1-D ISING MODELS, COMPOUND GEOMETRIC DISTRIBUTIONS AND SELFDECOMPOSABILITY*

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ABSTRACT. It is shown that the inverse of the partition function in 1-D Ising model, as a function of the external field, is a product of Fourier transforms of compound geometric distributions. These are random sums (randomly stopped random walk) with the probability of a success depending *only* on the interaction constant K between sites. Moreover, it is proved that those distribution belong to the Lévy class L of selfdecomposable probability measures, therefore they have the BDLP's, i.e., the background driving Lévy processes. It is important that the general structure of class L characteristic functions is well-known and that it is much more specific than the Lévy-Khintchine formula for infinite divisible variables.

Key words and phrases: 1-D Ising model; the partition function; com- pound geometric distributions; class L probability distributions; selfdecomposability property.

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1. An introduction.

The classical one-dimensional Ising model (in short: 1-D Ising) is far from any physical reality. Nevertheless it is usually presented as an introduction to more complex models, like the Onsager's solution to 2D Ising, and serves as the "testing grounds" for new hypotheses. It is the aim of this note to investigate the possible use of randomly stopped random walks (sums of independent identically distributed random variables) in this context. We will show that, in 1-D Ising, the inverse of the partition function, as a function of the external field, is a product of characteristic functions (Fourier transforms) of compound geometric distributions. In fact, those products are charactersitic functions of *selfdecomposbale (or Lévy class L) probability distributions*; cf. Appendix. One may expect that this point of view will provide new approaches to more complex physical models since the class L is quite rich and well understood in probability theory (also for Banach space valued selfdecomposable random variables). In particular, characteristic functions of class L probability distributions have known structure that is closed under multiplication and taking limits. Thus we have a general form of the free-energy functional. In the recent paper of De Coninck and Jurek (2000) has been introduced whole class of $\mathcal{L}$ Ising models via the property of selfdecomposability of class L probability measures. Moreover, Comets and Neveu (1995) gave new proof of the result of Aizenman, Lebowitz and Ruelle (1987) that in the Sherrington-Kirkpatrick model of spin glasses, in thermodynamical limit, one gets the log-normal distribution. But the log-normal distribution *is* also an element of the class L of selfdecomposable mesures. Furthermore recall that class L characteristic functions implicitly appeared in Newman (1974); for details see Section 4 below. Of course, the selfdecomposability implies the infinite divisibility. Infinite divisibility of Gibbs mesaures was studied, among others, by Waymire (1986). But our studies of selfdecomposability are somewhat indirectly related to Gibbs measures, namely via the inverse of the partition functions. Finally, at present we do not know any physical consequences of our analytical and probabilistic theorems.

2. The partition function in 1-D Ising model.

Consider N sites on a line with spin variables $\sigma_1, \sigma_2, \ldots, \sigma_N$, and assume the nearest-neighbour, pair-wise interactions. Let k_B denotes the Boltzman constant and T temperature, let J be the interaction between sites and H represents the external field. Using new variables (units)

$$K := J/k_B T \quad \text{and} \quad h := H/k_B T, \tag{1}$$

we write the corresponding *partition function* as

$$\widetilde{Z}_N(K,h) := \sum_{\sigma_1,\sigma_2,\ldots,\sigma_N} \exp[K\sum_{j=1}^{N-1}\sigma_j\sigma_{j+1} + h\sum_{j=1}^{N}\sigma_j], \tag{2}$$

where $\sigma_i = \pm 1, i = 1,2,\ldots,N$, and the sum is taken over all possible 2^N configurations of $\{\pm 1\}$. However, usally one "modifies" the partition function as follows:

$$Z_N(K,h) := \sum_{\sigma_1,\sigma_2,\ldots,\sigma_N} \exp[K\sum_{j=1}^{N}\sigma_j\sigma_{j+1} + \frac{1}{2}h\sum_{j=1}^{N}(\sigma_j + \sigma_{j+1})], \tag{3}$$

with the convention that

$$\sigma_{n+1} := \sigma_1 \quad (\textit{i.e., one considers } N \textit{ sites on a circle}). \tag{4}$$

Cf. Baxter (1982), pp.32-35. Thus each site has two neighbours and the exponent in (3) is a symmetric as a function in σ_j and σ_{j+1}. More precisely, defininig *the transfer matrix* $\mathbf{V} \ni (V(\sigma,\sigma'))$, where

$$V(\sigma,\sigma') := exp[K\sigma\sigma' + \frac{1}{2}h(\sigma+\sigma')],$$

$(\sigma,\sigma' \in \{\pm 1\})$, one may write that

$$Z_N = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_N} V(\sigma_1,\sigma_2)V(\sigma_2,\sigma_3)\ldots V(\sigma_{N-1},\sigma_N)V(\sigma_N,\sigma_1)$$
$$= tr\mathbf{V}^N. \tag{5}$$

Here *tr* denotes the trace functional (of a matrix) and the transfer matrix $\mathbf{V}$ is equal

$$\mathbf{V_{K,h}} = \mathbf{V} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}, \tag{6}$$

cf. Baxter(1982), p.32. Since $\mathbf{V}$ is a symmetric matrix it can be diagonalized (in the orthogonal base of its eigenvectors) and therefore we have

$$Z_N = tr\Big[\mathbf{P}\begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix}\mathbf{P}^{-1}\Big] = \lambda_1^N + \lambda_2^N, \tag{7}$$

where $\mathbf{P}$ is the matrix of (orthogonal) eigenvectors and λ_1 and λ_2 are the corresponding eigenvalues (these are the solutions to the characterstic polynomial equation

$$det(\mathbf{V} - \lambda I) = \lambda^2 - (2e^K \cosh h)\lambda + 2\sinh 2K = 0),$$

are given explicitly by the formula

$$\lambda_{1/2} = e^K \cosh h \pm \sqrt{e^{2K}\sinh^2 h + e^{-2K}}, \tag{8}$$

where + sign corresponds to λ_1 and $-$ to λ_2.

THEOREM 1. *For the transfer matrix* $\mathbf{V}_{K,h}$*, where* K *is the interaction variable and* h *is the external field, and* $N \geq 1$ *we have*

$$Z_N(K,h) = tr\mathbf{V}^N_{K,h} =$$
$$e^{NK}2^{-1}\binom{N}{2[N/2]}^{-1}\cosh^N h \prod_{j=1}^{[N/2]}\left(\frac{2}{1+\cos\theta_{j,N}} - \frac{1-e^{-4K}}{\cosh^2 h}\right)$$

where $\theta_{j,N} = \frac{2j-1}{N}\pi, j = 1, 2, \ldots, [N/2]$ *and* $[r]$ *denotes the integer part of real* r*.*

As a consequence of the above factorization we get the following main result:

THEOREM 2. *For any real* $K > 0$ *and any natural* $N \geq 1$*, the functions*

$$\phi_{N,K}(h) := \frac{Z_N(K,0)}{Z_N(K,h)} = \frac{tr\mathbf{V}^N_{K,0}}{tr\mathbf{V}^N_{K,h}}$$
$$= \frac{1}{\cosh^{N-2[N/2]} h}\prod_{j=1}^{[N/2]}\frac{1-\frac{1}{2}(1-e^{-4K})(1+\cos\theta_{j,N})}{\cosh^2 h - \frac{1}{2}(1-e^{-4K})(1+\cos\theta_{j,N})},$$

as functions of the external field h*, are characteristic functions of the class* L *distributions. Moreover, for* N *even they correspond to the sum of* $[N/2]$ *compound geometric distributions with the innovation distribution having the characteristic function* $1/\cosh^2 h$ *and the probabilities of a failure in Bernoulli trials are*

$$q_{j,N} := \frac{1}{2}(1-e^{-4K})(1+\cos\theta_{j,N});\ \theta_{j,N} = \frac{2j-1}{N}\pi,\ j = 1, 2, \ldots, [N/2].$$

In case of an odd N *, one needs to multiply the above by a characteristic function* $(\cosh h)^{-1}$.

The ratio of the partition functions in Theorem 2 has the following interpretation in terms of randomly stopped random walk; cf. formulae (9),(10) and (11) below.

COROLLARY 1. *Let* $Y,\quad X_1, X_2, \ldots,\quad G_{p_1}, G_{p_2}, \ldots$ *be a collection of independent random variables, where* Y *has the characteristc function* $1/\cosh h$, *further* $X_1, X_2, \ldots$ *are identically distributed with the characteristic function* $1/\cosh^2 h$ *and* $G_{p_1}, G_{p_2}, \ldots$ *have geometric distributions given by (9). Then the random variable*

$$\begin{aligned}&(N-2[N/2])Y + X_1 + X_2 + \ldots + X_{G_{p_1}} + \\ &X_{G_{p_1}+1} + X_{G_{p_1}+2} + \ldots + X_{G_{p_1}+G_{p_2}} + \ldots + \\ &\qquad\qquad X_{G_{p_1}+\ldots+G_{p_{[N/2]-1}}+1} + \ldots + X_{G_{p_1}+\ldots+G_{p_{[N/2]-1}}+G_{p_{[N/2]}}}\end{aligned}$$

with $p_j := 1-q_{j,N}, j = 1, 2, \ldots [N/2]$, *has the charactersitic function* $\phi_{N,K}(h)$ *from Theorem 2.*

3. Proofs and the compound geometric distributions.

We begin with the following auxilliary lemma.

LEMMA 1. *For a natural* $N \geq 2$, *the function*

$$f_N(x) := (1+\sqrt{1-x})^N + (1-\sqrt{1-x})^N$$

is a polynomial of $[N/2]$ *degree and admits a factorization*

$$f_N(x) = 2^{-1}\binom{N}{2[N/2]}^{-1}\prod_{j=1}^{[N/2]}\Big(\frac{2}{1+\cos\theta_{j,N}} - x\Big),$$

where $\theta_{j,N} = \frac{2j-1}{N}\pi, j = 1, 2, \ldots, [N/2]$.

Proof. Note that the zeros of the polynomial $f_N(x) = 0$ coincide with the solutions to the equation

$$\Big(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\Big)^N = -1 = e^{i\pi(1+2k)}, \quad \text{for } k = 0, \pm1, \pm2, \ldots.$$

Putting $\rho_k\colon = e^{i\pi(2k-1)/N}$, for $k = 1, 2, \ldots, N$, we see that the above equation has roots

$$x_k = \frac{4\rho_k}{1+2\rho_k+\rho_k^2} = \frac{4}{e^{-i\pi(2k-1)/N}+2+e^{i\pi(2k-1)/N}} = \frac{2}{1+\cos\theta_{k,N}},$$

for $k = 1, 2, \ldots, [N/2]$, which completes the proof.

$\square$

Proof of Theorem 1. Note that by formulae (5),(7),(8), the identity $\sinh^2 h = \cosh^2 h - 1$ and Lemma 1 we get

$$tr\mathbf{V}_{K,h}^N = \lambda_1^N + \lambda_2^N = (e^K \cosh h)^N f_N\Big(\frac{1-e^{-4K}}{\cosh^2 h}\Big).$$

Hence the formula in Theorem 1 follows, which completes the proof.

Before the proof of next result let us introduce the notion of compound geometric distribution. For a sequence $X_n, n \geq 1$, of independent copies of X (the innovation rv), the *random walk* is the sequence S_m given by

$$S_m = X_1 + X_2 + \ldots + X_m, \quad m = 1, 2, \ldots.$$

For $0 < p < 1$, let G_p be the *geometric random variable.* It is the *moment of the* first *success* in the Bernoulli trials (this is a sequence of independent identically distributed values of +1 (success) or 0 (failure), where $0 < p < 1$ is the probability of a success in one trial.) Hence

$$\mathcal{P}\{G_p = k\} = (1-p)^{k-1}p, \quad for \quad k = 1, 2, \ldots. \tag{9}$$

For a sequence of *i.i.d.* (X_n) and a geometric rv G_p independent of (X_n), we define the *geometrization of* X by

$$geom(p, X) := X_1 + X_2 + \ldots + X_{G_p}, \tag{10}$$

This is a random walk S_m stopped at the random time G_p. Its distribition is called *compound geometric distribution.* If ϕ_X denotes the *characteristic function* of rv X, that is

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{+\infty} e^{itx} dF(x),$$

where F is the cumulative distribution function of X, then

$$\phi_{geom(p,X)}(t) = \phi_X(t)\frac{1-q}{1-q\phi_X(t)}. \tag{11}$$

In the sequel we also write $geom(p, F)$ or $geom(p, \psi)$ for $geom(p, X)$ when F and ψ are the cumulative distribution function and the characteristic function of X, respectively.

LEMMA 2. *The characteristic function* $1/\cosh h$ *is in the class* L. *Also the two compond geometric distributions*

$$geom(p, 1/\cosh h) \quad and \quad geom(p, 1/\cosh^2 h) \quad are \quad in \quad class \quad L.$$

Proof. Let choose $\alpha \in (0, \pi/2)$ such that $\cos\alpha = q = 1 - p$. From (11) and the formula 1.438 in Gradshteyn and Ryzhik (1994) we get

$$\phi_{geom(p,(\cosh h)^{-1})}(t) = \frac{1-\cos\alpha}{\cosh t - \cos\alpha} = \prod_{k=-\infty}^{\infty} \Big[1 + \Big(\frac{t}{2k\pi + \alpha}\Big)^2\Big]^{-1}. \quad (12)$$

Let η_k be a sequence of i.i.d. *Laplace* rv's, (they have probability density $2^{-1}\exp[-|x|]$ and $x \in \mathbb{R}$ and characteristic functions $\phi_\eta(t) = (1+t^2)^{-1}$ that are in class L) and a_k be a sequence of square summable real numbers. Then series $\sum_{k=1}^{\infty} a_k \eta_k$ converges almost surely, say to S (or in probability , or in distribution) and S has probability distribution in class L because

$$\phi_S(t) = \prod_{k=1}^{\infty}[1 + a_k^2 t^2]^{-1} \quad \text{and} \quad (\cosh t)^{-1} = \prod_{k=1}^{\infty}\Big[1 + \frac{4t^2}{(2k-1)^2}\Big]^{-1};$$

Similarly from the formula 1.438 in Gradshteyn and Ryzhik (1994), taking for α the angle $\alpha + \pi$, we obtain

$$\frac{1+\cos\alpha}{\cosh t + \cos\alpha} = \prod_{k=-\infty}^{\infty}\Big[1 + \Big(\frac{t}{(2k+1)\pi + \alpha}\Big)^2\Big]^{-1}. \quad (13)$$

Hence for $q := \cos^2\alpha$ we conclude

$$\begin{aligned}\phi_{geom(p,1/\cosh^2 h)}(t) &= \frac{1-\cos^2\alpha}{\cosh^2 t - \cos^2\alpha} \\ &= \prod_{k=-\infty}^{\infty}\Big[1 + \Big(\frac{t}{2k\pi+\alpha}\Big)^2\Big]^{-1} \cdot \prod_{k=-\infty}^{\infty}\Big[1 + \Big(\frac{t}{(2k+1)\pi+\alpha}\Big)^2\Big]^{-1} \\ &= \prod_{k=-\infty}^{\infty}\Big[1 + \Big(\frac{t}{k\pi+\alpha}\Big)^2\Big]^{-1}, \quad (14)\end{aligned}$$

is a characteristic function from the class L. More precisely it is the characteristic function of the random series $\sum_{k=-\infty}^{\infty}(k\pi+\alpha)^{-1}\eta_k$. Thus the proof of Lemma 2 is complete. $\square$

Proof of Theorem 2. From Theorem 1 follows the formula for $\phi_{N,K}(h)$ as well as the formula for $q_{j,N}$ in the Bernoulli trials. From Lemma 2 we see that each term in the product is a class L characteristic function. More precisely it is the compound geometric distribution with the innovation characteristic function $1/\cosh^2 h \in L$. Consequently the semigroup property of L (closure under product of characteristic functions) gives $\phi_{N,K} \in L$, which completes the argument.

Proof of Corollary 1. Let X be a random variable with the probability distribution as $X_1, X_2, \ldots$. Let us consider only two "blocks" of the variable in Corollary 1, i.e., $X_1 + X_2 + ... + X_{G_{p_1}} + X_{G_{p_1}+1} + X_{G_{p_1}+2} + \ldots + X_{G_{p_1}+G_{p_2}}$ and denote it by T. Then using conditional expectation we have

$$\mathbb{E}[e^{itT}] = \mathbb{E}[\mathbb{E}[\mathbb{E}[e^{itT}|G_{p_1}]|G_{p_2}]] =$$
$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mathbb{E}[e^{it(X_1+\ldots+X_j+\ldots X_{j+k})}]\mathcal{P}\{G_{p_1}=j\}\mathcal{P}\{G_{p_2}=k\} =$$
$$\phi_{geom(p_1,X)}(t)\phi_{geom(p_2,X)}(t),$$

which together with Theorm 2 completes the proof.

4.Examples.

A. The Lee-Yang Theorem on the zeros of the partition function is an important tool in the rigorous study of phase transitions in lattice spin systems. Morever, it helps to prove inequalities for critical exponents as well the correlation inequalities. C.Newman (1974) proved the Lee-Yang property for quite general class of Ising models. He begins with a specific set $\mathcal{I}$ of *signed* measures μ, as the distributions of the spin variables, with prescribed form of the moment generating function. Then shows $\mathcal{I}$ has the Lee-Yang property, cf.Theorem 2.2., that $\mathcal{I}$ is convolution semigroup, Lemma 2.2., and finally that measures from closure of $\mathcal{I}$ have the Lee-Yang property as well. [Note here that the Lévy class L of selfdecomposable probability distributions is also a closed convolution semigroup !]. Newman gives examples of moment generating function $E_\mu(z)$ (this is his notation) from the class $\mathcal{I}$. However, the inverses of his many of partition functions, normalized to be 1 at zero, lead to the class L probability measures. In particular, by the formula (12) and the argument below it, we have that :

$$\frac{E_\mu(0)}{E_\mu(z)} = \frac{1}{\cosh z} \in L, \quad \frac{E_\mu(0)}{E_\mu(z)} = \frac{1-e^{-2J_{12}}}{\cosh(2z/a)-e^{-2J_{12}}} \in L,$$

for the Examples 2.1 and 2.3. The same claim is true for the Examples 2.4 and 2.5, from Newman (1975), but they are in the "closure" of $\mathcal{I}$, denoted by $\mathcal{N}$. More explicitly, for Example 2.5 we have

$$\frac{E_\mu(0)}{E_\mu(z)} = \frac{z}{\sinh z} \in L,$$

by Corollary 3 in [5].

Finally we would like to remark that another proof of Newman's result, concerning the Lee-Yang property, was given in Lieb and Sokal (1981), in their Appendix A .

B. Before proving Theorem 2 , our original aim was to work with the partition function $\widetilde{Z}_N(K, h)$ given by (2), thus concerning models without periodic boundary conditions. Using straightforward calculations one can obtain formulae :

$$\tilde{\phi}_{2,K}(h) := \frac{\tilde{Z}_2(K,0)}{\tilde{Z}_2(K,h)} = \frac{e^K + e^{-K}}{e^K \cosh 2h + e^{-K}} = geom(p_{(2)}, 1/\cosh^2 h) \in L, \quad (15)$$

by Lemma 2, where $p_{(2)} := (1 + e^{-2K})/2$. Similarly, for N=3 sites we get

$$\tilde{\phi}_{3,K}(h) := \frac{\tilde{Z}_3(K,0)}{\tilde{Z}_3(K,h)} = \frac{(e^K + e^{-K})^2}{e^{2K} \cosh 3h + (e^{-2K} + 2) \cosh h} = (\cosh h)^{-1} geom(p_{(3)}, (\cosh h)^{-2}) \in L, \quad (16)$$

by Lemma 2, where $p_{(3)} := p_{(2)}^2$. Finally, for N=4 (16 summands in (2)) "brute" calculations lead to the following:

$$\tilde{\phi}_{4,K}(h) := \frac{\tilde{Z}_4(K,0)}{\tilde{Z}_4(K,h)} = \frac{[(1 + e^{-2K})/2]^3}{\cosh^4 h - 2^{-1}(2 - e^{-2K} - e^{-4K}) + 8^{-1}(1 - e^{-2K} - e^{-4K} + e^{-6K})} = geom(p_{(4)}^{(1)}, (\cosh h)^{-2}) geom(p_{(4)}^{(2)}, (\cosh h)^{-2}) \in L, \quad (17)$$

by Lemma 2 and the semigroup property of L. Here the probabilities of the successes are given as follows

$$p_{(4)}^{(1)} := 1 - \frac{1}{4}(1 - e^{-2K})[2 + e^{-2K} + \sqrt{1 + (1 + e^{-2K})^2}],$$

and similarly

$$p_{(4)}^{(2)} := 1 - \frac{1}{4}(1 - e^{-2K})[2 + e^{-2K} - \sqrt{1 + (1 + e^{-2K})^2}].$$

So these are formulas similar to those in Theorem 2 (for the "modified" partition functions) but only for the case $N = 2, 3, 4$. However, for the ratios $\tilde{Z}_N(K, 0)/\tilde{Z}_N(K, h)$ one may expects that similar formulas hold true.

5.Concluding remarks.

a) For a finite number N of particles we showed that the inverse of the partition function can be expressed as a product of [N/2] Fourier transforms of the compound geometric distributions. The probability of a success (in one trial) depends *only* on the interaction constant and the geometric distributions have the same innovation characteristic function $(\cosh^2 h)^{-1}$. Furthermore, the characteristic functions in question correspond to randomly stopped random walks .

b) The inverses of the partition functions are characteristic functions from the class L of selfdecomposable probability measures. It is crucial that L forms a weakly closed convolution semigroup and each measure from L is a probability distribution of some random integral.

c) We have considered here only finite ensambles but because of b) at the thermodynamical limit we still have functions that are logarithms of selfdecomposable characteristic functions. Hence they are of the form (21) in the Appendix below, with appropriate BDLPs.

d) In the proof of Theorem 2 we have used essentially the transfer matrix $\mathbf{V}$ and its eigenvalues but its conculsion is given in terms of class L probability distributions. Similar statements we expect to have for $\tilde{Z}_N(K, h)$, as it is for $N = 2, 3, 4$. This raises the possibility of having a theory of lattice spin models without the algebraic language and, in particular, without the eigenvalues.

6. Appendix.

To make this presentation more selfcontained we recall some basic facts about the Lévy class L of selfdecomposable probability distributions. By definition *a class L distribution* is a limit distribution of a sequence

$$a_n(\xi_1 + \xi_2 + \ldots + \xi_n) + b_n, \quad a_n > 0, \quad b_n \in \mathbb{R},$$

where random variables $\xi_n' s$ *are independent but not necessarily identically distributed and the triangular array* $a_n \xi_k, 1 \leq k \leq n, n \geq 1$ *is uniformly infinitesimal, that is,* $\lim_{n\to\infty} max_{1\leq k\leq n} \mathcal{P}\{|a_n \xi_k| \geq \epsilon\} = 0,$

$$\textit{for any} \quad \epsilon > 0. \quad (18)$$

Then the first characterization of class L distributions is in terms of *the selfdecomposability property* of their charactersitic functions. Namely,

$$\phi \textit{ is class L char.f.} \quad \text{iff} \quad \forall (0 < c < 1) \exists (\text{char.f.} \phi_c) \ \phi(t) = \phi(ct)\phi_c(t). \quad (19)$$

In terms of rv's it means that rv X has probability distribution (or a charactersitic finction) in class L iff $X \stackrel{d}{=} cX + X_c$, for each $0 < c < 1$,where X_c is independent of X ; cf.Jurek and Mason (1993), Chapter 3. More importantly for class L distributions or rv's we have the following *random integral representation* (RIR):

$$X \in L \quad \text{iff} \quad X \stackrel{d}{=} \int_0^\infty e^{-s} dY(s), \tag{20}$$

where Y is a Lévy process unique in distribution with finite logarithmic moment $\mathbb{E}[\log(1 + |Y(1)|)] < \infty$. We refer to Y as the *background driving L évy process*. In short: Y is the BDLP of X. If ϕ is a charactersitic function of X and ψ of $Y(1)$ then from RIR we get that

$$\log \phi(t) \in L \quad \text{iff} \quad \log \phi(t) = \int_0^1 \log \psi(tu) u^{-1} du; \tag{21}$$

cf. Jurek and Mason (1993). The charactersitic function ψ is refered to as the *background driving rv* of ϕ. Allthough the stable laws (in particular, gaussian laws) are all in class L (they correspond to identically distributed summands in (18)), are basic examples of class L distribution , for *this* presentation, it is the Laplace distribution . It is a continous random variable , denoted by η, whose probability density is equal $2^{-1}\exp[-|x|]$ and its characteristic function is $\phi_\eta(t) = (1+t^2)^{-1}$. Since class L is closed under the convolution of distributions (i.e., multiplcation of charactersitic functions) and weak limits (i.e., point-wise convergence of charateristic functions) therefore all functions of the form $\prod_k (1+a_k^2 t^2)^{-1}$ are class L charactersitics functions if and only if $\sum_k a_k^2 < \infty$. Consequently, by (12) we see that $(\cosh h)^{-1}$ is class L characteristic function. Its BDLP is the Lévy process such that $Y(1)$ or the background driving rv ψ, has charatersitic function

$$\mathbb{E}[\exp itY(1)] = \psi(t) = \exp\Big[\int_0^\infty (\cos tx - 1)\frac{\pi}{4}\frac{\cosh(\pi x/2)}{\sinh^2(\pi x/2)}dx\Big]; \qquad (22)$$

cf. Jurek (1996), Corollary 4.

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