GENERALIZED LÉVY STOCHASTIC AREAS AND SELFDECOMPOSABILITY

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ABSTRACT. We show that a conditional characteristic function of generalized Lévy stochastic areas can be viewed as a product a selfdecomposable distribution (i.e., Lévy class L distribution) and its background driving characteristic function. This provides a stochastic interpretation for a ratio of some Bessel functions as well as examples of characteristic functions from van Dantzig class.

Key words and phrases: Bessel functions I_ν and J_ν; Lévy's stochastic areas; selfdecomposability property; s-selfdecomposability property; van Dantzig class $\mathcal{D}$.

1. An introduction. For a planar Brownian motion $\mathbf{B}_t = (Z_t, \tilde{Z}_t)$ and a stochastic area process

$$\mathcal{A}_u = \int_0^u Z_s d\tilde{Z}_s - \tilde{Z}_s dZ_s, \qquad u > 0,$$

Paul Lévy (1950) proved that its conditional characteristic function is of the form

$$\mathbb{E}[e^{it\mathcal{A}_u}|\mathbf{B}_u = a] = \frac{tu}{\sinh tu}\exp[-\frac{|a|^2}{2u}(tu\coth tu - 1)], \quad t \in \mathbb{R},$$

where $a \in \mathbb{R}^2$ and $u \geq 0$ are fixed. Thus, in particular,

$$\mathbb{E}[e^{it\mathcal{A}_u}|\mathbf{B}_u = (\sqrt{u}, \sqrt{u})] = \frac{tu}{\sinh tu}\exp[-(tu\coth tu - 1)], \quad t \in \mathbb{R}. \qquad (1)$$

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For another proof of Lévy stochastic area formula and related topics involving more stochastic argument we refer to Williams (1976) and Yor (1980).

In Jurek (2001), p.248, it was showed that in the formula (1) one has a product of the *class L characteristic function* $bt/\sinh bt$, ($b \in \mathbb{R}$ is a fixed parameter), and a characteristic function $\exp[-(bt\coth bt - 1)]$ of its *background driving random variable* that is from the class $\mathcal{U}$ of s-selfdecomposable distributions. Consequently, by Proposition 3, in Iksanov, Jurek and Schreiber (2003), we have that (1) is again in class L. In this note we prove that a similar result holds for a generalized Lévy stochastic areas; cf.Theorem 1. Here hyperbolic trigonometric functions are replaced by the modified Bessel function of first kind I_ν. As a by-product we get a formula for a ratio of such functions and provide examples of van Dantzig class $\mathcal{D}$; cf. Proposition 1, Corollaries 2-4 and the formula (12).

Bessel functions $I_\nu(t)$, as completely monotone functions in $\sqrt{t}$ or $\sqrt{\nu}$ or as functions of a pair (t, ν), appeared in many areas of probability and can be interpreted by some stochastic functionals; cf. Ismail and Kelker (1979), but in particular, cf. Pitman and Yor (1981) and references there-in. In this note we provide stochastic way of looking at $I_\nu(t)$, and their ratios, as function in $t \in \mathbb{R}$.

Our proofs here rely on different representations of Bessel functions and therefore for ease of reading, in the Appendix, we have collected all needed formulas. This also makes this paper more self-contained.

2. Selfdecomposable and s-selfdecomposable distributions.

Selfdecomposable random variables X (or their probability distribution or their characteristic functions ϕ) are defined as limits of normalized partial sums of infinitesimal independent random variables. Thus they are infinitely divisible, in short: $X \in ID$. Let L stand for the class of all either selfdecomposable random variables X or their probability distributions μ or their characteristic functions ϕ. Then

$$\phi \in L \quad \text{iff} \quad \forall (0 < c < 1) \exists (char.f.\ \psi_c) \;\; \phi(\cdot) = \phi(c\cdot)\psi_c(\cdot), \tag{2}$$

and this is so called *the selfdecomposability property.* One of the major characterizations of the class L, called *the random integral representation* is as follows

$$X \in L \quad \text{iff} \quad X \stackrel{d}{=} \int_{(0,\infty)} e^{-s} dY(s), \text{ with } E[\log(1 + |Y(1)|)] < \infty, \tag{3}$$

where $\stackrel{d}{=}$ means equality in distribution and $Y(\cdot)$ is a Lévy process called background driving Lévy process; in short: Y is BDLP of X or X is *driven* by Y; cf. Jurek and Mason (1993), Chapter 3. For BDLP identification purposes note that if ψ is the characteristic function of $Y(1)$ in (3) then

$$\psi(t) = \exp[t(\log \phi(t))'], \quad t \neq 0, \quad \text{and} \quad \psi(0) = 1. \tag{4}$$

In particular, class L characteristic functions are differentiable for $t \neq 0$ and ψ is infinitely divisible characteristic function with a finite logarithmic moment; cf. Jurek (2001), Proposition 3. Finally let us recall that $(L, *)$ is a closed topological subsemigroup of the semigroup ID.
Class L is quite rich and contains among others: all stable laws, gamma $\gamma_{\alpha,\lambda}$, Laplace (double exponential), t-Student, F-Fisher, log-normal, log-F, $\log|t|$, inverse Gaussian, Barndorff-Nielsen generalized hyperbolic, generalized gamma, etc.

Another class of infinitely divisible distributions needed here is the class $\mathcal{U}$ of *s-selfdecomposable probability measures.* One of the descriptions of $\mathcal{U}$ is as follows

$$\psi \in \mathcal{U} \ \text{ iff } \ \forall (0 < c < 1) \exists (\text{char.f. } \rho_c \in ID) \ \psi(t) = \psi^c(ct)\rho_c(t), \ t \in \mathbb{R}, \tag{5}$$

where the power is well defined as $\psi(t) \neq 0$. To the above we refer to as *the s-selfdecomposability probability.* From (5), in particular, we infer that $(\mathcal{U}, *)$ is also a closed convolution topological semigroup. In fact, we have the inclusions $L \subset \mathcal{U} \subset ID$. Similarly, for s-selfdecomposable distributions we have analogous random integral representation

$$X \in \mathcal{U} \ \text{ iff } \ X \stackrel{d}{=} \int_{(0,1)} s \, dY(s), \tag{6}$$

and as before we refer to Y as the BDLP for s-selfdecomposable X. From (6) we infer that if ϕ and ψ are characteristic functions of X and $Y(1)$, respectively, then

$$\psi(t) = \exp[(t \log \phi(t))'], \quad t \neq 0 \quad \text{and} \quad \psi(0) = 1.$$

For those and other relations between the classes L and $\mathcal{U}$ we refer to Jurek (1985) or Iksanov, Jurek and Schreiber (2003).

3. Stochastic areas and selfdecomposability. For $p > -1/2$, let us define a Gaussian process $\mathbf{V}^p$ and the corresponding generalized Lévy stochastic area as follows

$$\mathbf{V}_t^p = (V_t^p, \tilde{V}_t^p) := t^{-p} \int_0^t s^p \, d\mathbf{B}_s, \quad \text{and} \quad \mathcal{A}^p := \int_0^1 V_s^p \, d\tilde{V}_s^p - \tilde{V}_s^p \, dV_s^p,$$

where $\mathbf{B}_s$, $s \geq 0$, is the planar Brownian motion. In Biane and Yor (1987) (see also Yor (1989), p. 3052 or Duplantier (1989)) the following generalization of Lévy stochastic area formula was proved

$$\mathbb{E}[e^{i\lambda\mathcal{A}^p}|\mathbf{V}_1^p = a] = \frac{|\lambda|^\nu}{2^\nu \Gamma(\nu+1) I_\nu(|\lambda|)} \exp[-\frac{|a|^2}{2} |\lambda| \frac{I_{\nu+1}(|\lambda|)}{I_\nu(|\lambda|)}], \tag{7}$$

where $I_\nu(z)$ is the modified Bessel function of the index $\nu := p + 1/2 > 0$ and $a \in \mathbb{R}^2$ is fixed. In the 'products' in (7) we recognize a similar formula as in (1) and , moreover, the components have analogous interpretations. Namely we have the following result.

THEOREM 1. *(i) For $\nu > -1$, functions*

$$B_\nu(t) := \frac{t^\nu}{2^\nu \Gamma(\nu+1) \, I_\nu(t)}, \quad t \in \mathbb{R}, \quad \textit{are class L characteristic functions.}$$

More explicitly, these are the characteristic functions of random variables of the form

$$X_\nu := \sum_{k=1}^\infty z_{\nu,k}^{-1} \, \eta_k, \quad \textit{where} \quad \eta_k \quad \textit{are i. i. d. Laplace rv's,}$$

and $z_{\nu,k}$, $k = 1, 2, ...,$ are the zeros of the function $z^{-\nu} J_\nu(z)$. Here J_ν is the Bessel function of the first kind. Furthermore

$$\log B_\nu(t) = -\int_{\mathbb{R}\setminus\{0\}} (1 - \cos tx)(\sum_{k=1}^\infty e^{-|z_{\nu,k}||x|})/|x| dx.$$

(ii) Functions

$$b_\nu(t) := \exp\big[-t \, \frac{I_{\nu+1}(t)}{I_\nu(t)}\big], \quad t \in \mathbb{R}, \quad \textit{are class } \mathcal{U} \textit{ characteristic functions.}$$

Moreover, these are the background driving characteristic functions for B_ν, i.e., $\log b_\nu(t) = t\, d(\log B_\nu(t))/dt, \quad t \neq 0$. *Furthermore,*

$$(t/2)\, I_{\nu+1}(t) = I_\nu(t) \int_0^\infty (1-\cos tx)(\sum_{k=1}^\infty |z_{\nu,k}| e^{-|z_{\nu,k}|x})dx. \tag{8}$$

(iii) For all positive constants α and β, the functions $B_\nu^\alpha(t) \cdot b_\nu^\beta(t)$ are s-selfdecomposable (class $\mathcal{U}$ distributions). Moreover, for $0 < \beta \leq \alpha$ the resulting characteristic functions are selfdecomposable (class L characteristic functions).

(iv) For $\nu > -1/2$, functions $1/B_\nu(it) = \frac{\Gamma(\nu+1)J_\nu(t)}{(t/2)^\nu}, \quad t \in \mathbb{R}$, are characteristic functions corresponding to the probability densities

$$f_\nu(x) := \frac{\Gamma(\nu+1)}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\,(1-x^2)^{\nu-\frac{1}{2}} 1_{[-1,1]}(x).$$

These are not infinitely divisible characteristic functions.

Proof. From A.3(a) in the Appendix, we have

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_k^\infty \big(1+\frac{z^2}{z_{\nu,k}^2}\big), \quad z \in \mathbb{C}, \quad \nu > -1, \tag{9}$$

where $z_{\nu,k}$ are real zeros of the function $z^{-\nu}J_\nu(z)$, ordered accordingly to their absolute values; cf. also A.1(a) in the Appendix. Since Laplace (also called the double exponential) random variable η has characteristic function

$$\phi_\eta(t) = \int_{-\infty}^\infty e^{itx} 2^{-1} e^{-|x|}dx = (1+t^2)^{-1} = \exp \int_{\mathbb{R}\setminus\{0\}} (\cos\, tx - 1)\frac{e^{-|x|}}{|x|}dx,$$

we easily conclude that it is class L distribution. Furthermore,

$$B_\nu(t) = \prod_{k=1}^\infty \big(1+\frac{t^2}{z_{\nu,k}^2}\big)^{-1} = \exp \int_{\mathbb{R}\setminus\{0\}} (\cos tx - 1)(\sum_{k=1}^\infty e^{-|z_{\nu,k}||x|})/|x|\, dx \tag{10}$$

cf. Jurek (1996), p.177 for more details. Since class L is closed under dilations (multiplication of rv by scalars) and the weak convergence therefore X_ν has class L distribution as well, with the characteristic function $B_\nu(t)$,which proves the part (i).

Part (ii). By Proposition 3 in Jurek (2001) or see (4), for any class L characteristic function $\phi(t)$ we have that $\psi(t) := exp(td(log\phi(t))/dt)$ exists (for $t \neq 0$) and it is the characteristic function of BDLP. Since $z \frac{d}{dz} I_\nu(z) - \nu I_\nu(z) = z\, I_{\nu+1}(z)$, by A.8, we get

$$t\, d(\log B_\nu(t))/dt = \nu - t\, \frac{I'_\nu(t)}{I_\nu(t)} = -t\, \frac{I_{\nu+1}(t)}{I_\nu(t)} = \log b_\nu(t),$$

which proves that $b_\nu(t)$ are indeed characteristic functions of the BDLP for rv X_ν. Using (4) and Jurek (1996), we get the formula (8). Finally, since X_ν are given by series on independent Laplace rv therefore by Proposition 3 in Iksanov, Jurek and Schreiber (2003) to infer that $b_\nu(t)$ is from the class $\mathcal{U}$ of s-selfdecomposable distributions.

Part (iii). Since $L \subset \mathcal{U}$ and both classes are closed under taking positive powers therefore $B_\nu^\alpha(t) \cdot b_\nu^\beta(t) \in \mathcal{U}$, which proves the first part of (iii). Let $\alpha \geq \beta > 0$. Since $b_\nu(t) \in \mathcal{U}$ therefore $B_\nu(t) \cdot b_\nu(t) \in L$, by Theorem 1 in Iksanov, Jurek and Schreiber (2003). Since also $B_\nu^{\frac{\alpha}{\beta}-1}(t) \in L$ and L is a semigroup, we conclude that $B_\nu(t)^{\alpha/\beta} \cdot b_\nu(t) \in L$. Taking powers we get the second part of (iii).

For part (iv), using A.5 and part (i) of the theorem, we get

$$\frac{1}{B_\nu(it)} = \frac{1}{B(\frac{1}{2}, \nu + \frac{1}{2})} \int_{-1}^{1} e^{itx}(1 - x^2)^{\nu - \frac{1}{2}} dx, \tag{11}$$

where $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is Euler's beta function (Euler's integral of the first kind). Since, by A.1, $J_\nu(t)$ has real zeros therefore it can not be infinite divisible. This completes the proof of the Theorem.

For some related information on random variables of the form X_ν and the corresponding Dirichlet series we refer to Jurek (2000). Here, in particular, we have the following conditions on zeros of Bessel functions.

COROLLARY 1. *For zeros $z_{\nu,k}$, $k = 1, 2, ...,$ of the function $z^{-\nu} J_\nu(z)$ with $\nu > -1$, we have*

$$\sum_k |z_{\nu,k}| \int_{(|x|>1)} \log|x|\, e^{-|z_{\nu,k}||x|} dx < \infty,$$

$$\sum_k |z_{\nu,k}| \int_{(|x|>0)} \log(1 + |x|^2)\, e^{-|z_{\nu,k}||x|} dx < \infty.$$

[Both integrals can not be expressed in simple functions. The first involves the incomplete gamma function and the second sine and cosine integrals; cf. Gradshetyn and Ryzhik (1965) formulae 4.358 and 4.338, respectively.]

Proof. From (i) in Theorem 1 we infer that X_ν has Lévy spectral M of the form

$$M(A) = \sum_{k=1}^{\infty} \int_A e^{-|z_{\nu,k}||x|}\, |x|^{-1} dx, \quad \text{for all Borel subsets of } \mathbb{R} \setminus \{0\}.$$

And from random integral representation (3) or from (8) in Theorem 1(ii) we have

$$M(A) = \int_{(0,\infty)} N(e^s A) ds \quad \text{and} \quad N(A) = \sum_{k=1}^{\infty} \int_A |z_{\nu,k}| e^{-|z_{\nu,k}||x|} dx,$$

and N is a Lévy spectral measure of BDLP Y with finite logarithmic moment on $\{|x| > 1\}$. This gives the first condition in Corollary. For the second one we use the fact that

$$\int_{\{|x|>0\}} \frac{x^2}{1+x^2} M(dx) < \infty,$$

and the integration by parts formula.

REMARK 1. Biane and Yor (1987) expanded Brownian path along Legendre polynomials and computed conditional characteristic function of stochastic area given values of few first coefficients. In that case again we have formulas involving the products (7); cf. formula (4.10) in Yor (1989).

REMARK 2. The fractions $I_{\nu+1}/I_\nu$ of Bessel functions appear in the formula for the background driving characteristic functions b_ν for the selfdecomposable characteristic functions $B_\nu(t)$; cf. Theorem 1(ii). In Jurek (2001), in Example 2, it was found that functions

$$t \to \exp[-|t| \, \frac{K_{\nu-1}(|t|)}{K_\nu(|t|)} \,], \qquad t \in \mathbb{R},$$

involving fractions of modified Bessel functions K_ν, are background driving characteristic functions as well. This time they correspond to Student's t-distributions with 2ν degrees of freedom.

4. van Dantzig distributions and selfdecomposability.

Recall that an analytic characteristic function $\phi(z)$ defined in a strip $-a < \Im z < b$, for some $a, b, > 0$, belongs to *van Danztig class* $\mathcal{D}$, if $\psi(t) := 1/\phi(it)$ can be extended to a characteristic function; cf. Lukacs (1968). Equivalently,

$$\psi(t) \cdot \phi(it) = 1, \quad t \in \mathbb{R}, \quad (\text{van Dantzig pair } \ (\phi, \psi)). \tag{12}$$

If $\psi(t) = \phi(t)$ we say ϕ is *self-reciprocal characteristic function* in van Danzitg class $\mathcal{D}$. The three elementary examples of pairs (ψ, ϕ) are:

$$(\cos t, (\cosh t)^{-1}), \quad \Big(\frac{\sin t}{t}, \frac{t}{\sinh t}\Big) \ \text{ and } \ (e^{-ct^2}, e^{-ct^2}) \ \text{ with } \ c \geq 0.$$

The normal characteristic function is an example of self-reciprocal one. Here is another example.

PROPOSITION 1. *For $\nu > -1/2$, the ratios $\frac{J_\nu(t)}{I_\nu(t)}$, $t \in \mathbb{R}$, are non-infinitely-divisible self-reciprocal characteristic functions from van Danztig class $\mathcal{D}$.*

Proof. From Theorem 1, parts (i) and (iv), we have that the ratio is indeed a characteristic function. Since it has real zeros therefore $\frac{J_\nu(t)}{I_\nu(t)}$ can not be infinitely divisible. Its self-reciprocal property follows from the identities A.2 in the Appendix and the fact that B_ν are even functions; cf. in the Appendix formulae A.3 or A.3a. Thus the proof is complete.

COROLLARY 2. *All characteristic functions B_ν, for $\nu > -1/2$, belong to van Dantzig class $\mathcal{D}$.*

This is a consequence of Theorem 1, parts (i) and (iv). Here are more explicit formulae of class L and van Dantzig characteristic functions.

COROLLARY 3. *For $n = 0, 1, 2, ...$ and $t \in \mathbb{R}$, functions*

$$B_{n+\frac{1}{2}}(t) = \frac{((2n+1)!!)^{-1} \ t^{2n+1}}{\big(\sum_{\{k:\, n-k\, even\}} a_{k,n} t^{n-k}\big) \sinh t + \big(\sum_{\{k:\, n-k\, odd\}} a_{k,n} t^{n-k}\big) \cosh t}$$

where $a_{k,n} := (-1)^k \frac{(n+k)!}{2^k\, k!(n-k)!}$, $0 \leq k \leq n$, are characteristic functions of class L distributions that belong to van Dantzig class $\mathcal{D}$.

The proof is a combination of the formulae

$$I_{(n+\frac{1}{2})}(z) = \sqrt{\frac{2}{\pi}} z^{-(n+\frac{1}{2})}$$

$$\cdot\big[\big(\sum_{\{k:n-k\, even\}} a_{k,n}\, z^{n-k}\big)\,\sinh z + \big(\sum_{\{k:n-k\, odd\}} a_{k,n}\, z^{n-k}\big)\,\cosh z\big]$$

from A.6a and $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$, $n \geq 1$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, with the part (i) of Theorem 1.

COROLLARY 4. *For $n = 0, 1, 2, \ldots$ and van Dantzig class $\mathcal{D}$ characteristic functions $B_{n+\frac{1}{2}}(t)$, we have the equality*

$$B_{n+\frac{1}{2}}(t) = \frac{1}{(2n+1)!!\,\frac{d^n}{(t\,dt)^n}\big(\frac{\sinh t}{t}\big)}\ , t \in \mathbb{R},$$

where $\frac{d^n}{(t\,dt)^n}$ is n-times composition of the differential operator $f \to \frac{1}{t}\, f'(t)$.

This is a consequence of the identity $I_{n+\frac{1}{2}}(z) = z^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{d^n}{(z\,dz)^n}\big(\frac{\sinh z}{z}\big)$, cf. A.7a, in the Appendix, and the Theorem 1(i).

REMARK 3. Note that our class L characteristic functions $B_{\frac{1}{2}}(t), B_{\frac{3}{2}}(t)$ and $B_{\frac{5}{2}}(t)$ are exactly the examples (of the van Dantzig class $\mathcal{D}$) given in Lukacs (1968), on p.121, formulae (14.2a),(14.3a) and (14.4a). However, they were obtained there by different inductive procedures.

REMARK 4. It might be worthy to notice that Lukacs' 1968 examples of class $\mathcal{D}$ characteristic functions, given below his formula (13) on p. 120, are class L characteristic functions. The same applies to those given at the bottom of page 121 (convolutions of normal distributions with several gamma distributions). All in all it might be true that in each non self-reciprocal pair (ϕ, ψ) of characteristic functions satisfying van Dantzig condition (12) one of them is selfdecomposable. Or if it is not the case, then characterize that part of $\mathcal{D}$ that possesses this property.

REMARK 5. In search for examples of van Danztig pairs one may follow recent paper by de Meyer, Roynette, Vallois and Yor (2002). In Theorem 4.1 Authors have the equality

$$\mathbb{E}[\exp\, i\lambda \mathbf{B}_T]\,\mathbb{E}[\exp \frac{\lambda^2}{2} T] = 1, \quad \lambda \in \mathbb{R},$$

for a standard Brownian motion $\mathbf{B}_t, t \geq 0$, and some independent of it stopping times T.

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5. Appendix. Recall that the Bessel functions are the solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z}\frac{dZ_\nu}{dz} + (1 - \frac{\nu^2}{z^2})Z_\nu = 0.$$

Our main interest here is in the Bessel function of the first kind $J_\nu(z)$ and their modified versions $I_\nu(z)$, with $z \in \mathbb{C}$.

For ease of reading we collect here some formulas for Bessel functions. They are either taken directly from Gradshteyn and Ryzhik (1965), and in that case they are labelled by bold face numbers, or are combination of those.

A.1 $J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right), \quad \nu \neq -1, -2, -3, ...,$ **8.544**

where $z_{\nu,m}$ are zeros of the function $z^{-\nu} J_\nu(z)$, ordered by absolute value of their real parts.

A.1a For $\nu > -1$ all zeros $z_{\nu,m}$ are real numbers. **8.541**

A.2 $I_\nu(z) = e^{-\frac{\pi}{2} i\nu} J_\nu(e^{\frac{\pi}{2} i} z), \quad -\pi < \arg z \leq \pi/2,$ and
$I_\nu(z) = e^{\frac{3\pi}{2} i\nu} J_\nu(e^{-\frac{3\pi}{2} i} z), \quad \pi/2 < \arg z \leq \pi.$ **8.406**

A.3 $I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 + \frac{z^2}{z_{\nu,m}^2}\right), \quad \nu \neq -1, -2, -3, ...,$

This is derived from the factorization A.1 with a usage of A.2.

A.3a $I_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{z^{2k}}{k!\,\Gamma(\nu+k+1)},$ **8.445**

A.4 $J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{izt} (1-t^2)^{\nu-\frac{1}{2}}\, dt, \quad \Re\nu > -\frac{1}{2},$ **8.41**(10)

A.5 $I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{\pm zt} (1-t^2)^{\nu-\frac{1}{2}}\, dt, \quad \Re\nu > -\frac{1}{2},$ **8.431**(1)

A.6 $I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)! z^k} \left[(-1)^k e^z + (-1)^{n+1} e^{-z}\right],$ **8.467**

with $a_{k,n} := (-1)^k \frac{(n+k)!}{2^k\, k!(n-k)!}$, $0 \leq k \leq n$, which may be rewritten as

A.6a $I_{\pm(n+\frac{1}{2})}(z) = \sqrt{\frac{2}{\pi}} z^{-(n+\frac{1}{2})}$

$\cdot \left[\left(\sum_{\{k:n-k\, even\}} a_{k,n}\, z^{n-k}\right) \sinh z + \left(\sum_{\{k:n-k\, odd\}} a_{k,n}\, z^{n-k}\right) \cosh z\right]$

A.7 $J_{n+\frac{1}{2}}(z) = (-1)^n\, z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z\,dz)^n}\left(\frac{\sin z}{z}\right)$ **8.463**(1)

[$\frac{d^n}{(z\,dz)^n}$ is the n-times composition of an operator $f \to \frac{1}{z} f'(z)$].

A.7a $I_{n+\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z\,dz)^n}\left(\frac{\sinh z}{z}\right)$

It is a consequence of the formula A.7 and A.2.

A.8 $z\, \frac{d}{dz}\, I_\nu(z) - \nu\, I_\nu(z) = z I_{\nu+1}(z)$ **8.486**(4)

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