Invariant measures under random integral mappings and marginal distributions of fractional Lévy processes.*

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Abstract. It is shown that some convolution semigroups of infinitely divisible measures are invariant under certain random integral mappings. We characterize the coincidence of random integrals for s-selfdecomposable and selfdecomposable distributions. Some applications are given to the moving average fractional Lévy process (MAFLP).

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Abbreviated title: Invariant measures under random integral mappings

Let us recall that *the moving average fractional Lévy process* (in short, MAFLP) $(Z(t), t \in \mathbb{R})$ is given as follows

$$Z(t) := \int_{\mathbb{R}} \big((t-s)_+^\alpha - (-s)_+^\alpha\big) dY_\nu(s), \quad t \geq 0, \tag{1}$$

where $(Y_\nu(t), t \in \mathbb{R})$ is a Lévy process in $\mathbb{R}^d$, ν is the probability distribution of the Lévy process at time $t = 1$, the parameter α is from the interval $(0, 1/2)$

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and $a_+ := \max(0, a)$ is the positive part of a. We study here how the laws of $Z(t)$, in (1), are related to the law ν. This problem for selfdecomposable measures ν with finite variance was investigated in Cohen-Maejima (2011). Our approach to these questions is more general with completely different proofs and based on the so-called *random integral representation (or random integral mapping)*. This is a technique that represents an infinitely divisible distribution, say ρ, as a law of a random integral of the following form:

$$\rho = I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\Big(\int_{(a,b]} h(t)dY_\nu(r(t))\Big), \qquad (\star)$$

where $(a,b] \subset \mathbb{R}^+$, $h : \mathbb{R}^+ \to \mathbb{R}$, $Y_\nu(\cdot)$ is a Lévy process such that
$\mathcal{L}(Y_\nu(1)) = \nu$ and $r : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone time change, (2)

and its limit as $b \to \infty$; cf. Jurek (2011) for a review of the random integral mapping method and its application to characterizations of classes of infinitely divisible laws.

We investigate classes of probability measures that are invariant under random integral mappings $I^{h,r}_{(a,b]}$ (Proposition 1), then we characterize those generalized s-selfdecomposable measures that are, indeed, selfdecomposable ones (Proposition 2), and finally we specify our results to the moving average fractional Lévy processes (MAFLP).

We have decided to present our results for $\mathbb{R}^d$-valued random vectors. However, readers interested in infinite dimensional probability will notice that the proofs given below do not depend on the dimension and are valid for any real separable Banach or Hilbert space; cf. the concluding remark at the end of this paper.

1. Notations and definitions. Let ID and $ID_{\log}$ denote the class of all infinitely divisible probability measures on $\mathbb{R}^d$ and those that integrate the logarithmic function $\log(1 + ||x||)$, respectively. Further, let $*$ and $\Rightarrow$ stand for the convolution and the weak convergence of measures, respectively. Thus $(ID, *, \Rightarrow)$ becomes a closed convolution subsemigroup of the semigroup of all probability measures $\mathcal{P}$ (on $\mathbb{R}^d$).

Let $(Y_\nu(t), t \geq 0)$ denote a Lévy process, i.e., a stochastic process with stationary independent increments, starting from zero, and with paths that are continuous from the right and with finite left limits (in short: cadlag), such that ν is its probability distribution at time 1: $\mathcal{L}(Y_\nu(1)) = \nu$, where ν can be any ID probability measure. Throughout the paper $\mathcal{L}(X)$ will denote the probability distribution of an $\mathbb{R}^d$-valued random vector X. Furthermore, for a Borel probability measures μ its *characteristic function* $\hat{\mu}$ is defined as

$$\hat{\mu}(y) := \int_{\mathbb{R}^d} e^{i<y,x>}\mu(dx), \;\; y \in \mathbb{R}^d,$$

where $< \cdot, \cdot >$ denotes the scalar product. For measures $\mu \in ID$ their characteristic functions admit the following *Lévy-Khintchine representation*:

$\hat{\mu}(y) = e^{\Phi(y)},\ y \in \mathbb{R}^d$, and the Lévy exponents Φ are of the form

$$\Phi(y) = i < y, a > -\frac{1}{2} < y, Ry > + \int_{\mathbb{R}^d \setminus \{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M(dx), \quad (3)$$

where a is a *shift vector*, R is a *covariance operator* corresponding to the Gaussian part of μ, 1_B is the indicator function of the unit ball B and M is a *Lévy spectral measure*. Since there is a one-to-one correspondence between a measure $\mu \in ID$ and the triple a, R and M in (3) we will write $\mu = [a, R, M]$.

For h of bounded variation, cadlag Lévy process Y and monotone r we define here *the random integral* $(\star)$ by formal integration by parts formula, that is as follows:

$$\int_{(a,b]} h(t)dY(r(t)) := h(b)Y(r(b)) - h(a)Y(r(a)) - \int_{(a,b]} Y(r(t)-)dh(t)$$
$$= h(b)[Y(r(b)) - Y(r(a))] - \int_{(a,b]} (Y(r(t)-) - Y(r(a)))dh(t), \quad (4)$$

where $Y(r(t)-)$ denotes the left-hand limit. Consequently, for the partition $a = t_0 < t_1 < t_2 < ... < t_n = b$, the random integral (4) can be approximated by the Riemann-Stieltjes sums

$$h(b)[Y(r(b)) - Y(r(a))] - \sum_{j=1}^{n} Y(r(t_j))(h(t_j) - h(t_{j-1}))$$
$$= \sum_{j=1}^{n} h(t_j)(Y(r(t_j) - Y(r(t_{j-1})))). \quad (5)$$

Random integrals over half-lines are defined as limits almost surely (equivalently as limits in distribution or in probability) over finite intervals $(a, b]$ as $b \to \infty$; cf. Jurek and Vervaat (1983).

2. Results and proofs. For the three parameters in (2) (i.e., the functions h, r and the interval $(a, b]$), let $\mathcal{D}^{h,r}_{(a,b]}$ denote *the domain of definition* of the mapping $I^{h,r}_{(a,b]}$. That is, the set of those $\nu \in ID$ (Lévy processes $(Y_\nu(t), t \geq 0)$) such that the integral (2) is well defined. Then *the random integral mapping*

$$I^{h,r}_{(a,b]} \ :\ \mathcal{D}^{h,r}_{(a,b]} \longrightarrow ID, \quad (6)$$

is a homomorphism between the corresponding convolution semigroups. This is so, because approximating the integral ($\star$) (in (2)) by the Riemann-Stieltjes sums (5) we get

$$\log(\widehat{I^{h,r}_{(a,b]}(\nu)})(y) = \int_{(a,b]} \log\widehat{\nu}(h(t)y)dr(t), \ y \in \mathbb{R}^d; \tag{7}$$

cf. for more details Jurek-Vervaat (1983), Lemma 1.1 or Jurek and Mason (1993), Chapter 3.

Furthermore, from (7), we infer the following properties:

$$I^{h,r}_{(a,b]}(\nu_1)*I^{h,r}_{(a,b]}(\nu_2) = I^{h,r}_{(a,b]}(\nu_1*\nu_2), \quad I^{h,r}_{(a,b]}(T_u\nu) = I^{uh,r}_{(a,b]}(\nu) = T_u\big(I^{h,r}_{(a,b]}(\nu)\big) \tag{8}$$

$$I^{h,r}_{(a,b]}(\nu^{*s}) = (I^{h,r}_{(a,b]}(\nu))^{*s} = (I^{h,\,sr}_{(a,b]}(\nu)), \ I^{h,r}_{(a,b]\cup(b,c]}(\nu) = I^{h,r}_{(a,b]}(\nu) * I^{h,r}_{(b,c]}(\nu) \tag{9}$$

$$\text{if} \ \ \nu_n \Rightarrow \nu \ \ \text{then} \ \ I^{h,r}_{(a,b]}(\nu_n) \Rightarrow I^{h,r}_{(a,b]}(\nu), \tag{10}$$

where T_u is the dilation, i.e., $T_u(x) := ux$, $u \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $s > 0$. [Replacing the dilation T_u by a matrix (or a bounded linear operator) A in (8) we get $A(I^{h,r}_{(a,b]}(\nu)) = I^{h,r}_{(a,b]}(A\nu)$.]

Proposition 1. *Let $\mathcal{K}$ be a closed convolution subsemigroup of the semigroup ID (of all infinitely divisible measures) that is also closed under dilations and convolution powers (i.e., if $a \in \mathbb{R}$ and $\nu \in \mathcal{K}$ then $T_a\nu \in \mathcal{K}$ and for $c > 0$ also $\nu^{*c} \in \mathcal{K}$). Then if $\nu \in \mathcal{K} \cap \mathcal{D}^{h,r}_{(a,b]}$ then $I^{h,r}_{(a,b]}(\nu) \in \mathcal{K}$.*

The same holds also for improper random integrals (over half-lines or lines) provided they are well-defined.

Proof. The summands in (5) are independent and since $\nu = \mathcal{L}(Y(1)) \in \mathcal{K}$ we get that

$$\mathcal{L}[h(t_j)(Y(r(t_j) - Y(r(t_{j-1}))] = T_{h(t_j)}(\mathcal{L}(Y(1))^{*(r(t_j)-r(t_{j-1}))}) \in \mathcal{K},$$

if $r(t_j) - r(t_{j-1}) \geq 0$. Similarly, $\mathcal{L}[-h(t_j)(Y(r(t_{j-1}) - Y(r(t_j))] \in \mathcal{K}$ when $r(t_j)-r(t_{j-1}) \leq 0$. Closenesses and semigroup property of $\mathcal{K}$ guarantees that $I^{h,r}_{(a,b]}(\nu) \in \mathcal{K}$, which gives the proof of Proposition 1 for finite intervals $(a, b]$.

Since integrals on half-lines are given as weak limits of those over $(a, b]$ as $b \to \infty$ and $\mathcal{K}$ is closed in weak topology we get Proposition 1 for half-lines, which completes the proof.

Moreover, we have that

Corollary 1. *Domains of definition $\mathcal{D}^{h,r}_{(a,b]}$ of random integrals $I^{h,r}_{(a,b]}$ are examples of semigroups $\mathcal{K}$ from Proposition 1.*

For the purpose of this note we will consider two specific random integral mappings and their corresponding domains and ranges. Firstly, for $\beta > 0$ and $\nu \in ID$, let us define

$$I_{(0,1]}^{t,t^{\beta}}(\nu) \equiv \mathcal{J}^{\beta}(\nu) := \mathcal{L}\big(\int_{(0,1]} t\, dY_{\nu}(t^{\beta})\big), \quad \text{and} \quad \mathcal{U}_{\beta} := \mathcal{J}^{\beta}(ID). \qquad (11)$$

To the distributions from the semigroups $\mathcal{U}_{\beta}$ we refer to as *the generalized s-selfdecomposable distributions*; cf. the comment 4.2 below.

Secondly, for $\nu \in ID_{\log}$ let us put

$$I_{(0,\infty)}^{e^{-t},t}(\nu) \equiv \mathcal{I}(\nu) := \mathcal{L}\big(\int_{(0,\infty)} e^{-s}\, dY_{\nu}(s)\big) \quad \text{and} \quad L := \mathcal{I}(ID_{\log}). \qquad (12)$$

The distributions from the semigroup L are called *selfdecomposable* ones or *Lévy class L distributions*; cf. Jurek-Vervaat (1983), Theorem 2.3 or Jurek-Mason (1993), Chapter III; cf. the comments 4.3 and 4.4 below.

Between the classes L, $\mathcal{U}_{\beta}$ ($\beta > 0$), the class $\mathcal{G}$ of all Gaussian measures and the class $\mathcal{S}$ of all stable probability measures we have the following proper inclusions:

$$\mathcal{G} \subset \mathcal{S} \subset L \subset \mathcal{U}_{\beta} \subset ID, \quad \text{i.e.,} \quad \mathcal{I}(ID_{\log}) \subset \mathcal{J}^{\beta}(ID)\,. \qquad (13)$$

Example 1. *The classes L (of the selfdecomposable distributions), $\mathcal{U}_{\beta}$ (of the generalized s-selfdecomposable distributions) and $\mathcal{G}$ (of the Gaussian measures) are examples of the above class $\mathcal{K}$. Also the Urbanik class L_{∞}, that coincides with the smallest closed convolution semigroup generated by all stable distributions, is an example of the class $\mathcal{K}$; cf. Urbanik (1973), or Jurek (2004).*

From the inclusions in (13) we get that all selfdecomposable measures are generalized s-selfdecomposable ones whenever $\beta > 0$. With the notations $[a, R, M]$, described in (3), we give conditions for the converse claim.

Proposition 2. *Let $\nu = [b, S, N] \in ID$ and $\rho = [a, R, M] \in ID_{\log}$. Then the following conditions are equivalent:*

(i) $\mathcal{J}^{\beta}(\nu) = \mathcal{I}(\rho)$, i.e., a generalized s-selfdecomposable measure is in fact a selfdecomposable one;

(ii) $\nu = \rho^{*1/\beta} * \mathcal{I}(\rho)$ and, consequently, $\nu \in ID_{\log}$;

(iii) $\mathbb{R}^d \ni y \to \exp \beta\big[\log \hat{\nu}(y) - \beta \int_0^1 \log \hat{\nu}(t\,y) t^{\beta-1} dt\big]$ is a Fourier transform of an $ID_{\log}$ measure;

(iv) $\int_0^1 \big(N(A) - N(s^{-1}A)\big)s^{\beta-1}ds \geq 0$ *for all Borel sets* A *such that* $0 \notin A$ *and* $\int_{(||x||>1)} \log||x||N(dx) < \infty$.

For the proof we need the following

Lemma 1. *(i) If* $\nu = [b, S, N]$ *and* $\mathcal{J}^\beta(\nu) = [b^{(\beta)}, S^{(\beta)}, N^{(\beta)}]$ *then*

$$b^{(\beta)} := \frac{\beta}{\beta+1}\,(b + \int_{(||x||>1)} x\,||x||^{-1-\beta}N(dx)\,); \quad S^{(\beta)} := \tfrac{\beta}{2+\beta}\,S;$$
$$N^{(\beta)}(A) := \int_0^1 N(t^{-1/\beta}A)\,dt \, = \beta\int_0^1 N(s^{-1}A)s^{\beta-1}ds \ \ \textit{for each } A \in \mathcal{B}_0$$

(ii) If $\mu = [a, R, M]$ *and* $\mathcal{I}(\mu) = [a^\sim, R^\sim, M^\sim]$ *then*

$$a^\sim := a + \int_{(||x||>1)} x\,||x||^{-1}M(dx); \quad R^\sim := \frac{1}{2}\,R$$
$$M^\sim(A) := \int_0^\infty M(e^t A)dt = \int_0^1 M(t^{-1}A)\,t^{-1}dt \quad \textit{for each } A \in \mathcal{B}_0;$$

cf. Czyżewska-Jankowska and Jurek (2011), Lemma 2 (for part (i)) and Jurek-Vervaat (1983) p. 250 (for part (ii)) for more details.

Proof of Proposition 2. $(i) \equiv (ii)$. Let us put $\Phi(y) := \log\hat{\nu}(y)$ and $\Psi(y) := \log\hat{\rho}(y)$, i.e., they are the corresponding Lévy exponents. Then using (7), (11) and (12) we infer that (i) is equivalent to the following identity

$$\beta\int_0^1 \Phi(ty)t^{\beta-1}dt = \int_0^\infty \Psi(e^{-s}y)ds = \int_0^1 \Psi(ty)\frac{dt}{t}, \quad \text{for all y} \in \mathbb{R}^d. \quad (14)$$

Putting into above sy for y, where $s \in \mathbb{R}$ varies and y is fixed, and then substituting $w := st$ we get

$$\int_0^s \Phi(wy)w^{\beta-1}dw = \beta^{-1}s^\beta \int_0^s \Psi(wy)\frac{dw}{w}.$$

Differentiating with respect to s and then putting $s = 1$ we arrive at

$$\Phi(y) = \int_0^1 \Psi(wy)\frac{dw}{w} + \beta^{-1}\Psi(y), \quad \text{for all y}, \quad (15)$$

and after exponentiating both sides we get the equality (ii) in terms of Fourier transforms.

Conversely, starting with (15) and substituting ty for y and then integrating both sides over the unit interval with respect to dt^β we arrive at

$$\beta \int_0^1 \Phi(ty)t^{\beta-1}dt = \int_0^1 \Psi(ty)\frac{dt}{t}$$

which means that $\mathcal{J}^\beta(\nu) = \mathcal{I}(\rho)$.

[(i) and (ii)]$\Rightarrow$ (iii). Substituting $\mathcal{J}^\beta(\nu)$ for $\mathcal{I}(\rho)$ in (ii) and then taking Fourier transforms on both sides we get that $\hat{\rho}$ has the form as in (iii).

$(iii) \Rightarrow (iv)$. Assume $\rho \in ID_{\log}$ has Fourier transform given by (iii). Then $\rho^{*1/\beta} \in ID_{\log}$ and, by Lemma 1, its Lévy spectral measure is of the form

$$N(A) - N^{(\beta)}(A) = \beta \int_0^1 (N(A) - N(t^{-1}A)\, t^{\beta-1}dt \geq 0 \ \text{ for all } \ A \in \mathcal{B}_0$$

which is the claim (iv).

$(iv) \Rightarrow (i)$. Multiplying (iv) by β, and using the notation from Lemma 1 (i), we have that $0 \leq N^{(\beta)} \leq N$. Consequently, $N - N^{(\beta)}$ is a Lévy spectral measure with finite log-moment; cf. Czyżewska-Jankowska and Jurek (2011), Lemma 2 (ii). [In case of $\mathbb{R}^d$ or Hilbert space H that is trivial because Lévy measures are characterized by the integrability condition (21).]

Using Lemma 1(ii) we will check how $N - N^{(\beta)}$ behaves under mapping $\mathcal{I}$ or under the operation $\sim$. Note that

$$(\beta(N - N^{(\beta)}))^{\sim}(A) = \beta(\int_0^1 N(t^{-1}A)t^{-1}\, dt - \int_0^1 \int_0^1 N(t^{-1/\beta}s^{-1}A)dt\, s^{-1}ds)$$
$$= \beta(\int_0^1 N(t^{-1}A)t^{-1}\, dt - \int_0^1 (\int_0^s N(w^{-1}A)\beta w^{\beta-1}dw)s^{-(\beta+1)}ds)$$
$$= \beta(\int_0^1 N(t^{-1}A)t^{-1}\, dt - \int_0^1 N(w^{-1}A)w^{\beta-1}(w^{-\beta} - 1)dw)$$
$$= \beta \int_0^1 N(w^{-1}A)w^{\beta-1}dw = N^{(\beta)}(A).$$

Similarly, using Lemma 1 (ii), for the Gaussian covariance operator we have

$$(\beta(S - S^{(\beta)}))^{\sim} = (2\beta(\beta+2)^{-1}S)^{\sim} = \beta(\beta+2)^{-1}S = S^{(\beta)}.$$

Finally, applying (ii) in Lemma 1 for the shift vectors we get

$$(\beta(b-b^{(\beta)}))^{\sim}=\beta(b-b^{(\beta)})+\beta\int_{(||x||>1)}\frac{x}{||x||}N(dx)-\beta\int_{(||x||>1)}\frac{x}{||x||}N^{(\beta)}(dx)$$

$$=\beta(b-b^{(\beta)})+\beta\int_{(||x||>1)}\frac{x}{||x||}N(dx)-\beta^2\int_0^1\int_{(||x||>1)}\frac{x}{||x||}N(s^{-1}dx)s^{\beta-1}ds$$

$$=\beta(b-b^{(\beta)})+\beta\int_{(||x||>1)}\frac{x}{||x||}N(dx)-\beta\int_{(||w||>1)}\frac{w}{||w||}\int_{||w||^{-1}}^1\beta s^{\beta-1}dsN(dw)$$

$$=\beta(b-b^{(\beta)})+\beta\int_{(||x||>1)}\frac{x}{||x||}N(dx)-\beta[\int_{(||w||>1)}\frac{w}{||w||}N(dw)-\int_{(||w||>1)}\frac{w}{||w||^{1+\beta}}N(dw)]$$

$$=\beta(b+\int_{(||w||>1)}\frac{w}{||w||^{1+\beta}}N(dw)\ -b^{(\beta)})=\beta((\beta+1)\beta^{-1}b^{(\beta)}-b^{(\beta)})=b^{(\beta)}.$$

All in all we have that $\rho=[\,\beta(b-b^{(\beta)}),\beta(S-S^{(\beta)}),\beta(N-N^{(\beta)})\,]\in ID_{\log}$ and $\mathcal{I}(\rho)=\mathcal{J}^{\beta}(\nu)$, which completes the proof of $(iv)\Rightarrow(i)$ and thus the proof of Proposition 2.

Here we have the above condition (i) from Proposition 2 in terms of the triples from Lévy -Khintchine representation :

Corollary 2. *In order that* $\mathcal{J}^{\beta}([b,S,N])=\mathcal{I}([a,R,M])$ *it is necessary and sufficient that*

$$b=(\beta+1)\beta^{-1}\,a+\int_{(||x||>1)}x\,||x||^{-1}M(dx)\quad and\quad S=(\beta+2)(2\beta)^{-1}\,R$$

$$and\quad N(A)=\beta^{-1}M(A)+\int_0^1 M(t^{-1}A)t^{-1}dt\quad for\ all\quad A\in\mathcal{B}_0.$$

3. The case of MAFLP. Now we will specify our considerations to the case of MAFLP $Z(t)$ given in (1). First of all, note that similarly as in (1), for a Lévy process $(Y_\nu(t),t\geq 0)$, putting

$$U^{(\nu)}(t):=\int_{-\infty}^0((t-s)^{\alpha}-(-s)^{\alpha})dY_\nu(s)\text{ and }V^{(\nu)}(t):=\int_0^t(t-s)^{\alpha}dY_\nu(s)\quad(16)$$

we get that

$$Z(t)=U^{(\nu)}(t)+V^{(\nu)}(t)\ \text{ and the summands are independent.}\qquad(17)$$

This is so because a two-sided Lévy process (i.e., with time index in $\mathbb{R}$) is defined by taking independent copies of Lévy processes on both half-lines; cf. Marquardt (2006), p. 1102.

Furthermore, using the invariance principle for Lévy processes, that is, the property that for each fixed positive t we have

$$\big(-Y_\nu(-s), 0 \le s \le t\big) \stackrel{d}{=} \big(Y_\nu(s), 0 \le s \le t\big) \stackrel{d}{=} \big(Y_\nu(t) - Y_\nu(t-s)), 0 \le s \le t\big)$$

(the equality in distribution of three Lévy processes) we infer that

$$U^{(\nu)}(t) \stackrel{d}{=} \int_0^\infty \big((t+s)^\alpha - s^\alpha\big) dY_\nu(s), \qquad V^{(\nu)}(t) \stackrel{d}{=} \int_0^t s^\alpha dY_\nu(s). \tag{18}$$

Of course, from (18) and 11) we have that

$$I^{s,\, s^{1/\alpha}}_{(0,1]}(\nu) = I^{s^\alpha, s}_{(0,1]}(\nu) = \mathcal{L}(V^{(\nu)}(1)) \in \mathcal{U}_{1/\alpha} \quad \text{and} \quad 2 < 1/\alpha.$$

Then for $t > 0$, the above with (8), (9) and Example 1 (for the class $\mathcal{U}_\beta$) give

$$\begin{aligned} T_{t^\alpha}\big[\big(I^{s^\alpha, s}_{(0,1]}(\nu)\big)^{*t}\big] &= T_{t^\alpha}\big[I^{s^\alpha, t\, s}_{(0,1]}(\nu)\big] \\ &= I^{(t\,s)^\alpha, t\, s}_{(0,1]}(\nu) = I^{s^\alpha, s}_{(0,t]}(\nu) = \mathcal{L}(V^{(\nu)}(t)) \in \mathcal{U}_{1/\alpha}, \end{aligned} \tag{19}$$

and consequently we get

Corollary 3. *For all infinitely divisible measures ν, probability distributions of $V^{(\nu)}(t)$ are in the class $\mathcal{U}_{1/\alpha}$ of generalized s-selfdecomposable probability measures with $1/\alpha > 2$.*

In Jurek (1988) there are given different characterizations of the classes $\mathcal{U}_\beta$.

In (18) integrals $U^{(\nu)}(t)$ over half-line are defined as limits, i.e.,

$$U^{(\nu)}(t) = \lim_{b\to\infty} U^{(\nu),b}(t) := \lim_{b\to\infty} \int_{(0,b]} \big((t+s)^\alpha - s^\alpha\big) dY_\nu(s) \tag{20}$$

a.s. or in distribution. Because of (16) and (18), $U^{(\nu),\,b}(t)$ and $V^{(\nu)}(t)$ are stochastically independent and $\lim_{b\to\infty}[U^{(\nu),b}(t) + V^{(\nu)}(t)] = Z(t)$.

Since an integral $U^{(\nu),\,b}(t)$ is of the form $I^{h,r}_{(a,b]}$ we may apply Proposition 1 and get properties of marginal distributions of MAFLP summarized as follows:

Corollary 4. *Let $\mathcal{K}$ be a closed convolution semigroup of infinitely divisible measures that is also closed under dilations and convolution powers (i.e., if $c > 0$ and $\nu \in \mathcal{K}$ then $T_c\nu \in \mathcal{K}$ and $\nu^{*c} \in \mathcal{K}$). Then*

(a) if $\nu \in \mathcal{K}$ then $\mathcal{L}[U^{(\nu),b}(t) + V^{(\nu)}(t)] \in \mathcal{K}$ for all $t > 0$;

(b) if $\nu \in \mathcal{K}$ and MAFLP $Z(.)$ is well defined then its marginal distributions $\mathcal{L}(Z(t)) \in \mathcal{K}$ for all $t > 0$;

If we restrict our consideration to a real separable Hilbert spaces then the Lévy-Khintchine representation (3) is strengthened by the following characterization:

$$[\text{M is a Lévy spectral measure}] \ \text{ iff } \ [\int_H (1 \wedge ||x||^2) M(dx) < \infty]; \qquad (21)$$

cf. Parthasarathy (1967), Chapter VI. Consequently, part (b) in Corollary 4 for subclasses $\mathcal{K} \subset ID(H)$ can be improved as follows: if $\nu \in \mathcal{K}$ and has finite variance then MAFLP $Z(.)$ is well defined and $\mathcal{L}(Z(t)) \in \mathcal{K}$ for all $t > 0$. For this we may repeat the proof from Euclidean spaces.

Remark 1. The part (b)of Corollary 4 for $\mathcal{K} = L$ (the class of selfdecomposable measures) on $\mathbb{R}^d$ with finite variance was also shown in Cohen and Maejima (2011). However, note that their example in Proposition 3.3 has infinite variance and thus it does not serve the purpose they intended to, that is, to have that $\mathcal{J}^\beta(\nu) \in L$ with $\nu \notin L$. To have that phenomena one has to take $\nu = \rho^{*1/\beta} * \mathcal{I}(\rho) \notin L$; cf. Proposition 2 (ii).

4. Bibliographical comments and concluding remarks.

4.1. Cohen and Maejima (2011) defined the integral (1) in the same way as it was in Marquardt (2006); see also the reference therein. In particular they worked in the framework of Lévy processes with finite variance, square integrable functions and Euclidean spaces. However, using the formal integration by parts we are able to define random integrals for larger class of integrands h and Lévy processes Y. Moreover, still having the crucial equality (7).

4.2. The classes $\mathcal{U}_\beta$ were already introduced in Jurek (1988) as the limiting distributions for certain sums of independent variables. The terminology has its origin in the fact that distributions from the class $\mathcal{U}_1 \equiv \mathcal{U}$ were called *s-selfdecomposable distribution* (the "*s*-", stands here for *the shrinking operations* that were used originally in the definition of $\mathcal{U}$); cf. Jurek (1981), (1985), (1988) and references therein.

4.3. In classical probability theory selfdecomposability is usually defined via some decomposability property or as a certain class of limit distributions. However, since Jurek-Vervaat (1983) we know that the class L coincides with the class of distributions of random integrals given in (9). Hence it is used here as its definition.

4.4. It might be of interest to recall here that many classical distributions in mathematical statistics such as gamma, t-Student, Fisher F etc. are in the class L but, of course, they are not stable; cf. the survey article Jurek (1997) or Jurek-Yor (2004) or the book by Bondesson (1992).

Concluding remark. Last but not least, to enhance the potential readability of this note we presented our results for the Euclidean space $\mathbb{R}^d$. However, our methods and proofs are applicable for processes and random variables with values in any real separable Banach space. For an exposition of probability on Banach spaces see Araujo-Giné (1980) or Ledoux-Talagrand (1991) and for a case of Hilbert spaces we recommend Parthasarathy (1967), Chapter VI. In particular, the crucial Lévy-Khintchine representation (3) holds true in the generality of separable infinite dimensional Banach spaces. But there is no integrability criterium like (21) for Lévy (spectral) measures.

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