $\xi' \in F^P_i$ for some $i = 1, \dots, m$, then $\xi' \in S^P_{n+1}$ as well, so $\xi \in S^P_{n+1}$, which ends the proof of claim 2.

Continuing with the proof of condition (5), we need to show that $A_{n+1} \subset \tau(S_{n+1})$. By condition (iv) and Lemma 2.3.1, there is a set $H \subset \mathbb{R}$ such that $H \cap P_n = \emptyset$, $\tau(H) = A_{n+1}$, and $D(2^{-j}\xi) \geq 1$ for every $j \geq 0$ and $\xi \in H$. Therefore all we have to prove is $\tau(H) \subset \tau(S_{n+1})$, i.e. $H \subset S^P_{n+1}$.

We split the argument into two cases; first, we consider all $\xi \in H$ such that for every $j \geq 0$, $D(2^{-j}\xi) \geq n+1$, i.e. the set $B := H \cap \bigcap_{j=0}^\infty 2^j A^P_{n+1}$. We will show that $B \subset S^P_{n+1}$. Let $\xi \in B$. As we mentioned before $\bigcup_{j=1}^\infty 2^j Q = \mathbb{R}$, therefore $2^{-j}\xi \in Q$ for some $j \geq 1$. Condition (1) implies that $Q \subset P_n$; hence, we can consider $j_0 = \min\{j \in \mathbb{N}\colon 2^{-j}\xi \in P_n\}$. By our choice of H we have $H \cap P_n = \emptyset$, so we can see that $2^{-j_0+1}\xi \in 2P_n \setminus P_n$. Moreover, since $2^{-j_0+1}\xi \in A^P_{n+1}$, we obtain $2^{-j_0+1}\xi \in (2P_n \setminus P_n) \cap A^P_{n+1} = \tilde{F}_1 \subset S^P_{n+1}$. In this way we proved that $B \subset \bigcup_{k=0}^\infty 2^k S^P_{n+1}$. Therefore, $B = \bigcup_{k=0}^\infty (B \cap 2^k S^P_{n+1}) \subset \bigcup_{k=0}^\infty (H \cap \bigcap_{j=0}^k 2^j A^P_{n+1} \cap 2^k S^P_{n+1})$. But claim 2 implies that for $k \geq 0$

$$\bigcap_{j=0}^{k} 2^j A^P_{n+1} \cap 2^k S^P_{n+1} \subset S^P_{n+1}, \tag{2.3.2}$$

hence $B \subset S^P_{n+1}$.

The second case deals with the set $C := H \setminus \bigcap_{j=0}^{\infty} 2^j A_{n+1}^P$. We still wish to show that $C \subset S_{n+1}^P$. If $\xi \in C$ then $\xi \in H \subset A_{n+1}^P$, so we can find $j_0 \geq 0$ such that $\xi \in \bigcap_{j=0}^{j_0} 2^j A_{n+1}^P$ and $\xi \notin 2^{j_0+1} A_{n+1}^P$. To prove that $\xi \in S_{n+1}^P$ it suffices to show that $\xi \in 2^{j_0} S_{n+1}^P$, and then use formula (2.3.2) with $k = j_0$. To see why $\xi \in 2^{j_0} S_{n+1}^P$ observe that $D(2^{-j_0}\xi) \geq n+1$. Therefore, by the consistency equation, i.e. condition iii) we obtain

$$n + 2 \leq 1 + D(2^{-j_0}\xi) = D(2^{-j_0-1}\xi) + D(2^{-j_0-1}\xi + \pi). \tag{2.3.3}$$

Let us set $K = D(2^{-j_0-1}\xi)$, then $\tau(2^{-j_0-1}\xi) \in \bigcap_{k=1}^{K} A_k$. Moreover, since $\xi \notin 2^{j_0+1} A_{n+1}^P$ we obtain $K \leq n$, so by (5) we have $2^{-j_0-1}\xi \in \bigcap_{k=1}^{K} S_k^P$, i.e. $2^{-j_0-1}\xi + 2p_k\pi \in S_k$, where $p_k \in \mathbb{Z}$ are distinct by (4) and $k = 1, \ldots, K$. Similarly, set $L = D(2^{-j_0-1}\xi + \pi)$ and $L' = \min(L, n)$. Using again (4) we can find distinct $r_l \in \mathbb{Z}$ such that $2^{-j_0-1}\xi + \pi + 2r_l\pi \in S_l$, where $l = 1, \ldots, L'$.

In this way we obtain

$$2^{-j_0}\xi + 2\pi(2p_k) \in 2S_k \subset 2P_n \quad \text{for } k = 1, \ldots, K \tag{2.3.4}$$

and

$$2^{-j_0}\xi + 2\pi(2r_l + 1) \in 2S_l \subset 2P_n \quad \text{for } l = 1, \ldots, L'. \tag{2.3.5}$$

We claim that this gives us at least $n + 1$ distinct elements of $2P_n$. To see that these elements are distinct it is enough to observe that $2p_k \neq 2r_l + 1$ for any k, l. What remains to be verified is that $K + L' \geq n + 1$. Since $\xi \in H$ we obtain that $K \geq 1$, this, together, with formula (2.3.3), implies that in fact $K + L' \geq n + 1$.

By property (3) of the induction hypothesis, at least one of the elements given in (2.3.4) and (2.3.5) must lie in the complement of P_n. So, for some integer N, $2^{-j_0}\xi + 2N\pi \in 2P_n \setminus P_n$. In addition, since $2^{-j_0}\xi \in A_{n+1}^P$, we have $2^{-j_0}\xi + 2N\pi \in A_{n+1}^P$. Therefore, $2^{-j_0}\xi + 2N\pi \in \tilde{F}_1 \subset S_{n+1}^P$, i.e. $\xi \in 2^{j_0} S_{n+1}^P$, which completes the proof of condition (5).

To recap, we have defined sets $\{S_i\}_{i=1}^{\infty}$ such that if we set $P_n = \bigcup_{i=1}^{n} S_i$, then the following conditions hold:

(1) $Q \subset P_1$,

(2) $P_n \subset 2P_n$ for $i = 1, \ldots, n$,

(3) $\tau|_{S_i}$ is injective for $i = 1, \ldots, n$,

(4) $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and

(5) $\tau(S_i) = A_i$ for $i = 1, \ldots, n$.

Define $S = \bigcup_{i=1}^{\infty} S_i$. We claim that S satisfies properties (a)-(d) listed above. To show that $S \subset 2S$, it suffices to show that $S_i \subset 2S$ for every $i \in \mathbb{N}$, but this is immediate from condition (2).

To show that $\lim_{n\to\infty} \mathbf{1}_S(2^{-n}\xi) = 1$ almost everywhere, we note that $Q \subset S$, and since $\lim_{n\to\infty} \mathbf{1}_Q(2^{-n}\xi) = 1$, we obtain $\lim_{n\to\infty} \mathbf{1}_S(2^{-n}\xi) = 1$.

To prove $D(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi)$, it suffices to show that the equality holds for all $\xi \in [-\pi, \pi]$. By condition (4) we have $\mathbf{1}_S = \sum_{i=1}^{\infty} \mathbf{1}_{S_i}$, therefore

$$D'(\xi) := \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi) = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{\infty} \mathbf{1}_{S_i}(\xi + 2k\pi).$$

From (3) follows that $\sum_{k\in\mathbb{Z}} \mathbf{1}_{S_i}(\xi + 2k\pi) = \mathbf{1}_{\tau(S_i)}(\xi)$ for $\xi \in [-\pi, \pi]$. Therefore, using (5) we obtain

$$D'(\xi) = \sum_{i=1}^{\infty} \mathbf{1}_{\tau(S_i)}(\xi) = \sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\xi) = D(\xi).$$

Finally, we show that $|S| = 2\pi$. By condition (i) we have

$$|S| = \int_{-\infty}^{\infty} \mathbf{1}_S(\xi)\, d\xi = \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi)\, d\xi = \int_{-\pi}^{\pi} D(\xi)\, d\xi = 2\pi,$$

which completes the proof of the theorem. ♣

An immediate consequence of the proof of Theorem 2.3.2 is the following

Corollary 2.3.3. *If D is a dimension function of some wavelet then there exists an MSF wavelet ψ such that $D_\psi = D$.*

The theorem we just presented gives us an algorithm for constructing wavelet sets if the dimension function is given

Algorithm 2.3.4. *Assume that D is a function given which satisfies the assumptions of Theorem 2.3.2. Denote $A_k = \{\xi \in [-\pi, \pi] : D(\xi) \geq k\}$.*

1^o *Fix a measurable set $Q \subset \mathbb{R}$, such that $Q \subset 2Q$, $\lim_{n\to\infty} \mathbf{1}_Q(2^{-n}\xi) = 1$ a.e., $\tau|_Q$ is injective and $D(\xi) \geq 1$ for $\xi \in Q$.*

2^o *Let $E_1 = Q$. For $n \in \mathbb{N}$ define $\tilde{E}_{m+1} = (2E_m \setminus \bigcup_{i=1}^m E_i^P) \cap A_1^P$, and choose any measurable $E_{m+1} \subset \tilde{E}_{m+1}$ such that $\tau(E_{m+1}) = \tau(\tilde{E}_{m+1})$ and $\tau|_{E_{m+1}}$ is injective. Let $S_1 = \bigcup_{m=1}^\infty E_m$.*

3^o *If S_i are constructed for $1 \leq i \leq n$, let $P_n = \bigcup_{i=1}^n S_i$. Define $\tilde{F}_1 = (2P_n \setminus P_n) \cap A_{n+1}^P$ and $\tilde{F}_{m+1} = (2F_m \setminus \bigcup_{i=1}^m F_i^P) \cap A_{n+1}^P$, where again $F_{m+1} \subset \tilde{F}_{m+1}$ such that $\tau(F_{m+1}) = \tau(\tilde{F}_{m+1})$ and $\tau|_{F_{m+1}}$ is injective. Let $S_{n+1} = \bigcup_{m=1}^\infty F_m$.*

4^o *Let $S = \bigcup_{n=1}^\infty S_n$, then $W = 2S \setminus S$ is a wavelet set with dimension function equal to D.*

If the function D is bounded by n then $S_i = \emptyset$ for $i > n$. In particular, the algorithm greatly simplifies in case when $D \equiv 1$ and then yields MRA wavelet sets.

2.4 Other properties of the dimension function

Let us denote by $\mathcal{D}$ the set of all functions which are dimension functions of some wavelet. Theorem 2.3.2 provides some information about $\mathcal{D}$, but we are not aware of a general constructive procedure which would allow us to produce all dimension functions. As a result, further investigation of $\mathcal{D}$ relies on other techniques and studying examples. Below, we provide several examples of dimension functions and present some other properties of the dimension function.

In the MRA case, the dimension function is identically 1. The first non-trivial dimension function was discoveried by Journé.

EXAMPLE 2.4.1. *The Journé dimension function.*

Consider the Journé wavelet $\hat{\psi} = \mathbf{1}_{[-\frac{32}{7}\pi,-4\pi]\cup[-\pi,-\frac{4}{7}\pi]\cup[\frac{4}{7}\pi,\pi]\cup[4\pi,\frac{32}{7}\pi]}$. Then its dimension function is given by

$$D_\psi(\xi) = \begin{cases} 2 & \text{for } x \in [-\frac{2}{7}\pi, \frac{2}{7}\pi] \\ 1 & \text{for } x \in [-\pi, -\frac{6}{7}\pi] \cup [-\frac{2}{7}\pi, -\frac{4}{7}\pi] \cup [\frac{2}{7}\pi, \frac{4}{7}\pi] \cup [\frac{6}{7}\pi, \pi] \\ 0 & \text{for } x \in [-\frac{4}{7}\pi, -\frac{6}{7}\pi] \cup [\frac{4}{7}\pi, \frac{6}{7}\pi], \end{cases}$$

EXAMPLE 2.4.2. *A non-symmetric dimension function.*

Another nontrivial example of a dimension function which is bounded by two can be obtained by computing the dimension function of an MSF wavelet with the support $W = [-\frac{8}{3}\pi, -2\pi] \cup [-\pi, -\frac{2}{3}\pi] \cup [\frac{2}{5}\pi, \frac{2}{3}\pi] \cup [\frac{8}{3}\pi, 3\pi] \cup [6\pi, \frac{32}{5}\pi]$ (this wavelet set is considered by X. Dai and D.R. Larson in [DL]). Then for $\hat{\psi} = \mathbf{1}_W$ we have

$$D_\psi(\xi) = \begin{cases} 2 & \text{for } x \in [-\frac{2}{3}\pi, -\frac{2}{5}\pi] \cup [\frac{2}{3}\pi, \frac{4}{5}\pi] \\ 1 & \text{for } x \in [-\pi, -\frac{4}{5}\pi] \cup [-\frac{2}{5}\pi, \frac{2}{5}\pi] \cup [\frac{4}{5}\pi, \pi] \\ 0 & \text{for } x \in [-\frac{4}{5}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{5}\pi, \frac{2}{3}\pi], \end{cases}$$

which yields a non-symmetric dimension function.

EXAMPLE 2.4.3. *Dimension functions with arbitrarily large supremums.*

The following example of the dimension function was discovered by B. Behera and S. Madan in [BMa]. Let n be a positive integer and $\epsilon = \frac{\pi}{2^{n+1}-1}$. Define $T_1 = [-2\epsilon, 2\epsilon]$ and for $i = 2, \dots, n$, define $T_i = [-2^i\epsilon, -2^{i-1}\epsilon] \cup [2^{i-1}\epsilon, 2^i\epsilon]$. Finally, define $U = [-\pi + \epsilon, -2^n\epsilon] \cup [2^n\epsilon, \pi - \epsilon]$ and $V = [-\pi, -\pi + \epsilon] \cup [\pi - \epsilon, \pi]$. Then,

$$D_n(\xi) = \begin{cases} n - i + 1 & \text{for } \xi \in T_i, 1 \le i \le n \\ 0 & \text{for } \xi \in U \\ 1 & \text{for } \xi \in V. \end{cases}$$

is a dimension function of a wavelet. Notice that when $n = 1$, we obtain the MRA dimension function, and when $n = 2$, we obtain the Journé dimension function.

One might think that letting n go to ∞ in the above example would yield an unbounded dimension function. However, this is not the case. Indeed, the limit function would be $D_\infty(\xi) = i$ for all $\xi \in [-\frac{\pi}{2^i}, -\frac{\pi}{2^{i+1}}] \cup [\frac{\pi}{2^{i+1}}, \frac{\pi}{2^i}]$. In this limit, the zero set of D_∞ contains $[-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$, so $\Delta = [-\frac{\pi}{2}, \frac{\pi}{2}]$, and condition (iv) of Theorem 2.3.2 is not satisfied. The function D_∞ also is an example of a function which satisfies conditions (i) through (iii) of Theorem 2.3.2, but not condition (iv). In particular, condition (iv) is not independent of the other conditions, at least for unbounded dimension functions.

However, in spite of the setback above, unbounded dimension functions do exist as was shown by D. Speegle (see [BRS]).

The following fact shows that band limited wavelets have a bounded dimension function

FACT 2.4.4. *If* ψ *is an orthonormal wavelet and* $\operatorname{supp}\hat{\psi} \subset [-2n\pi, 2n\pi]$ *for some* $n \in \mathbb{N}$ *then* $D_\psi \leq n$.

PROOF. Let $s(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$. By the support condition on $\hat{\psi}$ we have $\operatorname{supp} s \subset [-n\pi, n\pi]$. Therefore (2.1) implies that $s \leq \mathbf{1}_{[-n\pi,n\pi]}$ and we obtain

$$D_\psi(\xi) = \sum_{k\in\mathbb{Z}} s(\xi + 2k\pi) \leq \sum_{k\in\mathbb{Z}} \mathbf{1}_{[-n\pi,n\pi]}(\xi + 2k\pi) \equiv n. \quad \clubsuit$$

It is natural to ask if the above estimate is optimal. The Meyer's construction of wavelets can be viewed as the origin of such a question. The construction concentrates on obtaining MRA wavelets ψ whose Fourier transform is smooth and compactly supported. The fact that the Meyer's wavelets have to satisfy $\operatorname{supp}\hat{\psi} \subset [-\frac{8}{3}\pi, \frac{8}{3}\pi]$ was mind-boggling until it was explained by the following observation due to G. Weiss

THEOREM 2.4.5. *If ψ is an orthonormal wavelet and* $\operatorname{supp}\hat{\psi} \subset [-\frac{8}{3}\pi, \frac{8}{3}\pi]$ *then* $D_\psi \leq 1$.

In other words, every wavelet, for which the above condition on the support holds, must be an MRA wavelet.

This observation shows already that the estimate given in Fact 2.4.4 is not optimal. A question if the symmetric interval $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ is the largest one which allows us to estimate the dimension function by 1, was posed by G. Weiss. The following proposition provides a positive answer to this question

PROPOSITION 2.4.6. *For every $\epsilon > 0$, there exists an MSF wavelet ψ such that* $\operatorname{supp}\hat{\psi} \subset [-\frac{8}{3}\pi, \frac{8}{3}\pi + \epsilon]$ *and* $\|D_\psi\|_\infty > 1$.

PROOF. Without the loss of generality we can assume that $0 < \epsilon \leq \frac{2\pi}{3}$. Let $L = [-\frac{8\pi}{3}, -\frac{4\pi}{3}]$ and $R = [\frac{2\pi}{3}, \frac{4\pi}{3}]$. We can use Corollary 2.4 of Chapter 7 [HW] to check that $L \cup R$ is a wavelet set. Now let

$$A_0 = \left[-\frac{5\pi}{3} + \frac{\epsilon}{8}, -\frac{5\pi}{3} + \frac{\epsilon}{2}\right], \qquad B_0 = \frac{1}{4}A_0,$$

$$A_n = B_{n-1} - 2\pi, \qquad B_n = 2^{-(n+2)}A_n.$$

Then the wavelet set W defined by

$$W = \big(L \setminus \bigcup_{n=0}^{\infty} A_n\big) \cup \bigcup_{n=0}^{\infty} B_n \cup \left[\frac{\pi}{3} + \frac{\epsilon}{8}, \frac{\pi}{3} + \frac{\epsilon}{2}\right] \cup \left[\frac{2\pi}{3} + \epsilon, \frac{4\pi}{3}\right] \cup \left[\frac{8\pi}{3}, \frac{8\pi}{3} + \epsilon\right]$$

gives us an MSF wavelet with the desired properties.

First let us prove that W is in fact a wavelet set. To do so we will use the notion of dilation and translation equivalency of sets. We say that $A, B \subset \mathbb{R}$ are *dilation equivalent* if there exist a partition $\{A_l : l \in \mathbb{Z}\}$ of A, a partition $\{B_l : l \in \mathbb{Z}\}$ of B, and an integer-valued sequence $\{j_l : l \in \mathbb{Z}\}$ such that $A_l = 2^{k_l} B_l$ for $l \in \mathbb{Z}$. Similarly, the sets A, B are *translation equivalent* if there exist a partition $\{A_l : l \in \mathbb{Z}\}$ of A, a partition $\{B_l \colon l \in \mathbb{Z}\}$ of B, and an integer-valued sequence $\{j_l \colon l \in \mathbb{Z}\}$ such that $A_l = B_l + 2j_l\pi$ for $l \in \mathbb{Z}$.

From Corollary 2.4 of Chapter 7 [HW] follows that W is a wavelet set if and only if it is dilation and translation equivalent to any other wavelet set. We will show that W is dilation and translation equivalent to the wavelet set $L \cup R$. For $0 < \epsilon \leq \frac{2\pi}{3}$ we have

$$2\left[\frac{\pi}{3} + \frac{\epsilon}{8}, \frac{\pi}{3} + \frac{\epsilon}{2}\right] \cup \left[\frac{2\pi}{3} + \epsilon, \frac{4\pi}{3}\right] \cup \frac{1}{4}\left[\frac{8\pi}{3}, \frac{8\pi}{3} + \epsilon\right] = R.$$

Moreover, by induction, $A_n \subset L$ for all $n \in \mathbb{N} \cup \{0\}$; therefore,

$$\big(L \setminus \bigcup_{n=0}^{\infty} A_n\big) \cup \bigcup_{n=0}^{\infty} 2^{n+2} B_n = \big(L \setminus \bigcup_{n=0}^{\infty} A_n\big) \cup \bigcup_{n=0}^{\infty} A_n = L.$$

Since the sets A_n, $n \in \mathbb{N} \cup \{0\}$ are disjoint and the same is true for the sets B_n, $n \in \mathbb{N} \cup \{0\}$, we proved that W is dilation equivalent to $L \cup R$. To check the translation equivalence let us observe that

$$\left[\frac{2\pi}{3} + \epsilon, \frac{4\pi}{3}\right] \cup \left(\left[\frac{8\pi}{3}, \frac{8\pi}{3} + \epsilon\right] - 2\pi\right) = R$$

and

$$\big(L \setminus \bigcup_{n=0}^{\infty} A_n\big) \cup \bigcup_{n=0}^{\infty} (B_n - 2\pi) \cup \left(\left[\frac{\pi}{3} + \frac{\epsilon}{8}, \frac{\pi}{3} + \frac{\epsilon}{2}\right] - 2\pi\right)$$

$$= \big(L \setminus \bigcup_{n=0}^{\infty} A_n\big) \cup \bigcup_{n=0}^{\infty} A_{n+1} \cup A_0 = L.$$

To prove that $\|D_\psi\|_\infty > 1$ it is enough to notice that the set W containes the interval $[\frac{8\pi}{3}, \frac{8\pi}{3} + \epsilon]$ as well an interval $[-\frac{8\pi}{3}, -\frac{8\pi}{3} + \delta]$, where δ is a positive number which depends on ϵ. Therefore, we obtain that $D_\psi(\xi) \geq 2$ for $\xi \in [\frac{4\pi}{3}, \frac{4\pi}{3} + \min(\frac{\epsilon}{2}, \frac{\delta}{4})]$. ♣

The above theorem together with Theorem 2.4.5 explains the appearance of the interval $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ in the Meyer's construction.

Since we proved that the interval $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ is optimal for estimating the dimension function by 1, it is of interest to investigate if there exist symmetric intervals which are optimal for estimating the dimension function by n. The methods of G. Weiss used to prove Theorem 2.4.5 can be applied to prove the following

THEOREM 2.4.7. *If ψ is an orthonormal wavelet and* $\operatorname{supp}\hat{\psi} \subset [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$ *for some $n \in \mathbb{N}$, then $D_\psi \le n$.*

PROOF. Since dimension function is 2π-periodic it is enough to consider two cases: $|\xi| \le \frac{2}{3}\pi$ and $\frac{2}{3}\pi \le \xi \le \frac{4}{3}\pi$. The condition on the support of $\hat{\psi}$ implies that in the first case we have $\hat{\psi}(2^j(\xi + 2k\pi)) = 0$ for $k \ne 0$ and $j \ge n$. Therefore, using (2.1.2) and (2.1.3), we obtain

$$D_\psi(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 + \sum_{j=1}^{n-1}\sum_{k\ne 0} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \le n,$$

for $|\xi| \le \frac{2}{3}\pi$.

In the second case we have $\hat{\psi}(2^j(\xi + 2k\pi)) = 0$ for $j \ge n+1$ and every integer k, so (2.1.3) implies that

$$D_\psi(\xi) = \sum_{j=1}^{n}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \le n,$$

for $\frac{2}{3}\pi \le \xi \le \frac{4}{3}\pi$. ♣

To prove that the intervals of Theorem 2.4.7 are optimal we shall use the following theorem due to E. Ionascu and C. Pearcy (see [IP])

THEOREM 2.4.8. *A measurable set $U \subset \mathbb{R}$ is a subset of a wavelet set if and only if the following two conditions hold: (i) there is a set $A \supset U$ such that $\tau|_A$ and*

$d|_A$ are injective, and $d(A) = [-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi)$, and (ii) there is a set $B \supset U$ such that $\tau|_B$ and $d|_B$ are injective and $\tau(B) = [-\pi, \pi)$. Moreover, in this case, there is a wavelet set W satisfying $U \subset W \subset A \cup B$.

Instead of giving an explicite wavelet as we did in Proposition 2.4.6 we will apply the above theorem to prove the existence of certain wavelets, which justify that the intervals of Theorem 2.4.7 are optimal. We have the following:

PROPOSITION 2.4.9. *For every $\epsilon > 0$ and $n \in \mathbb{N}$, there exists an MSF wavelet ψ such that* $\operatorname{supp} \hat{\psi} \subset [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ *and $\|D_\psi\|_\infty > n$.*

PROOF. Similarly as in Proposition 2.4.6 we can assume that $0 < \epsilon \leq \delta$ for some $\delta = \delta(n) > 0$. Moreover, since Proposition 2.4.6 deals with the case $n = 1$, we shall only consider $n \geq 2$.

Let $U = [-\frac{2^{n+2}}{3}\pi, -\frac{2^{n+2}}{3}\pi + \epsilon] \cup [\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$. In order to use Theorem 2.4.8 we have to find appropriate sets A and B. Let $A' = [-\frac{2^{n+2}}{3}\pi + \epsilon, \frac{2^{n+1}}{3}\pi] \cup [\frac{2^{n+1}}{3}\pi + \frac{\epsilon}{2}, \frac{2^{n+2}}{3}\pi]$. Clearly, if δ is small enough, there exist an $m \in \mathbb{N}$, such that $2^{-m}A' \subset [-\pi, \pi]$ and $2^{-m}A' \cap \tau(U) = \emptyset$. Therefore, if we let $A = 2^{-m}A' \cup U$, then the condition (i) of Theorem 2.4.8 is satisfied.

To find a suitable set B, observe that $\tau|_U$ is injective, if δ is small enough. Let $B' = \tau^{-1}([-\pi, \pi) \setminus \tau(U)) \cap [\frac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ and $B = B' \cup U$. Since $2(\frac{2^{n+2}}{3}\pi + \epsilon - 2\pi) \geq \frac{2^{n+2}}{3}\pi + \epsilon$ for $n \geq 2$, the condition (ii) of Theorem 2.4.8 holds for B.

From Theorem 2.4.8 it follows, that there exists a wavelet set W, such that $U \subset W \subset A \cup B$. Since $A \cup B \subset [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$, the MSF wavelet $\psi = \check{\mathbf{1}}_W$ satisfies $\operatorname{supp} \hat{\psi} \subset [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$. Moreover, from $U \subset W$, it follows that $D_\psi(\xi) \geq n + 1$ for $\xi \in [\frac{2}{3}\pi, \frac{2}{3}\pi + \frac{\epsilon}{2^{n+1}}]$, which concludes the proof. ♣

As we saw, every dimension function presented in Example 2.4.3 attains all integer values between zero and its supremum. The following fact conjectured by

M. Bownik and proven by D. Speegle shows that this is not an accident.

THEOREM 2.4.10. *Let ψ be an orthonormal wavelet with dimension function D_ψ, and let N be an integer greater than 1. If there is a set $A \subset \mathbb{R}$ such that its Lebesgue measure $|A| > 0$ and $D_\psi(\xi) \geq N$ for all $\xi \in A$, then there is a set $B \subset \mathbb{R}$ such that $|B| > 0$ and $D_\psi(\xi) = N - 1$ for all $\xi \in B$.*

PROOF. By the consistency equation (see Theorem 2.3.2 (iii))

$$D_\psi(2\xi) = D_\psi(\xi) + D_\psi(\xi + \pi) - 1,$$

it follows that $D_\psi(2\xi) \geq D_\psi(\xi) - 1$ a.e.. Suppose that $|\{\xi \in \mathbb{R}\colon D_\psi(\xi) = N-1\}| = 0$. Then, the inequality implies that for almost every $\xi \in 2A$, $D_\psi(\xi) \geq N$. Repeating this argument, we obtain $D_\psi(\xi) \geq N$ for all $\xi \in 2^j A$ and all $j \geq 1$. So, by the 2π-periodicity of D_ψ, we have that for every $j \geq 1$ and $\xi \in \tau(2^j A)$, $D_\psi(\xi) \geq N$; that is $D_\psi(\xi) \geq N$ for $\xi \in \bigcup_{j=1}^\infty \tau(2^j A)$. From the lemma below, it follows that $|\bigcup_{j=1}^\infty \tau(2^j A)| = 2\pi$, hence

$$\int_{-\pi}^{\pi} D_\psi(\xi)\, d\xi \geq N \Big| \bigcup_{j=1}^{\infty} \tau(2^j A) \Big| = 2N\pi > 2\pi,$$

which contradicts the fact that $\int_{-\pi}^{\pi} D_\psi(\xi)\, d\xi = 2\pi$ (see Theorem 4.1 (i)). ♣

LEMMA 2.4.11. *If $A \subset \mathbb{R}$ is a set of positive measure, then $|\bigcup_{j=1}^\infty \tau(2^j A)| = 2\pi$.*

PROOF. First, note that the lemma is obvious if A contains an interval. For the general case, let $x \neq 0$ be a Lebesgue point of A. Let $\delta > 0$, and see that there is a positive $\epsilon < x$ such that $|A \cap (x - \epsilon, x + \epsilon)| > (1 - \delta)2\epsilon$ and $2^n \epsilon = \pi$ for some $n \in \mathbb{Z}$.

It follows that for the set $B := 2^n\big(A \cap (x - \epsilon, x + \epsilon)\big)$, we have $B = 2^n A \cap (2^n x - 2^n \epsilon, 2^n x + 2^n \epsilon) \subset [2^n x - \pi, 2^n x + \pi)$. In particular, $|B| = |\tau(B)|$.

Finally, we note that

$$|\tau(2^n A)| \geq |\tau(B)| = |B| = 2^n |A \cap (x - \epsilon, x + \epsilon)| > 2^n (1 - \delta) 2\epsilon = (1 - \delta)2\pi.$$

Since $\delta > 0$ was arbitrary, it is clear that $|\bigcup_{j=1}^{\infty} \tau(2^j A)| = 2\pi$. ♣

The last fact we are going to present implies that there are "many" dimension functions - enough to connect all of the examples we have presented so far.

FACT 2.4.12. *The set $\mathcal{D}$ is arcwise connected in the $L^1(\mathbb{T})$ topology.*

PROOF. Let $D, D' \in \mathcal{D}$. By Corollary 2.3.3 we can find MSF wavelets ψ, ψ' with corresponding wavelet sets W, W' such that $D_\psi = D$ and $D_{\psi'} = D'$. It is easy to justify that $\|D - D'\|_{L^1(\mathbb{T})} \leq \|\mathbf{1}_W - \mathbf{1}_{W'}\|^2_{L^2(\mathbb{R})}$. In fact

$$\|D - D'\|_{L^1(\mathbb{T})} = \int_{-\pi}^{\pi} \left| \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \left(\mathbf{1}_W(2^j(\xi + 2k\pi)) - \mathbf{1}_{W'}(2^j(\xi + 2k\pi))\right) \right| d\xi \leq$$

$$\int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\mathbf{1}_W(2^j(\xi + 2k\pi)) - \mathbf{1}_{W'}(2^j(\xi + 2k\pi))| d\xi = \|\mathbf{1}_W - \mathbf{1}_{W'}\|^2_{L^2(\mathbb{R})}.$$

The arcwise connectivity of $\mathcal{D}$ follows from [Sp], where it was shown that the set of all characteristic functions of wavelet sets is arcwise connected in the $L^2(\mathbb{R})$ norm. ♣

We can also consider $\sqrt{\mathcal{D}} = \{\sqrt{D} \colon D \in \mathcal{D}\}$ with the $L^2(\mathbb{T})$ norm. The advantage of this comes from the fact that for $f, g \in L^2(\mathbb{R})$ we have $\|\sqrt{D_f} - \sqrt{D_g}\|_{L^2(\mathbb{T})} \leq \sqrt{2\pi}\|f - g\|_{L^2(\mathbb{R})}$, where D_f and D_g are defined by formula (2.6). In particular if $\psi_n \to \psi$ in $L^2(\mathbb{R})$, where ψ_n and ψ are orthonormal wavelets, then $\sqrt{D_{\psi_n}} \to \sqrt{D_\psi}$ in $L^2(\mathbb{T})$. The connectivity of $\sqrt{\mathcal{D}}$ in the $L^2(\mathbb{T})$ norm follows again from Corollary 2.3.3 and [Sp].

As an immediate consequence of Fact 2.4.12 we obtain the following:

COROLLARY 2.4.13. *The set $\mathcal{D}$ is uncountable.*

Chapter 3

MRA WAVELETS

3.1 Construction of MRA wavelets via scaling sets

Two functions whose translates and dilates form an orthonormal basis of $L^2(\mathbb{R})$ were well known even before wavelet theory appeared. We now call these functions the Haar wavelet and the Shannon wavelet. Unfortunately neither of these function is nice enough for all we want to do with wavelets today. The Haar wavelet is not a continuous function and the Shannon wavelet has poor behavior at infinity. The question of the existence of nicer functions which give rise to an orthonormal basis of $L^2(\mathbb{R})$ has become of much importance. The multiresolution analysis theory introduced by S. Mallat gives a relatively easy method for constructing wavelets via scaling functions and low–pass filters. It was used, for example, in the celebrated Daubechies construction of compactly supported wavelets with a given degree of smoothness. Although the formulas for obtaining a wavelet from a multiresolution analysis are straightforward, it is not obvious how to construct an MRA structure. Basically it all comes down to finding a function $g \in L^2(\mathbb{R})$ satisfying the following conditions

(S1) $g(2\xi) = m(\xi)g(\xi)$ a.e. $\xi \in \mathbb{R}$, where m is a 2π–periodic measurable function,

(S2) $\lim_{j\to\infty} |g(2^{-j}\xi)| = 1$ a.e. $\xi \in \mathbb{R}$,

(S3) $\sum_{k\in\mathbb{Z}} |g(\xi + 2k\pi)|^2 = 1$ a.e. $\xi \in \mathbb{R}$

(see Theorem 1.7.1). Then $\varphi = \check{g}$ is a scaling function and a corresponding wavelet ψ can be obtain by the well known equation $\hat{\psi}(\xi) = e^{i\xi/2}\overline{m(\xi/2+\pi)}g(\xi/2)$. In [C] A. Cohen showed how to construct such functions g. He gives conditions on a 2π-periodic function m, which are sufficient to imply that $g(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi)$ satisfies (S1)-(S3). This result was extended by M. Papadakis, H. Šikić and G.Weiss, who found sufficient and necessary conditions on a 2π-periodic function m, which guarantee that m is such a function known as low–pass filter (see [PSW]). These conditions are helpful for constructing low–pass filters but much work in this direction needs to be done.

In the following section we present a slightly different approach to the problem of constructing MRA wavelets. Instead of starting from a low–pass filter we begin by defining a function g on a small subset of $\mathbb{R}$ (scaling set). Then we extend it to the whole real line and appropriately modify it to obtain a function that satisfies (S1)-(S3). What is interesting in this approach is that this constructive procedure allows us to obtain all MRA wavelets.

To start our algorithm we need a scaling set[5], which is defined as follows:

Definition 3.1.1. *A measurable set $S \in \mathbb{R}$ is called a scaling set if*

(SS1) $S \subset 2S$ *a.e.,*

(SS2) $\lim_{j\to\infty} \mathbf{1}_S(2^{-j}\xi) = 1$ *a.e.* $\xi \in \mathbb{R}$,

(SS3) $\sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + 2k\pi) = 1$ *a.e.* $\xi \in \mathbb{R}$.

The theory of scaling sets is well developed and many researchers found their own prescriptions for obtaining such sets. In [BMM] a constructive procedure for obtaining all MSF wavelets (hence also scaling sets) is given. Bounded scaling sets can be constructed by a procedure presented in [PSW]. An easy way of constructing some scaling sets is also included in [BRS].

The construction we are going to describe is based on a scaling set and an appropriately chosen function, we call these two objects an *MRA–couple.*

Definition 3.1.2. *An MRA–couple (S, f) consists of a scaling set S and a measurable function f defined on S, which satisfies the following conditions*

(T1) $f(\xi) \neq 0$ *a.e.* $\xi \in S$,

[5]We remind the reader that more about scaling sets was said in §2.2 . Here we are using Theorem 2.2.2 to formulate the definition of scaling sets

(T2) $|f(2\xi)| \le |f(\xi)|$, *for* $\xi \in \frac{S}{2}$,

(T3) $\lim_{j\to\infty} |f(2^{-j}\xi)| = 1$ *a.e.* $\xi \in S$.

Note, that because $S \subset 2S$, conditions (T2) and (T3) are well defined; moreover, these conditions imply that $|f(\xi)| \le 1$ a.e. $\xi \in S$. Equipped with an MRA–couple we can proceed with the following construction

ALGORITHM 3.1.3. *Let* (S, f) *be an MRA–couple*

Alg1. Define a function $m_0(\xi) = \frac{f(2\xi)}{f(\xi)}$, *for* $\xi \in \frac{S}{2}$.

Alg2. Extend m_0 *to* $\frac{S}{2} + \pi$ *so that* $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$ *a.e.* $\xi \in \frac{S}{2}$; *that is,*

$$m_0(\xi) = \alpha(\xi)\sqrt{1 - |m_0(\xi - \pi)|^2}, \qquad \textit{for } \xi \in \frac{S}{2} + \pi,$$

where α *is a measurable unimodular function defined on* $\frac{S}{2} + \pi$. *Then extend* m_0 2π*-periodically to* $\mathbb{R}$.

Alg3. Extend f *to* h *defined on* $\mathbb{R}$, *which satisfies* $h(2\xi) = m_0(\xi)h(\xi)$ *a.e.* $\xi \in \mathbb{R}$. *This extension is unique and*

$$(3.1.1) \qquad h(\xi) = \begin{cases} f(2^{-n}\xi)\prod_{j=1}^{n} m_0(2^{-j}\xi), & \textit{for } \xi \in 2^n S \setminus 2^{n-1}S, \quad n \in \mathbb{N} \\ f(\xi) & \textit{for } \xi \in S. \end{cases}$$

Alg4. Define $\sigma_h(\xi) = \sum_{k\in\mathbb{Z}} |h(\xi + 2k\pi)|^2$, $\xi \in \mathbb{R}$. *Then* $0 < \sigma_h(\xi) \le 1$ *and* $g(\xi) = \frac{h(\xi)}{\sqrt{\sigma_h(\xi)}}$ *satisfies (S1)-(S3), that is* $\varphi(\xi) = \check{g}(\xi)$ *is a scaling function and* $m(\xi) = \frac{m_0(\xi)\sqrt{\sigma_h(\xi)}}{\sqrt{\sigma_h(2\xi)}}$ *its low–pass filter.*

Moreover, if $f(\xi) \ge c > 0$ *a.e.* $\xi \in S$, *then* $\sigma_h(\xi) = 1$ *a.e., that is* $\varphi(\xi) = \check{h}(\xi)$ *and* $m(\xi) = m_0(\xi)$.

We will justify Algorithm 3.1.3 in several steps

PROOF OF ALG1. By (T1) $f(\xi) \neq 0$ a.e. $\xi \in S$ and by (SS1) $\frac{S}{2} \subset S$, therefore m_0 is well defined on $\frac{S}{2}$.

Proof of Alg2. First note that from (T2) it follows that $|m_0(\xi)| \le 1$. This allows us to define m_0 on $\frac{S}{2}+\pi$ by the formula from Alg2. Therefore, all we need to show is that $\frac{S}{2}$ and $\frac{S}{2}+\pi$ are disjoint and that the 2π translates of $\frac{S}{2}\cup(\frac{S}{2}+\pi)$ form a partition of $\mathbb{R}$. The disjointness follows from (SS3) because we have $S\cap(S+2\pi)=\emptyset$. Moreover, (SS3) also implies that

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{\frac{S}{2}\cup(\frac{S}{2}+\pi)}(\xi+2k\pi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_{\frac{S}{2}}(\xi+2k\pi) + \sum_{k\in\mathbb{Z}} \mathbf{1}_{\frac{S}{2}+\pi}(\xi+2k\pi)$$

$$= \sum_{k\in\mathbb{Z}} \mathbf{1}_S(2\xi+4k\pi) + \sum_{k\in\mathbb{Z}} \mathbf{1}_S(2\xi-2\pi+4k\pi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_S(2\xi+2k\pi) = 1;$$

hence, the 2π translates of $\frac{S}{2}\cup(\frac{S}{2}+\pi)$ form a partition of $\mathbb{R}$ and this guarantees that we can extend m_0 2π-periodically from $\frac{S}{2}\cup(\frac{S}{2}+\pi)$ to $\mathbb{R}$.

Proof of Alg3. Condition (SS2) implies that $\bigcup_{n=0}^{\infty} 2^n S = \mathbb{R}$ a.e.; therefore, the disjoint sum $S \cup \bigcup_{n=1}^{\infty} 2^n S \setminus 2^{n-1} S$ forms a partition of $\mathbb{R}$. This allows us to define

$$h(\xi) = \begin{cases} f(2^{-n}\xi)\prod_{j=1}^{n} m_0(2^{-j}\xi), & \text{for } \xi \in 2^n S \setminus 2^{n-1} S, \quad n \in \mathbb{N} \\ f(\xi) & \text{for } \xi \in S. \end{cases}$$

If $\xi \in 2^n S \setminus 2^{n-1} S$, $n \in \mathbb{N}$, then it is easy to see that $h(2\xi) = m_0(\xi)h(\xi)$. In fact, in this case $2\xi \in 2^{n+1} S \setminus 2^n S$ and therefore,

$$h(2\xi) = f(2^{-n-1}2\xi)\prod_{j=1}^{n+1} m_0(2^{-j}2\xi) = m_0(\xi) f(2^{-n}\xi)\prod_{j=1}^{n} m_0(2^{-j}\xi) = m_0(\xi)h(\xi).$$

If $\xi \in S$, then either $2\xi \in 2S \setminus S$ or $2\xi \in S$. In the first case we have $h(2\xi) = m_0(\xi)f(\xi) = m_0(\xi)h(\xi)$; in the second, $h(2\xi) = f(2\xi) = m_0(\xi)f(\xi) = m_0(\xi)h(\xi)$, because of the definition of $m_0(\xi)$ for $\xi \in \frac{S}{2}$.

To see that this extension is unique it is enough to observe that every function $\tilde{h}$ with the property $\tilde{h}(2\xi) = m_0(\xi)\tilde{h}(\xi)$ must satisfy $\tilde{h}(\xi) = \tilde{h}(2^{-n}\xi)\prod_{j=1}^{n} m_0(2^{-j}\xi)$,

$n \in \mathbb{N}$. Therefore, if in addition $\tilde{h}(\xi) = f(\xi)$ for $\xi \in S$, then $\tilde{h}$ is forced to be exactly of the form (3.1.1).

To justify the last step of Algorithm 3.1.3 we will need the following:

LEMMA 3.1.4. *Let g be a measurable function defined on $\mathbb{R}$, which satisfies $g(2\xi) = m(\xi)g(\xi)$ a.e., where m is a 2π-periodic measurable function. If $\lim_{j\to\infty} |g(2^{-j}\xi)| = 1$ a.e. and $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$ a.e., then $\sigma_g(\xi) = \sum_{k\in\mathbb{Z}} |g(\xi + 2k\pi)|^2 \le 1$ a.e. Moreover, if $|g(\xi)| \ge c > 0$ a.e. $\xi \in E$, where E is a measurable set satisfying (SS2) and (SS3), then $\sigma_g(\xi) = 1$ a.e.*

PROOF. Since $g(2\xi) = m(\xi)g(\xi)$ and $\lim_{j\to\infty} |g(2^{-j}\xi)| = 1$ we obtain that $|g(\xi)| = \prod_{j=1}^{\infty} |m(2^{-j}\xi)|$ with convergence a.e. For a fixed set E satisfying (SS2) and (SS3), let us define

$$g_n(\xi) = \mathbf{1}_E(2^{-n}\xi) \prod_{j=1}^{n} |m(2^{-j}\xi)|, \qquad \text{for } n \in \mathbb{N}.$$

Since E satisfies (SS2) we obtain $g_n(\xi) \to |g(\xi)|$ a.e.; moreover, it is easy to check that $g_{n+1}(2\xi) = m(\xi)g_n(\xi)$. We claim that $\sigma_{g_n}(\xi) = \sum_{k\in\mathbb{Z}} |g_n(\xi + 2k\pi)|^2 = 1$ a.e. In fact, for $n = 1$ we have

$$\sigma_{g_1}(\xi) = \sum_{k\in\mathbb{Z}} |g_1(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \mathbf{1}_E(\frac{\xi}{2} + k\pi)|m(\frac{\xi}{2} + k\pi)|^2$$

$$= |m(\xi)|^2 \sum_{k\in\mathbb{Z}} \mathbf{1}_E(\frac{\xi}{2} + 2k\pi) + |m(\xi+\pi)|^2 \sum_{k\in\mathbb{Z}} \mathbf{1}_E(\frac{\xi}{2} + \pi + 2k\pi)$$

$$= |m(\xi)|^2 + |m(\xi+\pi)|^2 = 1,$$

where we used (SS3) for the set E. If we assume that $\sigma_{g_n}(\xi) = 1$ a.e., then

$$\sigma_{g_{n+1}}(2\xi) = \sum_{k\in\mathbb{Z}} |g_{n+1}(2\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |m(\xi + k\pi)g_n(\xi + k\pi)|^2$$

$$= |m(\xi)|^2\sigma_{g_n}(\xi) + |m(\xi+\pi)|^2\sigma_{g_n}(\xi+\pi) = 1$$

and our claim follows by induction. By Fatou's lemma we obtain

$$\sigma_g(\xi) = \sum_{k\in\mathbb{Z}} \liminf_{n\to\infty} |g_n(\xi+2k\pi)|^2 \le \liminf_{n\to\infty} \sigma_{g_n}(\xi) = 1 \quad a.e.,$$

and this proves the first part of the lemma.

To see that $|g(\xi)| \ge c > 0$ a.e. $\xi \in E$ implies $\sigma_g(\xi) = 1$ a.e. it is enough to show that $g_n(\xi) \le \frac{1}{c}|g(\xi)|$ and then use the Dominated Convergence Theorem to obtain

$$\sigma_g(\xi) = \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} |g_n(\xi+2k\pi)|^2 = \lim_{n\to\infty} \sigma_{g_n}(\xi) = 1 \quad a.e.$$

Since we assume that $\frac{1}{c}|g(\xi)| \ge \mathbf{1}_E(\xi)$ a.e., we have

$$g_n(\xi) = \mathbf{1}_E(2^{-n}\xi)\prod_{j=1}^{n}|m(2^{-j}\xi)| \le \frac{1}{c}|g(2^{-n}\xi)|\prod_{j=1}^{n}|m(2^{-j}\xi)| = \frac{1}{c}|g(\xi)|,$$

which proves the needed estimate. ♣

Proof of Alg4. By (T1) $h(\xi) \neq 0$ a.e. $\xi \in S$; therefore, $\sigma_h(\xi) > 0$ a.e. $\xi \in \mathbb{R}$. Since $|m_0(\xi)| + |m_0(\xi+\pi)|^2 = 1$ a.e. and $h(2\xi) = m_0(\xi)h(\xi)$ a.e. we can apply Lemma 3.1.4 to obtain $\sigma_h(\xi) \le 1$ a.e. $\xi \in \mathbb{R}$. From Lemma 3.1.4 it follows that the condition $f(\xi) \ge c > 0$ for a.e. $\xi \in S$ implies that $\sigma_h(\xi) = 1$ a.e., because S, being a scaling set, must satisfy (SS2) and (SS3).

Finally, we have to show that g defined by $g(\xi) = \frac{h(\xi)}{\sqrt{\sigma_h(\xi)}}$ satisfies (S1)-(S3). First we check that $g(2\xi) = m(\xi)g(\xi)$, where m is a 2π-periodic function defined by $m(\xi) = \frac{m_0(\xi)\sqrt{\sigma_h(\xi)}}{\sqrt{\sigma_h(2\xi)}}$. In fact,

$$g(2\xi) = \frac{h(2\xi)}{\sqrt{\sigma_h(2\xi)}} = \frac{m_0(\xi)h(\xi)}{\sqrt{\sigma_h(2\xi)}} = \frac{m_0(\xi)\sqrt{\sigma_h(\xi)}}{\sqrt{\sigma_h(2\xi)}}g(\xi) = m(\xi)g(\xi),$$

and this establishes (S1) for g. Condition (S3) is clearly satisfied because

$$\sum_{k\in\mathbb{Z}}|g(\xi+2k\pi)|^2=\sum_{k\in\mathbb{Z}}\frac{|h(\xi+2k\pi)|^2}{\sigma_h(\xi)}=1 \quad a.e. \quad \xi\in\mathbb{R}.$$

This also implies that $|g(\xi)|\le 1$ a.e. Therefore, to check that (S2) holds for g it is enough to show that $\liminf_{j\to\infty}|g(2^{-j}\xi)|\ge 1$ a.e. $\xi\in\mathbb{R}$. Toward this end we will prove that $\lim_{j\to\infty}|h(2^{-j}\xi)|=1$ a.e. $\xi\in\mathbb{R}$. Indeed, since S satisfies (SS2) we conclude that for almost every $\xi\in\mathbb{R}$ there is an $n\in\mathbb{N}$ such that $2^{-j}\xi\in S$ for all $j\ge n$. Therefore,

$$\lim_{j\to\infty}|h(2^{-j}\xi)|=\lim_{j\to\infty}|f(2^{-j}\xi)|=1 \quad a.e.,$$

where we used (T3) for f. Since $\sigma_h(\xi)\le 1$ a.e., we obtain

$$\liminf_{j\to\infty}|g(2^{-j}\xi)|=\liminf_{j\to\infty}\frac{|h(2^{-j}\xi)|}{\sqrt{\sigma_h(2^{-j}\xi)}}\ge\liminf_{j\to\infty}|h(2^{-j}\xi)|=1 \quad a.e.,$$

which proves (S3) for g. That $g\in L^2(\mathbb{R})$ follows from (S3), since

$$\int|g(\xi)|^2\,d\xi=\sum_{k\in\mathbb{Z}}\int_{2k\pi}^{2(k+1)\pi}|g(\xi)|^2\,d\xi=\int_0^{2\pi}\sigma_g(\xi)\,d\xi=2\pi.$$

This finishes the proof of Alg4.

We have presented the proof of the validity of Algorithm 3.1.3 in several steps. We should point out that the algorithm yields a unique scaling function if we specify exactly the function α in Alg2. In the future, if necessary, we will indicate what function α we are using to obtain a scaling function from an MRA–couple.

Although the algorithm we presented may to seem tedious at first glance, all the formulas it contains are straightforward and it has several nice features. One of them is good control over the support of the function g it produces. The other is the simplicity of conditions (T1)-(T3) which allows us to construct many scaling functions based on one scaling set. Both these properties will be used in the next section to prove an interesting result concerning the problem of constructing continuous paths of wavelets.

As we mentioned before, Algorithm 3.1.3 can be used to obtain all scaling functions. The main difficulty in proving this is finding a scaling set contained in the support of the Fourier transform of a given scaling function. To show that this is always possible we will use the following:

LEMMA 3.1.5. *Let $A \subset \mathbb{R}$ be a set of finite measure. If K is a measurable subset of $\mathbb{R}$ which satisfies (SS3), then* $\lim_{n\to\infty} |\bigcup_{k\in\mathbb{Z}\setminus\{0\}}(A + 2^n 2k\pi) \cap K| = 0.$

PROOF. If A and K are bounded then it is easy to see that the above limit is zero. In the general case let us choose an $\epsilon > 0$ and $M \in \mathbb{N}$ such that $|A \cap [-M, M]^c| \leq \frac{\epsilon}{3}$ and $|K \cap [-M, M]^c| \leq \frac{\epsilon}{3}$. Such an M exists since A has finite measure and so does K, because (SS3) implies that $|K| = 2\pi$. Let $A_0 = A \cap [-M, M]^c$, $A_1 = A \cap [-M, M]$ and similarly $K_0 = K \cap [-M, M]^c$, $K_1 = K \cap [-M, M]$. We have

$$|\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n 2k\pi) \cap K| \leq |\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_1 + 2^n 2k\pi) \cap K_1| + |\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_1 + 2^n 2k\pi) \cap K_0|$$

$$+ |\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_0 + 2^n 2k\pi) \cap K_1| + |\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_0 + 2^n 2k\pi) \cap K_0|.$$

It is clear that there is an $N \in \mathbb{N}$ such that $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}}(A_1 + 2^n 2k\pi) \cap K_1| = 0$ for $n \geq N$. We also have $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}}(A_1 + 2^n 2k\pi) \cap K_0| \leq |K_0| \leq \frac{\epsilon}{3}$ and, similarly, $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}}(A_0 + 2^n 2k\pi) \cap K_0| \leq |K_0| \leq \frac{\epsilon}{3}$. Moreover, since K satisfies (SS3), we have

$$|\bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A_0 + 2^n 2k\pi) \cap K_1| \leq |\bigcup_{k\in\mathbb{Z}} (A_0 + 2k\pi) \cap K| \leq \sum_{k\in\mathbb{Z}} \int \mathbf{1}_{A_0}(\xi - 2k\pi) \mathbf{1}_K(\xi)\, d\xi$$

$$= \int_{A_0} \sum_{k\in\mathbb{Z}} \mathbf{1}_K(\xi + 2k\pi)\, d\xi = |A_0| \leq \frac{\epsilon}{3};$$

therefore, $|\bigcup_{k\in\mathbb{Z}\setminus\{0\}}(A+2^n 2k\pi)\cap K|\le\epsilon$ for $n\ge N$, which proves that the limit of the measures is zero. ♣

We will use this lemma to prove that the set we shall construct in the following result is a scaling set. Let us also recall that a 2π-periodization of a set $E\subset\mathbb{R}$ is defined by $E^P=\bigcup_{k\in\mathbb{Z}}(E+2k\pi)$.

LEMMA 3.1.6. *If a measurable set $S'\subset\mathbb{R}$ satisfies (SS1), (SS2) and*

$$\sum_{k\in\mathbb{Z}}\mathbf{1}_{S'}(\xi+2k\pi)\ge 1 \quad \textit{for a.e. } \xi\in\mathbb{R}, \tag{3.1.2}$$

then there exists a scaling set contained in S'.

PROOF. Since (3.1.2) holds for S' it is clear that we can find a measurable set $K'\subset S'$ which satisfies (SS3). Let $K_0=S'\cap[-\pi,\pi]$ and define $K=K_0\cup(K'\setminus K_0^P)$. Clearly K is a subset of S' which satisfies (SS3). Moreover, since (SS2) holds for S' we conclude that (SS2) holds for K_0 and, therefore, for K, which contains K_0. We shall modify the set K to obtain a scaling set $S\subset S'$.

For an integer $n\ge 0$ define

$$E_n=2^{-n}K\setminus\bigcup_{j=n+1}^{\infty}((2^{-j}K)^P\setminus 2^{-j}K).$$

We claim that $S=\bigcup_{n=0}^{\infty}E_n$ is a scaling set contained in S'. For $n\ge 0$ we have $E_n\subset 2^{-n}K\subset 2^{-n}S'\subset S'$, where the last inclusion follows from (SS1) for S'. This proves that $S\subset S'$ and it remains to be shown that S satisfies (SS1)-(SS3).

To prove the inclusion $S\subset 2S$ it is enough to check that $E_n\subset 2E_{n+1}$ for $n\ge 0$. Toward this end we observe that $2E^P\subset(2E)^P$, is true for every set $E\subset\mathbb{R}$. Consequently,

$$2E_{n+1}=2^{-n}K\setminus\bigcup_{j=n+2}^{\infty}(2(2^{-j}K)^P\setminus 2^{-j+1}K)\supset 2^{-n}K\setminus\bigcup_{j=n+2}^{\infty}((2^{-j+1}K)^P\setminus 2^{-j+1}K)$$

$$= E_n;$$

thus, (SS1) holds for S.

We observed that K satisfies condition (SS2). This implies that (SS2) for S will follow from the equality $\lim_{j\to\infty} \mathbf{1}_S(2^{-j}\xi) = 1$ a.e. $\xi \in K$. Let us fix a $\xi \in K$, since we already proved that $S \subset 2S$, the above equality is satisfied if we know that there is an $n \geq 0$ such that $2^{-n}\xi \in S$. Consider, for $n \geq 0$,

$$B_n = 2^n \bigcup_{j=n+1}^{\infty} ((2^{-j}K)^P \setminus 2^{-j}K).$$

and $B = \bigcap_{n=0}^{\infty} B_n \cap K$. If $\xi \notin B$, then $\xi \notin B_n \cap K$ for some $n \geq 0$. Since $\xi \in K$, this implies that $2^{-n}\xi \notin \bigcup_{j=n+1}^{\infty}(2^{-j}K)^P \setminus 2^{-j}K$; that is,

$$2^{-n}\xi \in 2^{-n}K \setminus \bigcup_{j=n+1}^{\infty} ((2^{-j}K)^P \setminus 2^{-j}K) = E_n \subset S.$$

That S satisfies (SS2) will follow if we show that B has measure zero. Clearly $|B| \leq \liminf_{n\to\infty} |B_n \cap K|$. Since K satisfies (SS3) we have

$$(2^{-j}K)^P \setminus 2^{-j}K = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{-j}K + 2k\pi)$$

for $j \geq 0$. Therefore, for $n \geq 0$

$$B_n = 2^n \bigcup_{j=n+1}^{\infty} \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{-j}K + 2k\pi) = \bigcup_{j=n+1}^{\infty} \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{n-j}K + 2^n 2k\pi)$$

$$= \bigcup_{j=1}^{\infty} \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (2^{-j}K + 2^n 2k\pi) = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} (A + 2^n 2k\pi),$$

where $A = \bigcup_{j=1}^{\infty} 2^{-j}K$. Since (SS3) for K implies that $|K| = 2\pi$ we conclude that $|A| \leq \sum_{j=1}^{\infty} |2^{-j}K| = 2\pi$ and, in particular, A is of finite measure. From Lemma

3.1.5 it follows that $\lim_{n\to\infty} |B_n \cap K| = 0$; which shows that B has measure zero and, hence, S satisfies (SS2).

Proving that S satisfies (SS3) is equivalent to showing that $(S+2k\pi)\cap S = \emptyset$ for $k \in \mathbb{Z}\setminus\{0\}$ and $S^P = \mathbb{R}$. To see that the intersection is empty it is enough to check that $(E_n + 2k\pi)\cap E_m = \emptyset$ for $k \in \mathbb{Z}\setminus\{0\}$, $n, m \geq 0$. Without loss of generality we can assume that $m \leq n$. If $m = n$ then $(E_n + 2k\pi)\cap E_n \subset (2^{-n}K + 2k\pi)\cap 2^{-n}K = 2^{-n}((K + 2^{n+1}k\pi)\cap K) = \emptyset$ for $k \in \mathbb{Z}\setminus\{0\}$ since K satisfies (SS3). In case $m < n$ we have $E_n + 2k\pi \subset (2^{-n}K)^P \setminus 2^{-n}K$ for $k \in \mathbb{Z}\setminus\{0\}$. In particular $E_n + 2k\pi \subset \bigcup_{j=m+1}^{\infty}((2^{-j}K)^P \setminus 2^{-j}K)$ and, therefore, $(E_n + 2k\pi)\cap E_m = \emptyset$ for $k \in \mathbb{Z}\setminus\{0\}$; this implies that the sets $S + 2k\pi$, $k \in \mathbb{Z}$ are disjoint.

The last step of the proof is to show that $S^P = \mathbb{R}$. Since K satisfies (SS3) it is enough to prove that $K \subset S^P$. Let

$$C_n = \bigcup_{j=n+1}^{\infty} (2^{-j}K)^P$$

for $n \geq 0$ and $C = \bigcap_{n=0}^{\infty} C_n \cap K$. Consider a point $\xi \in K$. If $\xi \notin C_0$, then $\xi \in K \setminus C_0 \subset E_0 \subset S$. Therefore, we can concentrate on the case when $\xi \in C_0$. Since $C_{n+1} \subset C_n$ for $n \geq 0$, it is clear that either $\xi \in C$ or there is an $n' \geq 1$ such that $\xi \in C_{n'-1} \setminus C_{n'}$. If the latter is satisfied then $\xi \in (2^{-n'}K)^P$; therefore, there is an $l \in \mathbb{Z}$ such that $\xi + 2l\pi \in 2^{-n'}K$. Moreover, since ξ does not belong to $C_{n'}$ neither does $\xi + 2l\pi$. This gives us $\xi + 2l\pi \in 2^{-n'}K \setminus C_{n'} \subset E_{n'} \subset S$. It follows that $\xi \in S^P$. We finish the proof by showing that C has measure zero. Indeed, $|C| \leq \liminf_{n\to\infty} |C_n \cap K|$. Since K satisfies (SS3) we have

$$|C_n \cap K| \leq \sum_{j=n+1}^{\infty} |(2^{-j}K)^P \cap K| = \sum_{j=n+1}^{\infty} |2^{-j}K| = 2^{-n}|K| = 2^{-n}2\pi,$$

proving that $|C| = 0$. We have thus shown that (SS3) holds for S and this ends the proof of the lemma. ♣

If φ is a scaling function and we denote $\operatorname{supp}\hat{\varphi}$ by S', then (S1),(S2) and (S3) for $\hat{\varphi}$ will give us (SS1),(SS2) and (3.1.2) for S' respectively. Thus, we obtain the following:

PROPOSITION 3.1.7. *For every scaling function φ there exists a scaling set contained in the support of $\hat{\varphi}$.*

The proposition we just proved is the key observation that allows us to show that every scaling function can be constructed via Algorithm 3.1.3. We will say that a scaling function φ is *associated with an MRA–couple* (S, f) if $\hat{\varphi}(\xi) = g(\xi)$, a.e., where g is the function defined in Alg4. The following result can be viewed as a characterization of scaling functions.

THEOREM 3.1.8. *Every scaling function is associated with an MRA–couple.*

PROOF. Let φ be a scaling function and m its low–pass filter. By Proposition 3.1.7 we can find a scaling set S contained in the support of $\hat{\varphi}$. It is also clear that there exist a unimodular measurable function β such that $m(\xi) = \beta(\xi)|m(\xi)|$ a.e. $\xi \in \mathbb{R}$. We claim that φ is associated with an MRA–couple (S, f), where $f \equiv \hat{\varphi}|_S$ and the function α from Alg2 is given by $\alpha \equiv \beta|_{\frac{S}{2}+\pi}$. To justify that (S, f) is an MRA–couple we have to check conditions (T1)-(T3) for f. The first one is obvious by our choice of S. Since $\hat{\varphi}$ satisfies (S1) and $|m(\xi)| \leq 1$ a.e., condition (T2) holds for f as well. Similarly (S2) for $\hat{\varphi}$ implies that f has to satisfy (T3). It is also clear that φ is associated with (S, f). In fact, by Alg1, Alg2 and our choice of α we obtain that m_0 constructed in Algorithm 3.1.3, is equal to m almost everywhere. Moreover, the function h defined in Alg3 must be equal to $\hat{\varphi}$ by the uniqueness of the extension considered in Alg3. This implies that $\sigma_h(\xi) = 1$ a.e., for $\hat{\varphi}$ has to satisfy (S3). Therefore, g defined in Alg4 must be equal to $\hat{\varphi}$ a.e.; that is, φ is associated with (S, f). ♣

REMARK. From Algoritm MRA3 it is clear that if a scaling function is associated with an MRA–couple (S, f) then $S \subset \operatorname{supp}\hat{\varphi}$. Conversely, from the proof of

Theorem 3.1.8 it follows that if $S \subset \operatorname{supp} \hat{\varphi}$, where S is a scaling set and φ is a scaling function, then φ is associated with an MRA–couple $(S, \hat{\varphi}|_S)$.

In the next section we will use the above theorem and Algorithm 3.1.3 to construct a continuous path of wavelets connecting a given MRA wavelet with an appropriate MSF wavelet.

3.2 Continuous paths of wavelets

One of the most famous wavelets are the Schwartz class wavelets constructed by Meyer. The idea he used to obtain these wavelets relied on smoothing the Fourier transform of the Shannon wavelet. Wavelets constructed in this way have smooth and compactly supported Fourier transform, that automatically forces them to belong to the Schwartz class. This construction was extended by E. Hernandéz, X. Wang and G. Weiss in [HWW1]. The paper contains a detailed procedure of smoothing some special MSF wavelets, that is, $\check{\mathbf{1}}_{[-2a,-a]\cup[2\pi-a,4\pi-2a]}$, where $0 < a < 2\pi$ (note that $a = \pi$ yields the Shannon wavelet). The authors characterized all wavelets whose Fourier transform is supported in the interval $S_\alpha = \left[-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha\right]$, where $0 < \alpha \le \pi$. The characterization gives a simple constructive procedure for obtaining all such wavelets, in particular, Meyer's wavelets. From their construction it was clear that the family of wavelets they obtained was connected in the $L^2(\mathbb{R})$ norm. To be more precise, every wavelet from this family can be connected by a continuous path within this family to the MSF wavelet $\check{\mathbf{1}}_{[-2a,-a]\cup[2\pi-a,4\pi-2a]}$ on which it is based, it is also clear that all the MSF wavelets we have mentioned above form a continuous path if a changes between 0 and 2π. This observation led to an interesting problem of connectivity of the whole set of wavelets. The same problem was posed independently by X. Dai and D. Larson (see Problem A in [DL]) and is considered as "perhaps the most important open question" in the theory of wavelets they develop.

The connectedness problem turned out to be hard to solve. First result concerning it is due to D. Speegle who showed in his Ph.D thesis that the family of MSF wavelets is path connected. Then a group of people led by D. Larson X. Dai and G. Weiss proved in [WUT] that the set of all MRA wavelets is path connected. Both these results had as their base some algorithms enabling construction of wavelets. It is, therefore, natural to see how Algorithm 3.1.3 can be applied to describe continuous paths of wavelets. However, before we do so, let us go back to the beginnings of wavelet theory and refer to yet another interesting problem.

MSF wavelets first appeared in a paper of X. Fang and X. Wang [FW]. We recall that the name MSF (minimally supported frequency) wavelets relates to the fact that these are the wavelets with the smallest possible measure of the support of their Fourier transform. Another name for the same object was given by X. Dai and D. Larson in [DL]. They called this class *s-elementary wavelets*, where the prefix "s" corresponds to "set". After finding out about "MSF" wavelets D. Larson asked if the word "minimally" can correspond not only to the measure but also to an inclusion of supports of wavelets, that is, must the support of the Fourier transform of a wavelet contain a wavelet set? This question appears as Problem 4.A in [L] and is still open. We will answer it positively in the case of MRA wavelets and then we will use Algorithm 3.1.3 to connect, with a path of wavelets, a given MRA wavelet ψ with an MSF wavelet whose wavelet set is contained in $\operatorname{supp}\hat{\psi}$. This simpler version of the mentioned problem surfaced naturally in G. Weiss' program on smoothing MSF wavelets, which will be discused in details in the next section.

Let ψ be an MRA wavelet with a scaling function φ and W be a wavelet set which gives rise to an MRA wavelet (we will call such a set an *MRA wavelet set*) such that $W \subset \operatorname{supp}\hat{\psi}$. It is clear that $S = \bigcup_{j=1}^{\infty} 2^{-j}W$ is a scaling set which is contained in $\operatorname{supp}\hat{\varphi} = \bigcup_{j=1}^{\infty} 2^{-j}\operatorname{supp}\hat{\psi}$. Therefore, the problem of finding a wavelet set in the support of the Fourier transform of a given MRA wavelet is related to the problem of finding a scaling set in the support of the Fourier transform of a given scaling function. By Proposition 3.1.7 we know that for every scaling function φ

such a scaling set S exists. Then, obviously, $W = 2S \setminus S$ is an MRA wavelet set but it is not immediate that W is contained in the support of the Fourier transform of a wavelet corresponding to φ. The following proposition shows, however, that this is the case.

PROPOSITION 3.2.1. *If ψ is an MRA wavelet with a scaling function φ and S is a scaling set contained in* $\operatorname{supp}\hat{\varphi}$, *then the MRA wavelet set $W = 2S \setminus S$ is contained in* $\operatorname{supp}\hat{\psi}$.

PROOF. We have $|\hat{\psi}(\xi)|^2 = |\hat{\varphi}(\frac{\xi}{2})|^2 - |\hat{\varphi}(\xi)|^2$. Therefore, $\operatorname{supp}\hat{\psi} = (2\operatorname{supp}\hat{\varphi}) \setminus D$, where $D = \{\xi \in \mathbb{R}\colon |\hat{\varphi}(\frac{\xi}{2})| = |\hat{\varphi}(\xi)|\}$. Since $S \subset \operatorname{supp}\hat{\varphi}$ all we have to prove is that $(2S \setminus S) \cap D = \emptyset$ or, equivalently, $2S \cap D \subset S$. Assume that $\xi \in 2S \cap D$, then $|\hat{\varphi}(\xi)| = |\hat{\varphi}(\frac{\xi}{2})| \neq 0$, because $\frac{\xi}{2} \in S$. This implies that $|m(\frac{\xi}{2})| = 1$, where m is the low–pass filter of φ. Therefore we must have $m(\frac{\xi}{2} + k\pi) = 0$ for all odd integers k; hence, for such k we obtain $\hat{\varphi}(\xi + 2k\pi) = 0$. We claim that this implies that $\xi \in S$. In fact since S satisfies (SS3) there is an $l \in \mathbb{Z}$ such that $\xi + 2l\pi \in S$; in particular $\hat{\varphi}(\xi + 2l\pi) \neq 0$, so l must be even. Obviously $\frac{\xi}{2} + l\pi \in \frac{S}{2} \subset S$ and $\frac{\xi}{2} \in S$, therefore, using (SS3) again we conclude that $l = 0$ and $\xi \in S$ follows. ♣

As a corollary of the above proposition and Proposition 3.1.7 we obtain the following:

PROPOSITION 3.2.2. *If ψ is an MRA wavelet then there exists an MRA wavelet set contained in* $\operatorname{supp}\hat{\psi}$.

Let ψ be an MRA wavelet, the corollary assures that we can find a wavelet set W contained in $\operatorname{supp}\hat{\psi}$. We want to connect ψ to an MSF wavelet corresponding to W with a continuous path of wavelets ψ_t, $0 \leq t \leq 1$. Moreover, we would like to do it in a "nice" way, that is, we want $\operatorname{supp}\hat{\psi}_t \subset \operatorname{supp}\hat{\psi}$. To achieve this goal we need to know what is the support of the Fourier transform of a scaling function constructed via Algorithm 3.1.3.

PROPOSITION 3.2.3. *If φ is a scaling function associated with an MRA–couple (S, f) and m is its low–pass filter, then*

$$(\operatorname{supp}\hat{\varphi})^c = \bigcup_{n=1}^{\infty}\big((2^n S\setminus 2^{n-1}S)\cap \bigcup_{j=1}^{n} 2^j (J_f+\pi)^P\big),$$

where $J_f = \{\xi \in \frac{S}{2}\colon |f(2\xi)| = |f(\xi)|\}$. Moreover, $(\operatorname{supp} m)^c = (J_f+\pi)^P$.

PROOF. It is clear that $\operatorname{supp}\hat{\varphi} = \operatorname{supp} h$, where h is defined in (3.1.1); that is,

$$h(\xi) = \begin{cases} f(2^{-n}\xi)\prod_{j=1}^{n} m_0(2^{-j}\xi), & \text{for } \xi \in 2^n S\setminus 2^{n-1}S, \quad n\in\mathbb{N} \\ f(\xi) & \text{for } \xi \in S. \end{cases}$$

We will identify the set where $h(\xi)$ is equal to zero. By (T1) we have $f(\xi)\neq 0$ a.e. $\xi\in S$. So we can see from the above formula that $h(\xi)=0$ if and only if $\xi \in 2^n S\setminus 2^{n-1}S$ and $\prod_{j=1}^{n} m_0(2^{-j}\xi) = 0$, where $n\in\mathbb{N}$. The product is zero if and only if $2^{-j}\xi \in Z_{m_0}$, where $Z_{m_0} = \{\xi\in\mathbb{R}\colon m_0(\xi)=0\}$ and $1\le j\le n$. Therefore, we obtain

$$(\operatorname{supp}\hat{\varphi})^c = \bigcup_{n=1}^{\infty}\big((2^n S\setminus 2^{n-1}S)\cap \bigcup_{j=1}^{n} 2^j Z_{m_0}\big).$$

We will finish the first part of the proof by showing that $Z_{m_0} = (J_f+\pi)^P$. By (T1) and Alg1 we have $m_0(\xi)\neq 0$ a.e. $\xi\in\frac{S}{2}$; moreover, $|m_0(\xi)|=1$ a.e. $\xi\in\frac{S}{2}$ if and only if $m_0(\xi+\pi)=0$ a.e. $\xi\in\frac{S}{2}$. Therefore, we obtain

$$Z_{m_0}\cap\big(\frac{S}{2}\cup(\frac{S}{2}+\pi)\big) = \{\xi\in\frac{S}{2}\colon |m_0(\xi)|=1\}+\pi = J_f+\pi.$$

Since m_0 is 2π-periodic and, as we recall, 2π translates of the set $\frac{S}{2}\cup(\frac{S}{2}+\pi)$ form a partition of $\mathbb{R}$, we must have $Z_{m_0} = (J_f+\pi)^P$. This gives us the desired formula for $\operatorname{supp}\hat{\varphi}$. By Alg4 we have $\operatorname{supp} m = \operatorname{supp} m_0 = (Z_{m_0})^c$; therefore, it follows that $(\operatorname{supp} m)^c = (J_f+\pi)^P$, which concludes the proof. ♣

We will say that an MRA wavelet is *associated with an MRA–couple* (S, f) if one of its scaling functions is associated with (S, f). We have the following

COROLLARY 3.2.4. *If ψ, ψ' are MRA wavelets associated with (S, f), (S, f') respectively and $J_f = J_{f'}$ (defined as in Proposition 3.2.3), then* $\operatorname{supp}\hat{\psi} = \operatorname{supp}\hat{\psi}'$.

PROOF. Let φ be a scaling function of ψ, that is associated with (S, f) and, similarly, let φ' be a scaling function of ψ', that is associated with (S, f'). Let m be a low–pass filter of φ and m' a low–pass filter of φ'. Since $J_f = J_{f'}$ Proposition 3.2.3 implies that $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\varphi}'$ and $\operatorname{supp} m = \operatorname{supp} m'$. Therefore, the formulae $|\hat{\psi}(2\xi)| = |m(\xi+\pi)||\hat{\varphi}(\xi)|$ and $|\hat{\psi}'| = |m'(\xi+\pi)||\hat{\varphi}'(\xi)|$ immediately give us the equality $\operatorname{supp}\hat{\psi} = \operatorname{supp}\hat{\psi}'$. ♣

Before we present the main result of this section we will develop a little simpler approach to the problem of connectivity of wavelets. We will use the following well known lemma.

LEMMA 3.2.5. *Let $f_n \in L^2(\mathbb{R})$, $n \in \mathbb{N}$ be a sequence converging almost everywhere to $f \in L^2(\mathbb{R})$. If $\|f_n\|_2 \to \|f\|_2$ then $f_n \to f$ in $L^2(\mathbb{R})$.*

PROOF. Let β be a measurable unimodular function such that $f(x) = \beta(x)|f(x)|$. Clearly, β is uniquely defined on $K = \operatorname{supp} f$ (we can put $\beta(x) = 1$ for $x \in K^c$). Similarly, let β_n be a measurable unimodular function such that $f_n(x) = \beta_n(x)|f_n(x)|$. By the almost everywhere convergence of f_n to f we obtain that $\beta_n(x)\mathbf{1}_K(x) \to \beta(x)\mathbf{1}_K(x)$ a.e. $x \in \mathbb{R}$.

To prove that $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ it is enough to show that $\limsup_{n\to\infty} \|f_n - f\|_2^2 \le 0$. Since $\|f_n\|_2 \to \|f\|_2$ we obtain

$$\limsup_{n\to\infty} \|f_n - f\|_2^2 = \limsup_{n\to\infty}(\|f_n\|_2^2 + \|f\|_2^2 - \langle f_n, f\rangle - \langle f, f_n\rangle)$$

$$= 2\|f\|_2^2 - \liminf_{n\to\infty}(\langle f_n, f\rangle + \langle f, f_n\rangle).$$

The convergence $f_n \to f$ in $L^2(\mathbb{R})$ will follow if we show that $\liminf_{n\to\infty}(\langle f_n, f\rangle + \langle f, f_n\rangle) \ge 2\|f\|_2^2$. We have

$$\langle f_n, f\rangle = \langle \beta_n|f_n|, \beta|f|\rangle = \langle |f_n|, |f|\rangle + \langle \beta_n|f_n|, (\beta - \beta_n)|f|\rangle.$$

We claim that

$$\lim_{n\to\infty}\langle \beta_n|f_n|,(\beta-\beta_n)|f|\rangle = 0. \tag{3.2.1}$$

In fact, by the Schwarz inequality

$$|\langle \beta_n|f_n|,(\beta-\beta_n)|f|\rangle| \le \|f_n\|_2\|(\beta-\beta_n)|f|\|_2 = \|f_n\|_2\|(\beta\mathbf{1}_K-\beta_n\mathbf{1}_K)|f|\|_2.$$

Clearly, $(\beta(x)\mathbf{1}_K(x)-\beta_n(x)\mathbf{1}_K(x))|f(x)|\to 0$ a.e., moreover,

$$|(\beta(x)\mathbf{1}_K(x)-\beta_n(x)\mathbf{1}_K(x))|f(x)||\le 2|f(x)|,$$

thus, $\lim_{n\to\infty}\|(\beta\mathbf{1}_K-\beta_n\mathbf{1}_K)|f|\|_2=0$ by the Dominated Convergence Theorem. This and the assumption $\|f_n\|_2\to\|f\|_2$ prove (3.2.1).

The desired inequality $\liminf_{n\to\infty}(\langle f_n,f\rangle+\langle f,f_n\rangle)\ge 2\|f\|_2^2$ follows easily from (3.2.1) and Fatou's lemma. Indeed,

$$\liminf_{n\to\infty}(\langle f_n,f\rangle+\langle f,f_n\rangle)=\liminf_{n\to\infty}(2\langle|f_n|,|f|\rangle+2\operatorname{Re}\langle\beta_n|f_n|,(\beta-\beta_n)|f|\rangle)$$

$$=\liminf_{n\to\infty}2\langle|f_n|,|f|\rangle\ge 2\|f\|_2^2. \quad ♣$$

The lemma we just proved allows us to conclude that every path of wavelets which is continuous pointwise almost everywhere (on the Fourier transform side) is also continuous in the $L^2(\mathbb{R})$ norm. To be more specific let us prove the following

Proposition 3.2.6. *If ψ_t, $0\le t\le 1$ is a family of wavelets such that the function $t\to\hat\psi_t(\xi)$ is continuous for a.e. $\xi\in\mathbb{R}$, then the mapping $t\to\psi_t$ is a continuous function from the interval $[0,1]$ into $L^2(\mathbb{R})$.*

Proof. Since the Fourier transform is an isometry of $L^2(\mathbb{R})$ it is enough to show that the mapping $t\to\hat\psi_t$ is a continuous function from the interval $[0,1]$ into $L^2(\mathbb{R})$. This is, in turn, equivalent to proving that

$$\lim_{t\to s}\|\hat\psi_t-\hat\psi_s\|_2=0 \tag{3.2.2}$$

for every $s \in [0,1]$. Let us fix an $s \in [0,1]$, and consider a sequence $\{t_n\} \subset [0,1]$, $n \in \mathbb{N}$ which converges to s. Clearly, $\|\hat{\psi}_{t_n}\|_2 = \|\hat{\psi}_s\|_2 = 2\pi$ and by our assumption $\hat{\psi}_{t_n}(\xi) \to \hat{\psi}_s(\xi)$ a.e. $\xi \in \mathbb{R}$, therefore we can apply Lemma 3.2.5 to obtain that $\lim_{t_n \to s} \|\hat{\psi}_{t_n} - \hat{\psi}_s\|_2 = 0$. Since we chose an arbitrary sequence converging to s, this implies that (3.2.2) is satisfied, which ends the proof. ♣

At this point we are ready to prove the main result of this section.

THEOREM 3.2.7. *Let ψ be an MRA wavelet, W an MRA wavelet set contained in the support of $\hat{\psi}$ and $\tilde{\psi}$ any MSF wavelet corresponding to W. Then there exists continuous (in $L^2(\mathbb{R})$) path of MRA wavelets $\{\psi_t\}$, $t \in [0,1]$ such that $\psi_0 \equiv \psi$, $\psi_1 \equiv \tilde{\psi}$ and* $\operatorname{supp} \hat{\psi}_t \subset \operatorname{supp} \hat{\psi}$ *for $t \in [0,1]$.*

REMARK. We point out, that if W is a wavelet set then the class of MSF wavelets corresponding to this wavelet set is arcwise connected in the $L^2(\mathbb{R})$–norm. Therefore, to prove the theorem, it is enough to exhibit a path as given above, with the endpoint ψ_1 not necessarily equal to $\tilde{\psi}$ but satisfying $|\hat{\psi}_1| = \mathbf{1}_W$.

PROOF OF THEOREM 3.2.7. Let ψ be an MRA wavelet and φ a scaling function associated to ψ with a low–pass filter m. We know that

$$\hat{\psi}(2\xi) = e^{i\xi} s(2\xi)\overline{m(\xi+\pi)}\hat{\varphi}(\xi), \tag{3.2.3}$$

where s is a measurable 2π periodic function. Let W be an MRA wavelet set contained in the support of $\hat{\psi}$ (recall that by Proposition 3.2.2 such a set always exists). We know that $S = \bigcup_{j=1}^{\infty} 2^{-j} W$ is a scaling set contained in the support of $\hat{\varphi}$. By Theorem 3.1.8 and following it remark we conclude that φ is associated with an MRA–couple (S, f), where $f \equiv \hat{\varphi}|_S$. Clearly there exists a measurable unimodular function β defined on S such that $f(\xi) = \beta(\xi)|f(\xi)|$ a.e. $\xi \in S$. Let

$$f_t(\xi) = \beta(\xi)(t + (1-t)|f(\xi)|) \tag{3.2.4}$$

for $t \in [0,1]$. Note that $f_0 \equiv f$ and $f_1(\xi) = \beta(\xi)$ a.e. $\xi \in S$.

We claim that (S, f_t) is an MRA–couple for every $t \in [0,1]$ (the case $t = 0$ is trivial since as we just mentioned $f_0 \equiv f$). To prove the claim we need to check that the conditions (T1)-(T3) are satisfied for every f_t. We have, for $t \in (0,1]$,

$$|f_t(\xi)| \geq t > 0, \tag{3.2.5}$$

which proves (T1) for f_t. Since f satisfies (T2) we obtain

$$|f_t(2\xi)| = t + (1-t)|f(2\xi)| \leq t + (1-t)|f(\xi)| = |f_t(\xi)|;$$

that is, (T2) holds for f_t, $t \in (0,1]$ as well. To check that $(T3)$ is satisfied for f_t we observe

$$\lim_{j\to\infty} |f_t(2^{-j}\xi)| = \lim_{j\to\infty} (t + (1-t)|f(2^{-j}\xi)|) = t + (1-t) = 1$$

for a.e. $\xi \in S$, which shows that (S, f_t) is an MRA–couple for every $t \in [0,1]$.

Since φ is associated with the MRA–couple (S, f) it can be constructed by Algorithm 3.1.3. Recall, that the appropriate unimodular function α which occurs in Alg2 is described in Theorem 3.1.8. We do not need this description, but we use precisely the same α to construct a path connecting ψ_0 and ψ_1.

Let m_0^t be the function defined in Alg1 and Alg2 corresponding to (S, f_t), where in Alg2 we use the α mentioned above. Let h_t be the function defined in (3.1.1) corresponding to (S, f_t). Clearly, from (3.2.5) and Alg4 follows that $\sigma_{h_t} \equiv 1$ for $t \in (0,1]$; moreover, by the same argument as in the proof of Theorem 3.1.8, we find out that $\sigma_{h_0} \equiv 1$ (because $h_0 \equiv \hat{\varphi}$ and $\hat{\varphi}$ satisfies (S3)). Therefore, the scaling function φ_t associated with the MRA–couple (S, f_t) is given by $\hat{\varphi}_t(\xi) = h_t(\xi)$ a.e. $\xi \in \mathbb{R}$ for $t \in [0,1]$. In addition the low–pass filter m^t of φ_t is equal to m_0^t almost everywhere.

For $t \in [0,1]$ define ψ_t by the formula

$$\hat{\psi}_t(2\xi) = e^{i\xi} s(2\xi)\overline{m^t(\xi+\pi)}\hat{\varphi}_t(\xi) = e^{i\xi} s(2\xi)\overline{m_0^t(\xi+\pi)} h_t(\xi), \tag{3.2.6}$$

where s is the same as in (3.2.3). It is clear that $\{\psi_t\}$, $t \in [0,1]$ is a family of MRA wavelets. Since $f_0 \equiv f$ we obtain $\varphi_0 \equiv \varphi$ and, therefore, $m^0 \equiv m$, which shows that $\psi_0 \equiv \psi$. We also have $f_1(\xi) = \beta(\xi)$ a.e. $\xi \in S$, therefore, by (3.1.1) we obtain $\hat{\varphi}_1(\xi) = h_1(\xi) = f_1(\xi) = \beta(\xi)$ a.e. $\xi \in S$. In particular $|\hat{\varphi}_1(\xi)| = 1$ a.e. $\xi \in S$, thus condition (S3) for $\hat{\varphi}_1$ implies that $|\hat{\varphi}_1(\xi)| = \mathbf{1}_S(\xi)$ a.e. $\xi \in \mathbb{R}$. From this it follows that $|\hat{\psi}_1(\xi)| = \mathbf{1}_W(\xi)$ a.e. $\xi \in \mathbb{R}$; hence, ψ_1 is an MSF wavelet corresponding to W.

To prove that the path of wavelets we constructed is continuous in the $L^2(\mathbb{R})$ norm it is enough to show that the function $t \to \hat{\psi}_t(\xi)$ is continuous for a.e. $\xi \in \mathbb{R}$ and then apply Proposition 3.2.6. Clearly the function $t \to f_t(\xi)$ is continuous a.e. $\xi \in S$ for $t \in [0,1]$. Therefore, by Alg1 and Alg2 we obtain that the function $t \to m_0^t(\xi)$ is continuous a.e. $\xi \in \mathbb{R}$ for $t \in [0,1]$. This, in turn, together with (3.1.1) implies that the function $t \to h_t(\xi)$ is continuous a.e. $\xi \in \mathbb{R}$ for $t \in [0,1]$. It is clear, therefore, that by (3.2.6) we obtain the desired continuity of $t \to \hat{\psi}_t(\xi)$ for a.e. $\xi \in \mathbb{R}$ and $t \in [0,1]$.

It remains to prove that $\operatorname{supp} \hat{\psi}_t \subset \operatorname{supp} \hat{\psi}$ for $t \in [0,1]$. If $t = 1$ we have $\operatorname{supp} \hat{\psi}_1 = W \subset \operatorname{supp} \hat{\psi}$. We will use Corollary 3.2.4 to show that $\operatorname{supp} \hat{\psi}_t = \operatorname{supp} \hat{\psi}$ for $t \in [0,1)$. By definition, ψ_t is associated with the MRA–couple (S, f_t) for $t \in [0,1]$. Recall that $J_{f_t} = \{\xi \in \frac{S}{2} \colon |f_t(2\xi)| = |f_t(\xi)|\}$. It is easy to check, that $J_{f_t} = J_f$ for $t \in [0,1)$. In fact, for such t and a.e. $\xi \in \frac{S}{2}$ we have

$$|f_t(2\xi)| = |f_t(\xi)| \iff t + (1-t)|f(2\xi)| = t + (1-t)|f(\xi)| \iff |f(2\xi)| = |f(\xi)|.$$

Therefore, by Corollary 3.2.4 we obtain that $\operatorname{supp} \hat{\psi}_t = \operatorname{supp} \hat{\psi}$ for $t \in [0,1)$, which concludes the proof. ♣

Remark. The key point in proving the continuity of the path of wavelets considered in the above theorem was noticing that $\sigma_{h_t} \equiv 1$ for $t \in [0,1]$. Without this the almost everywhere continuity of $t \to \sigma_{h_t}(\xi)$ would not be clear. In fact, there exists an MRA–couple (S, f) such that the corresponding σ_h is smaller then

1 almost everywhere. If we define f_t as in (3.2.4) then $\sigma_{h_t} \equiv 1$ for $t \in (0,1]$ and therefore $\lim_{t\to 0} \sigma_{h_t}(\xi) \neq \sigma_{h_0}(\xi) = \sigma_h(\xi)$.

In [WUT] it is shown that the class of all MRA wavelets is connected. In [Sp] it is shown that the class of all MSF wavelets is connected. Since the latter containes some MRA wavelets (for example, the Shannon wavelet) we see that the union of the MRA wavelets and the MSF wavelets is connected. Theorem 3.2.7 and the result in [Sp] gives us another proof of the connectivity of this union and we are not using the result in [WUT].

Although we justified the usefulness of Algorithm 3.1.3 it is not a good tool for constructing wavelets of the Meyer type; by this we mean those wavelets, whose Fourier transform is smooth and compactly supported. However, we can use the main ideas of the algorithm to consider the problem of smoothing MSF wavelets; and we will do so in the next section.

3.3 Smoothing MSF wavelets

Results concerning G. Weiss' program on smoothing MSF wavelets appeared in [HWW1] and [HWW2]. To be more precise, and yet keep our discussion simple, let us cite one of them

THEOREM 3.3.1. *Suppose $\psi \in L^2(\mathbb{R})$ and $b = |\hat{\psi}|$ has support contained in*

$$S_\alpha = \left[-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha\right], \quad 0 < \alpha \leq \pi.$$

Then ψ is an orthonormal wavelet if and only if it is an MRA wavelet and

(i) $b^2(\xi) + b^2(\frac{\xi}{2}) = 1$ *for a.e.* $\xi \in [4\pi - \frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$;

(ii) $b(\xi) = 1$ *for a.e.* $\xi \in [2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$;

(iii) $b^2(\xi) + b^2(\xi + 2\pi) = 1$ *for a.e.* $\xi \in [-\frac{4}{3}\alpha, -\frac{2}{3}\alpha]$

(iv) $b(\xi) = b(\frac{\xi}{2} + 2\pi)$ *for a.e.* $\xi \in [-\frac{8}{3}\alpha, -\frac{4}{3}\alpha]$

(v) $\quad \hat{\psi}(\xi) = e^{ip(\xi)} b(\xi)$*, with* $p(\xi)$ *satisfying*

$$p(\xi) + p(2(\xi - 2\pi)) - p(2\xi) - p(\xi - 2\pi) = (2n(\xi) + 1)\pi$$

for a.e. $\xi \in [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha] \cap \operatorname{supp} b \cap (\frac{1}{2} \operatorname{supp} b)$*, where* $n(\xi)$ *is an integer–valued measurable function.*

(vi) $\quad b(\xi) = 0$ *for a.e.* $\xi \in [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha] = H_\alpha$

From the above theorem it follows that the values of $|\hat{\psi}(\xi)|$ for $\xi \in [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$ completely determine all the values $|\hat{\psi}(\xi)|$ on $\mathbb{R}$; in fact, on $[2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$ the function $|\hat{\psi}|$ can be chosen to be any measurable function b with range in $[0, 1]$ and then conditions (i)–(vi) give an implicit prescription how to obtain all the values of $|\hat{\psi}|$ on $\mathbb{R}$. In particular if b is a smooth function on $(2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha)$, which is equal to 0 on a righthand side neighborhood of $2\pi - \frac{4}{3}\alpha$ and equal to 1 on a lefthand side neighborhood of $2\pi - \frac{2}{3}\alpha$, then $|\hat{\psi}(\xi)|$ is smooth on $\mathbb{R}$. If we choose a smooth function p for condition (v) (e.g. $p(\xi) = \xi/2$), then $\hat{\psi}$ will be smooth and supported in $S_\alpha \subset (-4\pi, 4\pi)$. Similar types of results were proven for wavelets whose Fourier transform is supported in $\tilde{S}_\beta = [-4\pi + \frac{4}{3}\beta, \frac{8}{3}\beta]$, $0 < \beta \leq \pi$ and provided another examples of wavelets whose Fourier transform is smooth and supported in $\tilde{S}_\beta \subset (-4\pi, 4\pi)$ (see [HWW1]).

As we mentioned before these results can be viewed as smoothing procedures for the class of MSF wavelets $\psi = \check{\mathbf{1}}_{[-2a,-a]\cup[2\pi-a,4\pi-2a]}$, where $0 < a < 2\pi$. Natural questions which arose in this program were: "Can we smooth other MSF wavelets? In particular, can we find wavelets whose Fourier transform is smooth and compactly supported but not within $(-4\pi, 4\pi)$?", "Is it true that every wavelet with smooth and compactly supported Fourier transform can be viewed as a smooth version of an MSF wavelet? More precisely, can we find for such a wavelet ψ a wavelet set W and a positive constant c such that $|\hat{\psi}(\xi)| \geq c$ for $\xi \in W$?" We will answer these questions below. However, before we do so, let us prove an interesting fact concerning the original construction of E. Hernandez, X. Wang and G. Weiss given in [HWW1].

In [Ba] L. Baggett applied the theory of the multiplicity function to prove that every wavelet ψ such that $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in [-a, b]$, where $a, b > 0$ and $a + b + \min(a, b) \geq 2\pi$, is associated with a multiresolution analysis. He pointed out a connection of this result to those obtained in [HWW1]. To understand this connection we recall that in [HWW1] it was proved that if the Fourier transform of a wavelet ψ is supported in $S_\alpha = [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, $0 < \alpha \leq \pi$ (or $\tilde{S}_\beta = [-4\pi + \frac{4}{3}\beta, \frac{8}{3}\beta]$, $0 < \beta \leq \pi$), then ψ is an MRA wavelet and $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in H_\alpha = [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha]$ (or a.e. $\xi \in \tilde{H}_\beta = [-2\pi + \frac{4}{3}\beta, \frac{2}{3}\beta]$). Therefore, if we put $a = \frac{2}{3}\alpha$ and $b = 2\pi - \frac{4}{3}\alpha$ (or $a = 2\pi - \frac{4}{3}\beta$, $b = \frac{2}{3}\beta$), then $a + b + \min(a, b) \geq 2\pi$, which gives a clear analogy with the observation of L. Baggett.

We will prove an extension of L. Baggett's result (new information about the support of $\hat{\psi}$ will be given) and then we will use it to prove that if the Fourier transform of a wavelet ψ vanishes on H_α (or $\tilde{H}_\beta$), then ψ is an MRA wavelet and $\operatorname{supp} \hat{\psi} \subset S_\alpha$ (or $\operatorname{supp} \hat{\psi} \subset \tilde{S}_\beta$).

Proposition 3.3.2. *If ψ is a wavelet such that $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in [-a, b]$, where $a, b > 0$ and $a + b + \min(a, b) \geq 2\pi$, then ψ is an MRA wavelet and* $\operatorname{supp} \hat{\psi} \subset [-4\pi + 2b, 4\pi - 2a]$.

Proof. We will show that the dimension function of a wavelet ψ satisfying the above assumption is positive almost everywhere, which will imply immediately that ψ must be an MRA wavelet. Then we will use the information that $\hat{\psi}$ vanishes on $[-a, b]$ to prove that its scaling function must be supported in $[-2\pi + b, 2\pi - a]$; therefore, the inclusion $\operatorname{supp} \hat{\psi} \subset [-4\pi + 2b, 4\pi - 2a]$ will follow.

Clearly our assumption $a + b + \min(a, b) \geq 2\pi$ is equivalent to saying that $2a + b \geq 2\pi$ and $a + 2b \geq 2\pi$. Because of the last inequality and since D_ψ is 2π–periodic, it is enough to show that $D_\psi(\xi) > 0$ for a.e. $\xi \in [-a, 2b]$ in order to obtain that D_ψ

is positive almost everywhere.

For a.e. $\xi \in [-a, b]$ we have

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \geq \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2 = \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1 > 0.$$

Therefore, it remains to prove that $D_\psi(\xi) > 0$ for a.e. $\xi \in [b, 2b]$. Let $E = \{\xi \in [b, 2b] \colon |\hat{\psi}(\xi)| = 1\}$. For a.e. $\xi \in [b, 2b] \setminus E$ we have

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \geq \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2$$

$$= \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 - |\hat{\psi}(\xi)|^2 = 1 - |\hat{\psi}(\xi)|^2 > 0.$$

Thus, we can concentrate on showing that $D_\psi(\xi) > 0$ for a.e. $\xi \in E$ or, equivalently, that $D_\psi(\xi) > 0$ for a.e. $\xi \in E - 2\pi$. This is the point where we will have to use the other inequality, that is, $2a + b \geq 2\pi$. Indeed, this inequality implies that for $\xi \in E - 2\pi$ we have $2^j \xi \in [-a, 0]$, where j is a negative integer, thus for a.e. $\xi \in E - 2\pi$ we obtain

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \geq \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2$$

$$= \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 - |\hat{\psi}(\xi)|^2 = 1 - |\hat{\psi}(\xi)|^2.$$

It is clear, however, that $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in E - 2\pi$ (simply because for those ξ's we have $|\hat{\psi}(\xi + 2\pi)| = 1$), therefore we obtain that $D_\psi(\xi) = 1 > 0$ for a.e. $\xi \in E - 2\pi$, which proves that $D_\psi(\xi) = 1$ almost everywhere; that is, ψ must be an MRA wavelet.

The first part of the proof allows us to conclude that there exists a scaling function φ associated with ψ. For a.e. $\xi \in [-a, b]$ we have

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2 = \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1.$$

Thus, if m is the low–pass filter of φ, we obtain $|m(\xi)| = 1$ for a.e. $\xi \in [-\frac{a}{2}, \frac{b}{2}]$. This implies that $m(\xi) = 0$ for a.e. $\xi \in [-\pi - \frac{a}{2}, -\pi + \frac{b}{2}] \cup [\pi - \frac{a}{2}, \pi + \frac{b}{2}]$ or, equivalently, $m(\frac{\xi}{2}) = 0$ for a.e. $\xi \in [-2\pi - a, -2\pi + b] \cup [2\pi - a, 2\pi + b]$. Since $\hat{\varphi}(\xi) = m(\frac{\xi}{2})\hat{\varphi}(\frac{\xi}{2})$, we obtain $\hat{\varphi}(\xi) = 0$ for a.e. $\xi \in [-2\pi - a, -2\pi + b] \cup [2\pi - a, 2\pi + b]$. Note that, because of (S2), this allows us to conclude that $a, b < 2\pi$. From our assumption $a + b + \min(a, b) \geq 2\pi$ it follows that $2(2\pi - a) \leq 2\pi + b$ and $2(-2\pi + b) \geq -2\pi - a$. Therefore, we have $\hat{\varphi}(\xi) = 0$ for a.e. $\xi \in [2(-2\pi + b), -2\pi + b] \cup [2\pi - a, 2(2\pi - a)]$. Since $2\pi - a, -2\pi + b > 0$, the scaling equation $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ implies that $\operatorname{supp} \hat{\varphi} \subset [-2\pi + b, 2\pi - a]$. That $\operatorname{supp} \hat{\psi} \subset [-4\pi + 2b, 4\pi - 2a]$ follows immediately from $|\hat{\psi}(2\xi)| = |m(\xi + \pi)\hat{\varphi}(\xi)|$. ♣

As we mentioned in the proof, the condition $a + b + \min(a, b) \geq 2\pi$ is equivalent to $2a + b \geq 2\pi$, $a + 2b \geq 2\pi$. It is also clear that if $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in [-a, b]$, then $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in [-c, d]$, where $c \leq a$, $d \leq b$. Therefore, we can always assume that $2a + b = 2\pi$ and $a + 2b \geq 2\pi$, or that $a + 2b = 2\pi$ and $2a + b \geq 2\pi$. If we consider the first case and put $a = \frac{2}{3}\alpha$, $\alpha > 0$, then $2a + b = 2\pi$ will give us $b = 2\pi - \frac{4}{3}\alpha$ and $a + 2b \geq 2\pi$ will give us $\alpha \leq \pi$. Therefore, by Proposition 3.3.2 if we assume that $\hat{\psi}(\xi) = 0$ for a.e. $\xi \in [-a, b] = [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha] = H_\alpha$ we will obtain that $\operatorname{supp} \hat{\psi} \subset [-4\pi + 2b, 4\pi - 2a] = [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha] = S_\alpha$. The other case gives us a similar result for $\tilde{H}_\beta$ and $\tilde{S}_\beta$. Combining this with results of [HWW1] gives us the following

Proposition 3.3.3. *If ψ is a wavelet and $0 < \alpha \leq \pi$ then the following are equivalent* *(i)* $\operatorname{supp}\hat{\psi} \subset S_\alpha = [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$.

(ii) $\hat{\psi}(\xi) = 0$ *for a.e.* $\xi \in H_\alpha = [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha]$.

(iii) ψ *is an MRA wavelet such that* $\operatorname{supp}\hat{\psi} \subset S_\alpha$, $\hat{\psi}(\xi) = 0$ *for a.e.* $\xi \in H_\alpha$ *and* ψ *can be constructed by the procedure described in (i)–(v) of Theorem 3.3.1.*

And a similar result holds for $\tilde{S}_\beta$ and $\tilde{H}_\beta$.

This concludes our contribution concerning the original construction of E. Hernandez, X. Wang and G. Weiss given in [HWW1]. Let us, therefore, consider following question: "Is it true that every wavelet with smooth and compactly supported Fourier transform can be viewed as a smooth version of an MSF wavelet?" First of all let us to point out that if ψ is such a wavelet, that is $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$, then by P. Auscher's argument ψ must be an MRA wavelet (if $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$, then D_ψ is continuous, therefore equal to 1). Recall that we already proved in Proposition 3.2.2 that for every MRA wavelet ψ we can find an MRA wavelet set contained in $\operatorname{supp}\hat{\psi}$. What we need, however, is something more: that is, we want to prove that if $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$, then there exists a wavelet set W and $c > 0$ such that

$$|\hat{\psi}(\xi)| \geq c \quad \text{for a.e. } \xi \in W \tag{3.3.1}$$

(note that the "almost everywhere" convention is required because we use it to define wavelet sets, nevertheless, if W is a finite sum of closed intervals, then $|\hat{\psi}(\xi)| \geq c$ for a.e. $\xi \in W$ implies that $|\hat{\psi}(\xi)| \geq c$ for every $\xi \in W$, since the closure of $W \setminus E$ is equal to W for every null set E).

We will employ methods similar to those of Section 3.1 and Section 3.2 to show that for every band–limited MRA wavelet ψ, we can find an MRA wavelet set on which $\hat{\psi}$ is bounded below.

Proposition 3.3.4. *If* φ *is a scaling function and* $\operatorname{supp}\hat{\varphi} \subset [-2n\pi, 2n\pi]$, $n \in \mathbb{N}$, *then there exists a scaling set* S *such that* $|\hat{\varphi}(\xi)| \geq \frac{1}{\sqrt{2n}}$ *for a.e.* $\xi \in S$.

PROOF. Let $S' = \{\xi \in \mathbb{R} \colon |\hat{\varphi}(\xi)| \geq \frac{1}{\sqrt{2n}}\}$. We want to apply Lemma 3.1.6 to conclude that there is a scaling set S contained in S'. To do so, we need to check that S' satisfies (SS1), (SS2) and (3.1.2). Since $|\hat{\varphi}(\xi)| \leq 1$, condition (S1) for $\hat{\varphi}$ implies that (SS1) holds for S'. Similarly (SS2) for S' follows from (S2) for $\hat{\varphi}$. We will use the fact that $\hat{\varphi}$ satisfies (S3) to prove that (3.1.2) holds for S'.

Since $\sum_{k\in\mathbb{Z}} \mathbf{1}_{S'}(\xi + 2k\pi)$ is a 2π–periodic and integer–valued function, it is enough to prove that the measure of

$$A = \{\xi \in [0, 2\pi] \colon \sum_{k\in\mathbb{Z}} \mathbf{1}_{S'}(\xi + 2k\pi) = 0\}$$

is equal to zero. If $\xi \in A$, then $\xi + 2k\pi \notin S'$ for $k \in \mathbb{Z}$, therefore $|\hat{\varphi}(\xi + 2k\pi)| < \frac{1}{\sqrt{2n}}$ for $\xi \in A$ and $k \in \mathbb{Z}$. Since $\operatorname{supp}\hat{\varphi} \subset [-2n\pi, 2n\pi]$ we obtain that for $\xi \in A$

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = \sum_{k=-n}^{n-1} |\hat{\varphi}(\xi + 2k\pi)|^2 < 2n\frac{1}{2n} = 1.$$

Thus (S3) for $\hat{\varphi}$ implies that A has measure zero, which ends the proof. ♣

If ψ is a band–limited MRA wavelet with a scaling function φ and W is an MRA wavelet set such that $|\hat{\psi}(\xi)| \geq c$ for a.e. $\xi \in W$, where $c > 0$, then the formula $|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2$ implies that φ is band–limited as well and $|\hat{\varphi}(\xi)| \geq c$ for a.e. $\xi \in S = \bigcup_{j=1}^{\infty} 2^{-j}W$, which, as we recall, is a scaling set. The following proposition shows that the converse holds as well.

PROPOSITION 3.3.5. *If ψ is an MRA wavelet with a scaling function φ and S is a scaling set such that $|\hat{\varphi}(\xi)| \geq c > 0$ for a.e. $\xi \in S$, then $|\hat{\psi}(\xi)| \geq c^2$ for a.e. $\xi \in W = 2S \setminus S$.*

PROOF. We have $|\hat{\psi}(\xi)| = |m(\frac{\xi}{2} + \pi)\hat{\varphi}(\frac{\xi}{2})|$, where m is the low–pass filter of φ. We shall show that $|\hat{\psi}(\xi)| \geq c^2$ for a.e. $\xi \in W = 2S \setminus S$. Clearly, it is enough to show that $|m(\frac{\xi}{2} + \pi)| \geq c$ and $|\hat{\varphi}(\frac{\xi}{2})| \geq c$ for a.e. $\xi \in W$. The latter is clearly satisfied because $\xi \in W$ implies that $\frac{\xi}{2} \in S$. To prove the former we first observe that the inequality $|\hat{\varphi}(\xi)| \leq 1$ which holds a.e. and the scaling equation allow us to concludes that $|m(\xi)| \geq c$ for a.e. $\xi \in \frac{S}{2}$. If $\xi \in W$, then we have $\frac{\xi}{2} + \pi \in (S \setminus \frac{S}{2}) + \pi$. From (SS1) and (SS3) it follows that the set $(S \setminus \frac{S}{2}) + \pi$ is 2π–translation congruent to $\frac{S}{2}$. Thus, we obtain $|m(\frac{\xi}{2} + \pi)| \geq c$, since m is 2π–periodic. ♣

PROPOSITION 3.3.6. *If ψ is an MRA wavelet and* $\operatorname{supp} \hat{\psi} \subset [-4n\pi, 4n\pi]$, $n \in \mathbb{N}$, *then there exists an MRA wavelet set W such that $|\hat{\psi}(\xi)| \geq \frac{1}{2n}$ for a.e. $\xi \in W$.*

PROOF. Let φ be a scaling function associated with ψ. Since $|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2$, we have that $\operatorname{supp} \hat{\varphi} \subset [-2n\pi, 2n\pi]$. By Proposition 3.3.4 there exists a scaling set S such that $|\hat{\varphi}(\xi)| \geq \frac{1}{\sqrt{2n}}$ for a.e. $\xi \in S$. Therefore, by Proposition 3.3.5 we obtain that $|\hat{\psi}(\xi)| \geq \frac{1}{2n}$ for a.e. $\xi \in W = 2S \setminus S$. ♣

From the above proposition it follows that every wavelet whose Fourier transform is smooth and compactly supported is, in some sense, a smoothed version of an MSF–MRA wavelet. It is natural to ask if the converse is true; that is, "Is it true, that for every MRA wavelet set W there is a wavelet ψ and $c > 0$ such that $\hat{\psi} \in \mathcal{C}_c^{\infty}(\mathbb{R})$ and $|\hat{\psi}(\xi)| \geq c$ for a.e. $\xi \in W$?" The answer to this question is negative and to prove it we will present a counterexample.

In [HWW2] an interesting example of an MRA wavelet set was considered: $W = [-\frac{8}{3}\pi, -2\pi] \cup [-\frac{4}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{4}{3}\pi] \cup [2\pi, \frac{8}{3}\pi]$. The authors of [HWW2] showed that their smoothing procedures applied to this wavelet set do not produce band–limited wavelets. We will use a similar reasoning to prove the following

PROPOSITION 3.3.7. *If ψ is an MRA wavelet such that $|\hat{\psi}|$ is continuous and $|\hat{\psi}(\xi)| \geq c$ for a.e. $\xi \in W = [-\frac{8}{3}\pi, -2\pi] \cup [-\frac{4}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{4}{3}\pi] \cup [2\pi, \frac{8}{3}\pi]$, where $c > 0$, then ψ is not band–limited.*

PROOF. Let us suppose that ψ is band–limited. If φ is a scaling function associated with ψ then from the formula $|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2$ we deduce that φ is also band–limited and that $|\hat{\varphi}|$ is continuous on $\mathbb{R}\setminus\{0\}$ (the sum is finite on every closed interval which does not contain the origin). If we put 1 as the value of $|\hat{\varphi}|$ at the origin, then condition (S2) for $\hat{\varphi}$ shall guarantee that $|\hat{\varphi}|$ will be continuous on $\mathbb{R}$, thus, we can assume that this is the case. The scaling set corresponding to W is $S=[-\frac{4}{3}\pi,-\pi]\cup[-\frac{2}{3}\pi,\frac{2}{3}\pi]\cup[\pi,\frac{4}{3}\pi]$. Clearly $|\hat{\varphi}(\xi)|\geq c$ for a.e. $\xi\in S$. Moreover, since $|\hat{\varphi}|$ is continuous and S is a finite sum of closed intervals we obtain that $|\hat{\varphi}(\xi)|\geq c$ for every $\xi\in S$. Therefore, if m is the low–pass filter of φ, then there is an $\epsilon>0$ such that $|m(\xi)|\geq\frac{c}{2}$ for a.e. $\xi\in(-\frac{2}{3}\pi-\epsilon,-\frac{2}{3}\pi+\epsilon)\cup(\frac{2}{3}\pi-\epsilon,\frac{2}{3}\pi+\epsilon)$. It is also clear that we can choose even smaller ϵ to assure that $|\hat{\varphi}(\xi)|\geq\frac{c}{2}$ for $\xi\in(\frac{2}{3}\pi-\epsilon,\frac{2}{3}\pi+\epsilon)$. Thus, for every $n\in\mathbb{N}$ and a.e. $\xi\in(\frac{2}{3}\pi-2^{-n}\epsilon,\frac{2}{3}\pi+2^{-n}\epsilon)$ we have

$$|\hat{\varphi}(2^n\xi)|=|\hat{\varphi}(\xi)|\prod_{j=1}^{n}|m(2^j\xi)|\geq\left(\frac{c}{2}\right)^{n+1},$$

where the last inequality follows from the fact that $2^j\frac{2}{3}\pi\equiv\pm\frac{2}{3}\pi$ modulo 2π. Since $|\hat{\varphi}|$ is continuous we obtain that $\hat{\varphi}(2^n\frac{2}{3}\pi)\neq 0$ for every $n\in\mathbb{N}$, thus φ is not band–limited, which gives us a contradiction. ♣

We showed that some MSF-MRA wavelets can not be smoothed, that is, for some MRA wavelet sets we can not find a band–limited smooth wavelet ψ such that (3.3.1) holds. It is of interest to find a class of MRA wavelet sets for which it is possible. To do so we will develop a different smoothing procedure from the one in [HWW2]. The procedure presented therein was based on smoothing low–pass filters. Our approach, which uses main ideas of Algorithm 3.1.3, is based on producing scaling functions whose Fourier transform belong to $\mathcal{C}_c^\infty(\mathbb{R})$. We have to prove, therefore, that for each such scaling function there exists an associated wavelet ψ such that $\hat{\psi}\in\mathcal{C}_c^\infty(\mathbb{R})$.

LEMMA 3.3.8. *If φ is a scaling function such that $\hat{\varphi}\in\mathcal{C}_c^\infty(\mathbb{R})$, then its low–pass filter belongs to the class $\mathcal{C}^\infty(\mathbb{R})$.*

PROOF. Recall, that the low–pass filter m of φ is given by

$$m(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k e^{ik\xi},$$

where

$$\alpha_k = \frac{1}{2} \int \varphi\left(\frac{x}{2}\right) \overline{\varphi\left(x + k\right)}\, dx.$$

We show that m is smooth by proving the estimate

$$|k^n \alpha_k| \leq C_n$$

for every $n \in \mathbb{N}$. By the Plancherel theorem we have

$$|k^n \alpha_k| = \left| \frac{1}{2\pi} \int \hat{\varphi}(2\xi)\overline{\hat{\varphi}(\xi)} k^n e^{-ik\xi}\, d\xi \right| = \left| \frac{1}{2\pi} \int \hat{\varphi}(2\xi)\overline{\hat{\varphi}(\xi)} \left(\frac{d}{d\xi}\right)^n \left(e^{-ik\xi}\right)\, d\xi \right|$$

$$= \left| \frac{1}{2\pi} \int \left(\frac{d}{d\xi}\right)^n \left(\hat{\varphi}(2\xi)\overline{\hat{\varphi}(\xi)}\right) e^{-ik\xi}\, d\xi \right| \leq \frac{1}{2\pi} \int \left| \left(\frac{d}{d\xi}\right)^n \left(\hat{\varphi}(2\xi)\overline{\hat{\varphi}(\xi)}\right) \right| d\xi = C_n.$$

Since $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$ we obtain that C_n is finite for every $n \in \mathbb{N}$. This ends the proof. ♣

From the above lemma it follows that if we consider a wavelet ψ defined by the usual formula $\hat{\psi}(2\xi) = e^{i\xi}\overline{m(\xi + \pi)}\hat{\varphi}(\xi)$, then $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$ (note that it is not clear if the condition $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$ implies the existence of a scaling function φ associated with ψ such that $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$). We can, therefore, concentrate on finding scaling functions φ, such that $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$. One of the ways to achieve this is by smoothing scaling functions of MSF wavelets. Proposition 3.3.7 indicates that it is not true that for every scaling set S we can find a scaling function φ such that $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$ and $|\hat{\varphi}(\xi)| \geq c > 0$ for a.e. $\xi \in S$ (consider, for example, $S = [-\frac{4}{3}\pi, -\pi] \cup [-\frac{2}{3}\pi, \frac{2}{3}\pi] \cup [\pi, \frac{4}{3}\pi]$). Therefore, our first goal is to find out which scaling sets are appropriate for a smoothing procedure.

Let us consider the Shannon wavelet set $W = [-2\pi, \pi] \cup [\pi, 2\pi]$ with the corresponding scaling set $S = [-\pi, \pi]$. This scaling set has the following interesting property. If we enlarge S a little bit, that is, if we take $S_\epsilon = (-\pi - \epsilon, \pi + \epsilon)$, then S_ϵ is still contained in $2S$ for small ϵ ($\epsilon \leq \pi$). Since every scaling set has to satisfy (SS1) we can expect that this can be true for some larger class of scaling sets. Clearly, the scaling set $[-\frac{4}{3}\pi, -\pi] \cup [-\frac{2}{3}\pi, \frac{2}{3}\pi] \cup [\pi, \frac{4}{3}\pi]$ from Proposition 3.3.7 does not have this property; however, we can slightly perturb this set to obtain $[-\frac{4}{3}\pi + a, -\pi] \cup [-\frac{2}{3}\pi - a, \frac{2}{3}\pi + a] \cup [\pi, \frac{4}{3}\pi - a]$ for $a \leq \frac{1}{3}\pi$, which is a family of scaling sets enjoying the above property. The importance of this property in connection to the problem of smoothing MSF wavelets was discovered by M. Bownik. We will use the Shannon scaling set to explain what is the role of this property in the process of smoothing the Shannon wavelet.

Let $S = [-\pi, \pi]$ and $S_\epsilon = (-\pi - \epsilon, \pi + \epsilon)$, where $0 < \epsilon \leq \pi$. Let f be a smooth function such that $f(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $f(\xi) = 0$ for $\xi \notin S_\epsilon$. Define

$$m_0(\xi) = \begin{cases} \frac{f(2\xi)}{f(\xi)} & \text{for } \xi \in \frac{S_\epsilon}{2} \\ 0 & \text{for } \xi \in S_\epsilon \setminus \frac{S_\epsilon}{2}. \end{cases} \tag{3.3.2}$$

We claim that if we choose ϵ small enough, then m_0 extends to a 2π–periodic function on $\mathbb{R}$ (note that we do not need $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$). Since m_0 is non trivial on $\frac{S_\epsilon}{2}$ we have to check that $\frac{S_\epsilon}{2} \cap (\frac{S_\epsilon}{2} + 2k\pi) = \emptyset$ for integer $k \neq 0$. This is the first place where we use the property $S_\epsilon \subset 2S$. Clearly, since $\frac{S_\epsilon}{2} \subset S$ and S satisfies (SS3), the intersection is empty. It remains to check that $\frac{S_\epsilon}{2} \cap ((S_\epsilon \setminus \frac{S_\epsilon}{2}) + 2k\pi) = \emptyset$ for $k \in \mathbb{Z}$. If $k = 0$ then it is obviously true. For $k \neq 0$ we can prove something more: that $\frac{S_\epsilon}{2} \cap (S_\epsilon + 2k\pi) = \emptyset$ if we choose ϵ small enough. This is the second place where we use the property $S_\epsilon \subset 2S$. Since $\frac{S_\epsilon}{2} \subset S$ we can choose ϵ small enough to assure that the distance of $\frac{S_\epsilon}{2}$ to the border of S is greater then ϵ. Thus, since $S \cap (S + 2k\pi) = \emptyset$ for $k \neq 0$, we will obtain that $\frac{S_\epsilon}{2} \cap (S_\epsilon + 2k\pi) = \emptyset$ for such ϵ. In our case the border of S is $\partial S = \{-\pi, \pi\}$, and the distance of $\frac{S_\epsilon}{2}$ to the border of S is equal to $\pi - (\frac{1}{2}\pi + \frac{\epsilon}{2})$. Therefore, we see that $\pi - (\frac{1}{2}\pi + \frac{\epsilon}{2}) \geq \epsilon$ for $\epsilon \leq \frac{1}{3}\pi$; that is, $S \cap (S + 2k\pi) = \emptyset$ for $k \neq 0$ and $\epsilon \leq \frac{1}{3}\pi$. This allows us to conclude that we can extend m_0 2π–periodically to $\mathbb{R}$, provided $\epsilon \leq \frac{1}{3}\pi$.

It is easy to see that f and the extended m_0 satisfy the scaling equation

$$f(2\xi) = m_0(\xi) f(\xi) \quad \text{for } \xi \in \mathbb{R}. \tag{3.3.3}$$

Indeed, if $\xi \notin S_\epsilon$ then $f(\xi) = 0$, moreover, since $S_\epsilon \subset 2S \subset 2S_\epsilon$, we obtain that $2\xi \notin S_\epsilon$; thus, $f(2\xi) = 0$, and both sides of (3.3.3) are equal to zero (note that we have used the property $S_\epsilon \subset 2S$ again). If $\xi \in S_\epsilon$, then (3.3.3) follows from (3.3.2).

Let $\sigma_f(\xi) = \sum_{k\in\mathbb{Z}} |f(\xi + 2k\pi)|^2$. Since f is a smooth function such that $f(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $f(\xi) = 0$ for $\xi \notin S_\epsilon$, we obtain, that σ_f is smooth and bounded above and below. Therefore, $\sqrt{\sigma_f}$ is also smooth and bounded above and below, which allows us to define $g(\xi) = \frac{f(\xi)}{\sqrt{\sigma_f(\xi)}}$ and conclude that g is smooth, moreover, it is clear that, $g(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $g(\xi) = 0$ for $\xi \notin S_\epsilon$.

We claim that g satisfies (S1)–(S3). It is easy to check that $g(2\xi) = m(\xi)g(\xi)$ a.e., where $m(\xi) = \frac{m_0(\xi)\sqrt{\sigma_f(\xi)}}{\sqrt{\sigma_f(2\xi)}}$ (see proof of Alg4), which establishes (S1). Since condition (S3) is clearly satisfied we can use Theorem 1.7 of §2.1 in [HW] to conclude that $g(0) \neq 0$ implies that $|g(0)| = 1$, which gives us (S2). There is also another way to prove that (S2) holds. We observed already that $\frac{S_\epsilon}{2} \cap (S_\epsilon + 2k\pi) = \emptyset$ for $k \neq 0$. Equivalently, $(\frac{S_\epsilon}{2} + 2k\pi) \cap S_\epsilon = \emptyset$ for $k \neq 0$. Since $0 \in \frac{S_\epsilon}{2}$, we have $2k\pi \notin S_\epsilon$ for $k \neq 0$, thus $f(2k\pi) = 0$ for such k. This gives us that $\sigma_f(0) = |f(0)|^2$, which proves that $|g(0)| = 1$.

Let φ be a scaling function defined by $\hat{\varphi}(\xi) = g(\xi)$. Since $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$, from Lemma 3.3.8 it follows that the low–pass filter of $\hat{\varphi}$ (denoted by m) is smooth. Therefore, $\hat{\psi}(\xi) = e^{i\xi/2}\overline{m(\xi + \pi/2)}\hat{\varphi}(\xi/2)$ defines an orthonormal wavelet, such that $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$. We also have that $\hat{\varphi}(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $\hat{\varphi}(\xi) = 0$ for $\xi \notin S_\epsilon$; in particular there exists a positive c, such that $|\hat{\varphi}(\xi)| \geq c$ for $\xi \in S$. By Proposition 3.3.5 we obtain that $|\hat{\psi}(\xi)| \geq c^2$ for $\xi \in W = 2S \setminus S$, which proves that $\hat{\varphi}$ is a smoothed version of an MSF wavelet (in our case the Shannon wavelet). It is also clear, that $\operatorname{supp}\hat{\psi} \subset 2S_\epsilon$, moreover, since $m(\frac{\xi}{2}+\pi) = 0$ for $\xi \in S_{-\epsilon} = [-\pi+\epsilon, \pi-\epsilon]$, we obtain that $\operatorname{supp}\hat{\psi} \subset 2S_\epsilon \setminus S_{-\epsilon} \subset W_{2\epsilon} = (-2\pi - 2\epsilon, -\pi + 2\epsilon) \cup (\pi - 2\epsilon, 2\pi + 2\epsilon)$.

The above example gives yet another method of smoothing the Shannon wavelet. The method may seem to be more complicated then the original construction of Y. Meyer as well as those included in [HWW1] and [HWW2]. The nice feature, however, of the procedure just presented is that it can be applied to a large class of scaling sets yielding band–limited wavelets with smooth Fourier transform.

Let us introduce necessary notation to state this result in larger generality. For a set $E \subset \mathbb{R}$ we define $E_\epsilon = \{\xi \in \mathbb{R}\colon \operatorname{dist}(\xi, E) < \epsilon\}$ and $E_{-\epsilon} = ((E^c)_\epsilon)^c$. The border of E is denoted by ∂E. We have the following:

THEOREM 3.3.9. *Assume that a scaling set S is a a finite sum of closed intervals and $S_\delta \subset 2S$ for some $\delta > 0$. Then for every positive $\epsilon \leq \frac{\delta}{3}$ there exists a scaling function φ such that $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$, $\hat{\varphi}(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $\hat{\varphi}(\xi) = 0$ for $\xi \notin S_\epsilon$. Moreover, there exists an MRA wavelet associated with φ such that $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$, $\hat{\psi}(\xi) \neq 0$ for $\xi \in W = 2S \setminus S$ and $\hat{\psi}(\xi) = 0$ for $\xi \notin W_{2\epsilon}$.*

PROOF. Let us recall some of the properties of S_ϵ we proved above.

$$\frac{S_\epsilon}{2} \cap \left(\frac{S_\epsilon}{2} + 2k\pi\right) = \emptyset \quad \text{for } k \in \mathbb{Z} \setminus \{0\}. \tag{3.3.4}$$

Since $\epsilon \leq \frac{\delta}{3}$ we have that $\operatorname{dist}(\frac{S_\epsilon}{2}, \partial S) \geq \epsilon$, thus

$$\frac{S_\epsilon}{2} \cap \left(\left(S_\epsilon \setminus \frac{S_\epsilon}{2}\right) + 2k\pi\right) = \emptyset \quad \text{for } k \in \mathbb{Z}. \tag{3.3.5}$$

We highlight the crucial points of the construction of φ omitting arguments already given during the presentation of the smoothing procedure of the Shannon wavelet.

Step 1. We start with a function $f \in \mathcal{C}^{\infty}(\mathbb{R})$ such that $f(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $f(\xi) = 0$ for $\xi \notin S_\epsilon$. Then we define

$$m_0(\xi) = \begin{cases} \frac{f(2\xi)}{f(\xi)} & \text{for } \xi \in \frac{S_\epsilon}{2} \\ 0 & \text{for } \xi \in S_\epsilon \setminus \frac{S_\epsilon}{2}. \end{cases} \tag{3.3.6}$$

From (3.3.4) and (3.3.5) it follows that m_0 extends uniquely to a 2π–periodic function on $\mathbb{R}$, which we call m_0 as well. Moreover, the scaling equation

$$f(2\xi) = m_0(\xi) f(\xi)$$

holds for $\xi \in \mathbb{R}$.

Step 2. The function

$$\sigma_f(\xi) = \sum_{k \in \mathbb{Z}} |f(\xi + 2k\pi)|^2$$

is smooth and bounded above and below. Define

$$g(\xi) = \frac{f(\xi)}{\sqrt{\sigma_f(\xi)}},$$

which belongs to the class $\mathcal{C}_c^{\infty}(\mathbb{R})$ and satisfies $g(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $g(\xi) = 0$ for $\xi \notin S_\epsilon$.

Step 3. Conditions (S1)–(S3) hold for g, therefore, $\hat{\varphi}(\xi) = g(\xi)$ defines a scaling function. Moreover, $\hat{\varphi} \in \mathcal{C}_c^{\infty}(\mathbb{R})$, $\hat{\varphi}(\xi) \neq 0$ for $\xi \in S_\epsilon$ and $\hat{\varphi}(\xi) = 0$ for $\xi \notin S_\epsilon$. Furthermore, the low–pass filter m of φ is smooth.

This provides us with an appropriate scaling function. We still need to prove that a wavelet ψ given by

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} \overline{m\left(\frac{\xi}{2} + \pi\right)} \hat{\varphi}\left(\frac{\xi}{2}\right)$$

satisfies the desired properties. Clearly, $\hat{\psi} \in \mathcal{C}_c^{\infty}(\mathbb{R})$. By Proposition 3.3.5 we have $\hat{\psi}(\xi) \neq 0$ for $\xi \in W$. We shall show that $\hat{\psi}(\xi) = 0$ for $\xi \notin W_{2\epsilon}$. Since

$W = 2S \setminus S$ we have $2S_\epsilon \setminus S_{-\epsilon} \subset W_{2\epsilon}$. If $\xi \notin W_{2\epsilon}$, then $\xi \notin 2S_\epsilon \setminus S_{-\epsilon}$. Let us consider two cases, $\xi \notin 2S_\epsilon$ and $\xi \in S_{-\epsilon}$. If the former holds, then $\hat{\varphi}(\frac{\xi}{2}) = 0$, thus $\hat{\psi}(\xi) = 0$. We will prove that the latter case implies that $m(\frac{\xi}{2} + \pi) = 0$. Indeed, since S satisfies (SS3) we have $S_{-\epsilon} \cap (S_\epsilon + 2k\pi) = \emptyset$ for $k \in \mathbb{Z} \setminus \{0\}$. In particular, $(S_{-\epsilon} + 2\pi) \cap (S_\epsilon + 4k\pi) = \emptyset$ for $k \in \mathbb{Z}$ or, equivalently, $(\frac{S_{-\epsilon}}{2} + \pi) \cap (\frac{S_\epsilon}{2} + 2k\pi) = \emptyset$ for $k \in \mathbb{Z}$. Therefore, if $\xi \in S_{-\epsilon}$, then $\frac{\xi}{2} + \pi + 2k\pi \notin \frac{S_\epsilon}{2}$ for $k \in \mathbb{Z}$. Since m vanishes precisely on the same set as m_0, from (3.3.6) it follows that $m(\frac{\xi}{2} + \pi) = 0$, thus $\hat{\psi}(\xi)$ is equal to zero as well. ♣

If ψ is an MRA–MSF wavelet such that $|\hat{\psi}(\xi)| = \mathbf{1}_W(\xi)$ and the scaling set of W satisfies the assumptions of Theorem 3.3.9, then we can smooth such a wavelet using the procedure presented above. As we mentioned before, all band–limited wavelets with smooth Fourier transform constructed in [HWW1] and [HWW2] have band within the interval $[-4\pi, 4\pi]$. Our construction can be used to obtain wavelets with a larger band. As an example consider an MRA wavelet set

$$W = \left[-\frac{1}{2}\pi, -\frac{1}{4}\pi\right] \cup \left[\pi, \frac{5}{4}\pi\right] \cup \left[\frac{7}{4}\pi, 2\pi\right] \cup \left[\frac{5}{2}\pi, 3\pi\right] \cup \left[\frac{13}{4}\pi, \frac{7}{2}\pi\right] \cup \left[6\pi, \frac{13}{2}\pi\right].$$

Its scaling set consists of three intervals

$$S = \left[-\frac{1}{4}\pi, \pi\right] \cup \left[\frac{5}{4}\pi, \frac{7}{4}\pi\right] \cup \left[3\pi, \frac{13}{4}\pi\right]$$

and it is easy to check that $S_\delta \subset 2S$ for $\delta = \frac{\pi}{4}$. From Theorem 3.3.9 it follows that for every $\epsilon \leq \frac{\pi}{12}$ there exists an MRA wavelet ψ whose Fourier transform is smooth, and such that $\hat{\psi}(\xi) \neq 0$ for $\xi \in W$ and $\hat{\psi}(\xi) = 0$ for $\xi \notin W_{2\epsilon}$. Therefore, we see that the band of ψ exceeds $\frac{13}{2}\pi$.

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