Random integral representations for free-infinitely divisible and tempered stable distributions*

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ABSTRACT. There are given sufficient conditions under which mixtures of dilations of Lévy spectral measures, on a Hilbert space, are Lévy measures again. We introduce some random integrals with respect to infinite dimensional Lévy processes, which in turn give some integral mappings. New classes (convolution semigroups) are introduced. One of them gives an unexpected relation between the free (Voiculescu) and the classical Lévy-Khintchine formulae while the second one coincides with tempered stable measures (Mantegna and Stanley) arisen in statistical physics.

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In probability theory and mathematical statistics the Fourier and Laplace transforms are the main tools used to prove weak limit theorems or to identify probability distributions. These are purely *analytic methods* used in other branches of mathematics as well. Jurek and Vervaat (1983) had introduced

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the method of random integral representations that allows to describe distributions as laws of some random integrals with respect to Lévy processes.

The aim of this note is to extend the random integral representation approach to more general schemes then those in Jurek (1985, 1988) and Jurek and Vervat (1983). As a consequence we will find such representations for two classes (semigropus) of measures. The first, a class $\mathcal{E}$, is "related" to the class of free infinitely divisible measures and the second, $\mathcal{TS}_\alpha$, coincides with the tempered stable distributions

Our results are mainly given in a generality of measures on a Hilbert space, with some digression to a case of Banach spaces. However, methods of proofs are dimensionless so they can be read for Euclidean spaces as well.

1. Introduction and terminology. Let E denotes a real separable Banach space, E' its conjugate space, $< \cdot, \cdot >$ the usual pairing between E and E', (this is just a scalar product in a case when E is a Hilbert or an Euclidean space), and $||.||$ the norm on E. The σ-field of all Borel subsets of E is denoted by $\mathcal{B}$, while $\mathcal{B}_0$ denotes Borel subsets of $E \setminus \{0\}$. By $\mathcal{P}(E)$ or simply by $\mathcal{P}$, we denote the (topological) semigroup of all Borel probability measures on E, with convolution "$*$" and the topology of weak convergence "$\Rightarrow$."

Recall that a measure $\mu \in \mathcal{P}$ is called *infinitely divisible* if for each natural $n \geq 2$ there exists $\nu_n \in \mathcal{P}$ such that $\nu_n^{\star n} = \mu$. The class $ID(E)$, of all infinitely divisible probability measures on E, is characterized by the Lévy-Khintchine formula. Namely,

$$\begin{aligned} &\mu \in ID \quad \text{iff} \quad \hat{\mu}(y) = \exp[\,\Phi(y)], \quad \text{where} \\ &\Phi(y) := i < y, a > -1/2 < Ry, y > + \\ &\qquad \int_{E\setminus\{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_{||x||\leq 1}(x)]M(dx), \quad y \in E'; \quad (1) \end{aligned}$$

Φ is called the *Lévy exponent* of $\hat{\mu}$ (cf. Araujo and Giné (1980), Section 3.6); we will write $\mu = [a, R, M]$, if $\hat{\mu}$ has the above form. For $\mu \in ID(E)$ one can define arbitrary positive convolution powers, namely if $\mu = [a, R, M] \in ID(E)$ then $\mu^{\star c} = [c \cdot a, c \cdot R, c \cdot M]$, for any $c \geq 0$, and $T_a\mu = [a_c, c^2R, T_cM]$, where $a_c = c\,a + c\int_E x[1_B(cx) - 1_B(x)]M(dx)$, where the mapping T_c is given by $T_cx = cx, x \in E$ and $T_cM(A) = M(c^{-1}A)$ for all $A \in \mathcal{B}_0$. Totality of all Lévy spectral measures on E will be denoted by $\mathcal{M}(E)$.

2. The λ -mixtures of Lévy spectral measures. Our main objective in this section is to provide a method of constructing Lévy spectral measures using the mixtures of T_tM. Namely, for a non-negative Borel measure λ on $\mathbb{R}^+ = (0,\infty)$ and a Borel measure M on $E\setminus\{0\}$ we define

$$M^{(\lambda)}(A) := \int_0^\infty (T_tM)(A)\lambda(dt) = \int_0^\infty \int_{E\setminus\{0\}} 1_A(tx)M(dx)\lambda(dt), \quad (2)$$

$A \in \mathcal{B}_0$; cf. Jurek (1990). In particular, we have that

$$\textit{if} \quad M^{(\lambda)} \in \mathcal{M}(E) \quad \textit{then} \quad M \in \mathcal{M}(E), \quad \int_0^\infty \min(1,t^2)\,\lambda(dt) < \infty \quad \textit{and}$$
$$\int_0^\infty M(\{x : ||x|| > t^{-1}\})\lambda(dt) = \int_{E\setminus\{0\}} \lambda(\{s : s > ||x||^{-1}\})M(dx) < \infty; \quad (3)$$

cf. Jurek (1990), Proposition 2. The converse implication to (3) is not completely settled.

PROPOSITION 1. *Suppose that λ and M are Borel measures on $(0,\infty)$ and E respectively, such that*

$$\int_{E\setminus\{0\}} \big[||x|| \int_0^{||x||^{-1}} t\,\lambda(dt) + \lambda(\{s : s > ||x||^{-1}\})\big]M(dx) =$$
$$\int_0^\infty \big[t \int_{\{0<||x||\le t^{-1}\}} ||x||M(dx) + M(\{||x| > t^{-1}\})\big]\lambda(dt) < \infty. \quad (4)$$

Then $M^{(\lambda)}$ and M are Lévy spectral measures on E and λ is a Lévy spectral measure on $(0,\infty)$.

Proof. From Araujo-Gine (1980), Theorem 6.3, we know that if a measure M integrates $\min(1,||x||)$ on a Banach space E then it is a Lévy spectral measure. Condition (4) is just that integrability condition for $M^{(\lambda)}$. [It was written in a form of a sum of two integrals to indicate two different behaviors of Lévy measures: on open neighborhoods of zero and on their complements.]

Finally, from the inequality $\min(1,t)\cdot\min(1,||x||) \le \min(1,t||x||)$, for all $t \ge 0$ and $x \in E$, and formula (3) we infer the remaining claims.

PROPOSITION 2. *Suppose that H is a real separable Hilbert space, and M and λ are Borel measures on H and on $(0,\infty)$, respectively. Then $M^{(\lambda)}$ is a Lévy spectral measure on H if and only if*

$$\int_{H\setminus\{0\}} \big[||x||^2 \int_0^{||x||^{-1}} t^2\,\lambda(dt) + \lambda(\{s : s > ||x||^{-1}\})]M(dx) =$$
$$\int_0^\infty \big[t^2 \int_{\{0<||x||\le t^{-1}\}} ||x||^2 M(dx) + M(\{||x| > t^{-1}\})\big]\lambda(dt) < \infty, \quad (5)$$

Moreover, M is a Lévy spectral measures on H and λ is a Lévy spectral measure on $\mathbb{R}$.

Proof. The proof follows the same way as in Proposition 1. However, we use the function $\max(1, ||x||^2)$ and the characterization of $ID(H)$ from Parthasarathy (1967),Chapter VI, Theorem 4.10.

Examples. A). Let e denotes the standard exponential distribution with the density $e^{-t}, t > 0$. Then

COROLLARY 1. *On any real separable Hilbert space H we have that $M^{(\mathrm{e})}$ is a Lévy spectral measure if and only if so is M.*

Proof. Let M be a Lévy spectral measure on H. Since we have that

$$g(||x||) := ||x||^2 \int_0^{||x||^{-1}} t^2 e^{-t} dt + \int_{||x||^{-1}}^\infty e^{-t} dt =$$
$$2||x||^2\big[1 - e^{-||x||^{-1}}(1 + ||x||^{-1})\big],$$

therefore $g(||x||) \le 2||x||^2$, for $||x|| \le 1$. On the other hand, $\lim_{||x||\to\infty} g(||x||) = 1$, which implies that $g(||x||) \le K$, for $||x|| > 1$. Consequently,

$$\int_{H\setminus\{0\}} \min(1, ||x||^2)M^{(e)}(dx) = \int_{\{0<||x||\le 1\}} g(||x||)M(dx)$$
$$+ \int_{\{||x||>1\}} g(||x||)M(dx) \le$$
$$2\int_{\{0<||x||\le 1\}} ||x||^2 M(dx) + K \int_{\{||x||>1\}} M(dx) < \infty$$

and Proposition 2 gives that $M^{(e)}$ is spectral measure.

Converse implication also follows from Proposition 2, and thus the proof is complete.

B). Let $\rho_\alpha(dt) := t^{-\alpha-1}e^{-t}dt$ be a measure on $(0,\infty)$.

COROLLARY 2. *On any Hilbert space H, if $M^{(\rho_\alpha)}$ is a Lévy spectral measure then so is M and*

$$0<\alpha<2, \int_{||x||>1} ||x||^\alpha M(dx)<\infty,$$
$$\text{and for all } s>0, \quad \int_{\{0<||x||\le 1\}} ||x||^\alpha e^{-s/||x||} M(dx)<\infty.$$

Conversely, if $0<\alpha<2$ and $\int_{\{||x||>0\}}||x||^\alpha M(dx)<\infty$ then $M^{(\rho_\alpha)}$ and M are Lévy spectral measures.

Proof. Let us introduce a function

$$h_\alpha(||x||) := ||x||^2 \int_0^{||x||^{-1}} t^{(2-\alpha)-1}e^{-t}dt + \int_{||x||^{-1}}^\infty t^{-\alpha-1}e^{-t}dt$$
$$= ||x||^\alpha\big[\int_0^1 t^{(2-\alpha)-1}e^{-t/||x||}dt + \int_1^\infty t^{-\alpha-1}e^{-t/||x||}dt\big].$$

From Proposition 2, we have that $\int_{H\setminus\{0\}} h_\alpha(||x||)M(dx)<\infty$. Consequently, $h_\alpha(||x||)<\infty$ for $M-a.a.x$. Consequently, $2-\alpha>0$ (from the first integral) and $\alpha>0$. Since we also have that

$$\int_{H\setminus\{0\}} h_\alpha(||x||)M(dx) = \int_0^1 t^{(2-\alpha)-1}\big(\int_{H\setminus\{0\}}||x||^\alpha e^{-t/||x||}M(dx)\big)dt$$
$$+\int_1^\infty t^{-\alpha-1}\big(\int_{H\setminus\{0\}}||x||^\alpha e^{-t/||x||}M(dx)\big)dt<\infty. \quad (6)$$

Hence, the function $t\to\int_{H\setminus\{0\}}||x||^\alpha e^{-t/||x||}M(dx)<\infty$ for almost all (Lebesgue measure) $t\in\mathbb{R}^+$. Its monotonicity gives that it is finite for all $t>0$. Finally,

$$e^{-1}\int_{\{||x||>1\}}||x||^\alpha M(dx) \le \int_{\{||x||>1\}}||x||^\alpha e^{-1/||x||}M(dx)<\infty,$$

which completes the proof of the first part. The converse part follows from the fact that $h_\alpha(||x||)\le 2((2-\alpha)\alpha)^{-1}||x||^\alpha$ and Proposition 2.

3. Random integral representations. Random integral representation method allows to represent a random variable, more precisely its probability distribution, as a probability distribution of random integrals; cf. Jurek-Mason (1993) and references therein.

Theorem 1. *Let $\lambda(\cdot)$ be a Borel measure on positive half-line that is finite on sets bounded away from zero and let $\Lambda(t) := \lambda(\{s > 0 : s > t\})$, $t > 0$. Further, let $Y(t), t \geq 0$, be a cadlag Lévy process with values in a Hilbert space H and $Y(1)$, an infinitely divisible random element, is described by a triple $[a, R, M]$. Then, in order that the limit*

$$I_{(\alpha,\beta]} := \int_{(\alpha,\,\beta]} t\,dY(\Lambda(t)) \to \int_{(0,\,\infty)} t\,dY(\Lambda(t)) =: I_{(0,\infty)}, \tag{7}$$

exits in distribution ,as $\alpha \downarrow 0$ and $\beta \uparrow \infty$, it is sufficient and necessary that

$$\int_0^\infty t\lambda(dt) < \infty, \ \textit{provided} \ \ a \neq 0; \int_0^\infty t^2\lambda(dt) < \infty, \ \textit{provided} \ \ R \neq 0;$$
$$\int_{\{0<||x||\leq 1\}} ||x|| \int_{||x||^{-1}}^\infty t\lambda(dt)M(dx) + \int_{\{||x||>1\}} ||x|| \int_0^{||x||^{-1}} t\lambda(dt)M(dx) < \infty;$$
$$\textit{and } M^{(\lambda)} \textit{ is a Lévy spectral measure.} \tag{8}$$

Furthermore, if the limit $I_{(0,\infty)}$ has representation $[a^{(\lambda)}, R^{(\lambda)}, M^{(\lambda)}]$ then

$$a^{(\lambda)} = (\int_0^\infty t\lambda(dt)) \cdot a + \int_0^\infty \int_{H\setminus\{0\}} [1_B(tx) - 1_B(x)]\, t\, x\, M(dx)\, \lambda(dt);$$
$$R^{(\lambda)} = (\int_0^\infty t^2\lambda(dt))\cdot R; \qquad M^{(\lambda)}(A) = \int_0^\infty \int_{H\setminus\{0\}} 1_A(tx)M(dx)\lambda(dt). \tag{9}$$

Proof. From the definition of random integrals $W_{(\alpha,\,\beta]} := \int_{(\alpha,\beta]} h(t)dY(\tau(t))$, where $h : (\alpha, \beta] \to \mathbb{R}$ and $\tau : (\alpha, \beta] \to \infty$ are deterministic functions, and τ is monotone one, and Y is a cadlag Lévy process, we have that

$$\mathbb{E}[e^{i<y,\, W_{(\alpha,\beta]}>}] = \exp \int_{(\alpha,\beta]} \Big(\log \mathbb{E}[e^{i<(\pm)h(t)\, y, Y(1)>}] \Big) (\pm)\, d\tau(t), \tag{10}$$

where one takes the sign "+" for nondecreasing τ and "-" for nonincreasing τ; cf. Jurek-Vervaat(1983) or Jurek-Mason (1993), Section 3.6. Hence using the Lévy-Khintchine formula (1) and taking $h(t) = t$ and $\tau = \Lambda$ in (11), we conclude that random element $I_{(\alpha,\beta]}$ has an infinitely divisible distribution with the triple $[a^{(\lambda)}_{(\alpha,\beta]}, R^{(\lambda)}_{(\alpha,\beta]}, M^{(\lambda)}_{(\alpha,\beta]}]$ given as follows

$$a^{(\lambda)}_{(\alpha,\beta]} = (\int_{(\alpha,\beta]} t\lambda(dt)) \cdot a + \int_{(\alpha,\beta]} t \int_{H\setminus\{0\}} [1_B(tx) - 1_B(x)]xM(dx)\lambda(dt);$$

$$R^{(\lambda)}_{(\alpha,\beta]} = (\int_{(\alpha,\beta]} t^2\lambda(dt)) \cdot R;$$

$$M^{(\lambda)}_{(\alpha,\beta]}(A) = \int_{(\alpha,\beta]} \int_{H\setminus\{0\}} 1_A(tx)\, M(dx)\, \lambda(dt) = M^{(\lambda|_{(\alpha,\beta]})}(A), \quad (11)$$

where the triple $[a, R, M]$ comes from the Lévy-Khintchine representation of the infinitely divisible random element $Y(1)$. As $\alpha \downarrow 0$ and $\beta \uparrow \infty$ then $M^{(\lambda)}_{(\alpha,\beta]} \uparrow M^{(\lambda)} \in \mathcal{M}(H)$, Gaussian covariance operators $R^{(\lambda)}_{(\alpha,\beta]} \to R^{(\lambda)}$. Finally, for the shift part note that

$$||[1_B(tx) - 1_B(x)]|| = 1 \text{ iff } \ 1 < ||x|| \le t^{-1} \ \text{ or } \ t^{-1} < ||x|| \le 1,$$

and the second summand for a shift vector in (10) exits as a Bochner integral on the product space $(0,\infty) \times (H \setminus \{0\})$. Consequently, $I_{(\alpha,\beta]}$ converges in distribution to $I_{(0,\infty)}$, by Parthasarathy (1968),Theorem 5.5, because $M^{(\lambda)}_{(\alpha,\beta]} \uparrow M^{(\lambda)} \in \mathcal{M}(H)$. Thus the proof is complete.

As in previous papers of Jurek&Vervaat (1983) or Jurek (1982, 1985, 1988) here we introduce the following *random integral mapping*

$$\mathcal{K}^{(\lambda)}(\mu) := \mathcal{L}(\int_0^\infty t\, dY_\mu(\Lambda(t))) \in ID \quad (12)$$

where $Y_\mu(t), t \ge 0$ is a cadlag Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu$ and $\Lambda(\cdot)$ is the cumulative distribution function or the tail function of λ — note that from Proposition 2, λ, as a Lévy spectral measure, is finite on any half-line (a,∞), $a > 0$.

COROLLARY 3. *For probability measures of the form $\mathcal{K}^{(\lambda)}(\mu)$ one has*

$$\mathbf{E}[e^{i<y, \int_0^\infty t\, dY_\mu(\Lambda(t))>}] = \exp \int_0^\infty \log \mathbf{E}[e^{it<y,Y_\mu(1)>}]\lambda(dt),$$

where $y \in E'$;(with possible minus signs as described in (10)). Furthermore, the random integral mapping $\mathcal{K}^{(\lambda)}(\mu)$ has the following algebraic properties

$$\mathcal{K}^{(\lambda)}(\mu_1 \star \mu_2) = \mathcal{K}^{(\lambda)}(\mu_1) \star \mathcal{K}^{(\lambda)}(\mu_2), \quad \mathcal{K}^{(\lambda_1+\lambda_2)}(\mu) = \mathcal{K}^{(\lambda_1)}(\mu) \star \mathcal{K}^{(\lambda_2)}(\mu)$$

One of the advantages of random integral representation is that it allows easily to incorporate space and time changes. Here is an example.

COROLLARY 4. *For $a \in \mathbb{R}$, $c > 0$ and a random integral $\int_{(\alpha,\beta]} h(t)dY(\tau(t))$, where $h : (\alpha, \beta] \to \mathbb{R}$ and $\tau : (\alpha, \beta] \to \infty$ are deterministic functions, and τ is monotone one, and Y is a cadlag Lévy process, we have*

$$\Big(\mathcal{L}\Big(a\int_{(\alpha,\beta]} h(t)dY(\tau(t))\Big)^{*c}\Big)\widehat{\ }(y) = \mathcal{L}\Big(\int_{(\alpha,\beta]} a\, h(t)dY(c\,\tau(t))\Big)(y) =$$

$$\exp\int_{(\alpha,\beta]} \log \mathbf{E}[e^{i\,a\,h(t)<y,Y_\mu(c)>}]d\tau(t), \quad y \in H. \quad (13)$$

It is also true for integrals over half line, provided they exist.

Proof. Use (10) and the fact that $\mathcal{L}(Y(c)) = (\mathcal{L}(Y(1)))^{*c}$.

4. Two applications of the random integral method.

A). *Free infinite divisibility.* From Example **A** we infer that $M^{(e)}$ is Lévy spectral measure (on H) whenever so is M. Furthermore by Theorem 1, formula (9), $R^{(e)} = 2\,R$, and

$$a^{(e)} = a + \int_{\{||x||>1\}} x(1 - e^{-||x||^{-1}}(1 + ||x||^{-1}))M(dx)$$
$$+ \int_{\{0<||x||\le1\}} x\, e^{-||x||^{-1}}(1 + ||x||^{-1}))M(dx) \quad (14)$$

exits in a Bochner sense. To this end note that $\lim_{s\to0} s\,e^{-s^{-1}}(1 + s^{-1}) = 0$ and $\lim_{s\to\infty} s(1 - e^{-s^{-1}}(1 + s^{-1})) = 0$.

Furthermore, for $r > 0$ and a Borel D of the unit sphere $S = \{x : ||x|| = 1\}$, let us define *the Lévy spectral function* $L_M(D; r)$ associated with the measure M as follows $L_M(D; r) = M(\{x : x\,||x||^{-1} \in D$ and $||x|| > r\})$. Then using (10) we get

$$L_{M^{(e)}}(D; r) = \int_0^\infty L_M(D; rt^{-1})\,e^{-t}dt = r\int_0^\infty L_M(D; s^{-1})\,e^{-rs}ds \quad (15)$$

for $r > 0$. Hence, $r^{-1}L_{M^{(e)}}(D; r), r > 0$, is a Laplace transform of (unique) function $L_M(D; s^{-1})$ and thus $M^{(e)}$ uniquely determines M. Hence, $a^{(e)}$ and M uniquely identifies a. All in all with Theorem 1 we conclude that

$$\mathcal{K}^{(e)} : ID \ni \mu \to \mathcal{L}(\int_0^\infty t\,dY_\mu(1 - e^{-t})) \in \mathcal{E} := \mathcal{K}^{(e)}(ID) \quad (16)$$

is well defined one-to-one random integral mapping, where $Y_\mu(t), t \geq 0$, is a cadlag Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu$. Consequently, we obtained a convolution subsemigroup $\mathcal{E} \subset ID$.

COROLLARY 5. *In order, for a function $g : H \to \mathbb{C}$, to be a characteristic function of a measure from the convolution semigroup $\mathcal{E}$, it is necessary and sufficient that*

$$g(y) = \exp\big[i < y, a > - < y, Ry > + \\ \int_{H\setminus\{0\}} \Big(\frac{1}{1 - i < y, x >} - 1 - i < y, x > 1_{\{||x|| \leq 1\}}\Big) M(dx)\big], \quad (17)$$

where $a \in H$, R is non-negative, self-adjoint, trace operator and M is a Borel measure that integrates the function $\min(1, ||x||^2)$ *over H. In fact, g is a characteristic of the measure $\mathcal{K}^{(e)}([a, R, M])$.*

One gets (17) by putting into (1) the triplet: the vector $a^{(e)}$, the covariance operator $R^{(e)}$ and the Lévy spectral measure $M^{(e)}$.

REMARK 1. One has two possibilities of looking at the class $\mathcal{E}$. Either, as a subset of ID with the triples $[a^{(e)}, R^{(e)}, M^{(e)}]$ and the kernel Φ from formula (1) or as a set of probability distributions given by triples [a,R,M] and but with a new kernel

$$\Phi_1(y) := \big[i < y, a > - < y, Ry > + \\ \int_{H\setminus\{0\}} \Big(\frac{1}{1 - i < y, x >} - 1 - i < y, x > 1_{\{||x|| \leq 1\}}\Big) M(dx)\big], \; y \in H. \quad (18)$$

Note that both kernels are additive in a, R and M, i.e., sums of those parameters correspond to the convolution of probability measures.

(A similar phenomenon was already noted in Jurek-Vervaat (1983) formula (4.3), pages 254-255, for the Lévy class L of selfdecomposable distributions.)

PROPOSITION 3. *Let $I^{(e)} := \int_0^\infty t dY(1 - e^{-t})$ and $\phi_{I^{(e)}}(y)$, and $\phi_{Y(1)}(y)$, $y \in H$ are characteristic functions of $I^{(e)}$ and $Y(1)$, respectively. Then*

$$\log \phi_{I^{(e)}}(y) = \int_0^\infty \log \phi_{Y(1)}(ty) e^{-t} dt, \\ \log \phi_{Y(1)}(y) = \mathfrak{L}^{-1}[s^{-1} \log \phi_{I^{(e)}}(s^{-1}y; x)]|_{x=1}, \quad (19)$$

where for each $y \in H$, $\mathfrak{L}^{-1}$ is the inverse of the Laplace transform of the function $s^{-1}\log\phi_{I^{(e)}}(s^{-1}y)$. Hence, the mapping $\mathcal{K}^{(e)} : ID(H) \to \mathcal{E}$ is an algebraic isomorphism between convolution semigroups and for its inverse $(\mathcal{K}^{(e)})^{-1}$ we have

$$((\mathcal{K}^{(e)})^{-1}(\rho))\widehat{\ }(y) = \exp \mathfrak{L}^{-1}[s^{-1}\log\hat{\rho}(s^{-1}y;\, x)]|_{x=1}, \quad y \in H.$$

Proof. From Corollary 3 we have

$$\log\phi_{I(e)}(y) = \int_0^\infty \log\phi_{Y(1)}(ty)e^{-t}dt.$$

Putting, for each fixed $y \in H$,

$$f_y(s) := \log\phi_{I^{(e)}}(sy) \text{ and } \; g_y(s) := \log\phi_{Y(1)}(sy), \text{ for } \; s \in \mathbb{R},$$

and using the above relation we get

$$f_y(s) = \int_0^\infty g_y(ts)e^{-t}dt, \quad f_y(s^{-1}) = s\int_0^\infty g_y(x)e^{-sx}dx, \; s > 0.$$

Consequently, $\frac{1}{s}\,f_y(\frac{1}{s}) = \mathfrak{L}[(g_y(x);s]$ is the Laplace transform evaluated at s. This completes the proof.

D. Voiculesu and others developed a theory of "a free probability". For our needs here let us recall briefly that with a probability measure μ, on a real line, one associates a complex valued function $V_\mu(z) := F_\mu^{-1}(z) - z, z \in \mathcal{D}$, where $\mathcal{D}$ is an appropriately selected domain in the complex upper half-plane and $F_\mu(z) := 1/G_\mu(z)$, where

$$G_\mu(z) := \int_{-\infty}^\infty \frac{1}{z-t}\,\mu(dt) \quad \text{is called the Cuchy transform of } \mu. \tag{20}$$

For two probability measures, on the real line, μ and ν, if the sum $V_\mu(z) + V_\nu(z)$ corresponds to another probability measure then we denote it by $\mu\Box\nu$. Hence one can introduce $\Box$-infinite divisibility and a semigroup $(ID(\mathcal{P}(\mathbb{R}),\Box)$. From Bercovici and Pata (1999) and Barndorff-Nielsen and Thornbjornsen (2002) we have that

$$\nu \in ID(\mathcal{P}(\mathbb{R}),\Box) \;\text{ iff }\; z\,V_\nu(z^{-1}) = az + \sigma^2z^2 + \int_{\mathbb{R}\setminus\{0\}}\Big(\frac{1}{1-zx} - 1 - zx1_{\{|x|\le 1\}}\Big)M(dx)],\; z \in \mathbb{C}^-, \tag{21}$$

where $a,\sigma \in \mathbb{R}$, and M integrates $\min(1,|x|^2)$ over $\mathbb{R}$, i.e., M is a Lévy spectral measure on real line.

COROLLARY 6. *A probability measure ν, on $\mathbb{R}$, is $\Box$-infinitely divisible if and only if there exist a unique reals a and σ^2 and a Lévy spectral measure M such that*

$$(it)\, V_\nu((it)^{-1}) = \log(\mathcal{K}^{(e)}(\mu))\widehat{\ }(t) = \log\Big(\mathcal{L}(\int_0^\infty s dY_\mu(1-e^{-s}))\Big)\widehat{\ }(t), \quad (22)$$

$t \in \mathbb{R}\setminus\{0\}$, where $(Y_\mu(t), t \geq 0)$ is a Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu = [a, \sigma^2, M]$. In other words, functions $t \to e^{it\, V_\nu((it)^{-1})}, t \in \mathbb{R}$, are characteristic functions, and the class of measures corresponding to them coincides with the class $\mathcal{E}$. Furthermore, for the Voiculescu transform V_ν, we have that $V_\nu(it) = it \log(\mathcal{K}^{(e)}(\mu))\widehat{\ }(-t^{-1})$, for $t \in \mathbb{R}\setminus\{0\}$.

Proof. Use Corollary 3 for $E = \mathbb{R}$ and then apply Theorem 1 with the formula (17).

REMARK 2. For any finite measure m, on a Hilbert or Banach space, let $e(m)$ denotes the compound Poisson measure. Since it is $*$-infinitely divisible, (i.e., $e(m) \in ID(H)$), we can insert it into a Lévy (compound Poisson) process $Y_{e(m)}(t), t \geq 0$. Consequently,

$$\text{if } \mathbf{G}_m(y) := \int_H \frac{1}{1 - i < y, x >} m(dx), \ y \in H, \ \text{ then}$$

$$\mathbf{F}_m(y) := \log(\mathcal{K}^{(e)}(e(m))\widehat{\ }(y) = \mathbf{G}_m(y) - m(H) = \int_H \frac{i < y, x >}{1 - i < y, x >} m(dx). \quad (23)$$

To see that equalities recall that $(e(m))\widehat{\ }(y) = \exp(\hat{m}(y) - m(H))$ and this with (13) and Corollary 3 give the above formula.

B). *Tempered stable probability measures.* Let us consider the example **B** from Section 2. Let us assume that

$$\int_H ||x||^\alpha M(dx) < \infty \ \text{ and } M \text{ is a Borel measure on } H.$$

In the sequel, by ID_α we denote those infinitely divisible whose Lévy spectral measures satisfy the above integrability condition. Consequently, from Corollary 2 we have that both $M^{(\rho_\alpha)}$ and M are Lévy spectral measures and from Theorem 1 we get $R^{(\rho_\alpha)} = \Gamma(2-\alpha)R$ is covariance operator of Gaussian measure. For the shift vector $a^{(\rho_\alpha)}$, we need three integrals; cf. formula (12). Firstly, note that

$$\int_{\{0<||x||\le 1\}} ||x|| \int_{||x||^{-1}}^{\infty} t^{-\alpha}e^{-t}dt\, M(dx) \le$$
$$\int_{\{0<||x||\le 1\}} ||x||^2 M(dx) \int_1^{\infty} t^{(2-\alpha)-1}e^{-t}dt\, M(dx) < \infty, \text{ for } 0<\alpha<2.$$

And secondly, note that

$$\int_{\{||x||>1\}} ||x|| \int_0^{||x||^{-1}} t^{-\alpha}e^{-t}dtM(dx) = \int_{\{||x||>1\}} ||x||\gamma(1-\alpha, ||x||^{-1})M(dx)$$
$$= \int_{\{||x||>1\}} ||x||^{\alpha}\Big(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\alpha+n)} \cdot \frac{1}{||x||^n}\Big)M(dx) < \infty, \text{ for } 0<\alpha<1.$$

Consequently, for $0<\alpha<1$, the random integral mapping

$$\mathcal{K}^{(\rho_\alpha)} : ID_\alpha \ni \mu \to \mathcal{L}(\int_0^{\infty} t\, dY_\mu(\Gamma(\alpha,t))) \in \mathcal{TS}_\alpha := \mathcal{K}^{(\rho_\alpha)}(ID_\alpha) \quad (24)$$

is well defined. In above $\gamma(\alpha,x)$ and $\Gamma(\alpha,x)$ denote the incomplete Euler's gamma functions, i.e.,

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}dt,\ x>0,\ (\Re\alpha>0);\ \ \Gamma(\alpha,x) = \int_x^{\infty} t^{\alpha-1}e^{-t}dt,\ x>0.$$

Measures from the class $\mathcal{TS}_\alpha$ are called *tempered stable distributions.*

Let us mention here that tempered stable processes are of importance in statistical physics as they exhibits different local and global behavior; (cf. Mantegna and Stanley (1994), Kaponen (1995); comp. Corollary 7 below.

PROPOSITION 4. *Assume that* $0<\alpha<1$. *Let* $I^{(\rho_\alpha)} := \int_0^\infty tdY(\Gamma(-\alpha,t))$ *and* $\phi_{I^{(\rho_\alpha)}}(y)$, *and* $\phi_{Y(1)}(y)$, $y\in H$, *are characteristic functions of* $I^{(\rho_\alpha)}$ *and* $Y(1)$, *respectively. Then*

$$\log\phi_{I^{(\rho_\alpha)}}(y) = \int_0^{\infty} \log\phi_{Y(1)}(ty)t^{-\alpha-1}e^{-t}dt,$$
$$\log\phi_{Y(1)}(y) = \mathfrak{L}^{-1}[s^{\alpha}\log\phi_{I^{(\rho_\alpha)}}(s^{-1}y;x)]|_{x=1}, \qquad (25)$$

where for each $y \in H$, $\mathfrak{L}^{-1}$ *is the inverse of the Laplace transform of the function*

$s^\alpha \log \phi_{I(\rho_\alpha)}(s^{-1}y)$. *Hence, the mapping* $\mathcal{K}^{(\rho_\alpha)} : ID(H_\alpha) \to \mathcal{TS}_\alpha$ *is an algebraic isomorphism between convolution semigroups. For its inverse* $(\mathcal{K}^{(\rho_\alpha)})^{-1}$ *we have*

$$((\mathcal{K}^{(\nu_\alpha)})^{-1}(\rho))\widehat{\;}(y) = \exp \mathfrak{L}^{-1}[s^\alpha \log \hat{\nu}(s^{-1}y;\, x)]|_{x=1}, \quad y \in H, \quad (\nu \in \mathcal{TS}_\alpha).$$

The proof is analogous to that of Proposition 3.

REMARK 3. The previous result is also true for $1 \leq \alpha < 2$ if one considers only Lévy processes with symmetric spectral measures M and shifts $a = 0$.

Here is yet another example of usefulness of the random integral representation method.

COROLLARY 7. *Let* $0 < \alpha < 1$ *and* $X := \int_0^\infty t\, dY(\Gamma(-\alpha, t)$ *be* $\mathbb{R}^d$*-valued random vector with* $\mathcal{L}(Y(1)) = [0, 0, M] \in ID_\alpha$. *Then*

$$(\mathcal{L}(s^{-1/\alpha}X)^{*s}) \Rightarrow \eta_\alpha, \;\; as \;\; s \to 0,$$

where η_α *denotes the strictly stable law with exponent* α

Proof. Using Corollary 4 with $a = s^{-1/\alpha}$ and $c = s$ we have

$$((\mathcal{L}(\frac{1}{s^{1/\alpha}}X)^{*s}))\widehat{\;}(y) = \exp \int_0^\infty s \, \log \mathbf{E}[\exp i\, t/s^{1/\alpha} < y, Y(1) >] t^{-\alpha-1} e^{-t} dt$$
$$= \exp \int_0^\infty \log \mathbf{E}[\exp iu < y, Y(1) >] u^{-\alpha-1} e^{-u\, s^{1/\alpha}} du \;\; (\text{as } s \to 0)$$
$$\to \exp \int_{\mathbb{R}^d \setminus \{0\}} \int_0^\infty [e^{iu<y,x>} - 1 - iu < y, x > 1_{||x|| \leq 1}(x)] u^{-\alpha-1} du\, M(dx) =$$
$$\exp[-c_\alpha \int_{\mathbb{R}^d \setminus \{0\}} |< y, x >|^\alpha (1 - i \, \tan(\pi\alpha/2)\, sign < y, x >) M(dx)],$$

where $c_\alpha > 0$. (The last equality is obtained via contour integration; see any book on stable laws.) Finally, the last formula is the characteristic function of a strictly α-stable probability measure on $\mathbb{R}^d$.

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