Measure valued cocycles from my papers in 1982 and 1983 and Mehler semigroups.

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Studying limit laws on Banach spaces, in the 80's, we had encountered probability measure valued equations. They "went" either without a name or as the cocycle equations. These days we are aware that some of those equations were well studied and are known as the *Mehler semigropus.* Since in the past our final goal was *the random integral representation*, therefore the solutions to measure valued equations were simply parts of proofs and were not exposed for their importance in themselves !

Below we review contents of some "old" papers and indicate their connections to the Mehler's semigroups.

[Papers [4] and [5] are from 1988 and 1996, respectively, but we added them here, for some completeness of the presentation.]

At the very end of these notes, Section C, we formulate a "new" result for the case of an arbitrary *one parameter group or semigroup* T_t, $t \in \mathbb{R}$, and discuss the "problems" that one encounters when we have only an arbitrary one parameter semigroup $T_t, t \geq 0$, and the corresponding cocycle equation or generalized Mehler semigroup:

$$\mu_{t+s} = \mu_t * T_t \, \mu_s, \quad \text{for all} \quad 0 < t, \, s < \infty.$$

A). `Norm continuous semigroups of operators.`

[1] *Random integral representations for selfdecomposable Banach space valued random variables,* by ZJJ and Wim Vervaat, **Z. Wahrscheinlichkeitstheorie verw. Gebiete 62, 247-262, 1983**.

In terms of rv's, one had studied equation for Banach space valued rv X:

$$\forall (t>0) \exists (rv)\, X_t \,(\text{independent of } X) \qquad X \stackrel{d}{=} e^{-t}X + X_t; \tag{1}$$

where the mapping $t \to \mathcal{L}(X_t)$ is weakly continues. For details cf. formula (3.1) on p.251. Applying above equality for t then u or at once for $t+u$ one arrives at

$$X_{t+u} \stackrel{d}{=} e^{-t}X_u + X_t, \quad \text{for all} \;\; t,u>0. \tag{2}$$

Solutions to (2) are obtained in two steps.

STEP 1. Kolmogorov's Consistency Theorem and (2) guarantee existence of a process $(Z(t), t \geq 0)$ with independent increments and cadlag path such that

$$Z(t+s) - Z(t) \stackrel{d}{=} e^{-t}X_s, \;\; \text{and in particular } Z(t) \stackrel{d}{=} X_t \;\; \text{for each } s,t \geq 0.$$

(completeness and separability of the space where rv take values on is essential here to achieve cadlag paths.). cf. (3.5), p. 252 and Appendix.

STEP 2. Using the above properties one shows that

$$Y(t) := \int_{(0,t]} e^s \, dZ(s), \;\; t \geq 0, \;\; \text{is a Lévy process.} \tag{3}$$

cf. (3.6), p. 253.

Finally the "chain rule" gives that

$$X_t \stackrel{d}{=} \int_{(0,\,t]} e^{-s} dY(s), \quad \text{for each } t \geq 0, \tag{4}$$

for a unique (in distribution) Lévy process Y, is the general solution to (2); cf. p. 253.

Conversely, simple calculations and properties of Lévy processes, show that X_t given by (4) satisfies (2) !!

[2] *An integral representation of operator-selfdecomposable random variables*, by ZJJ, **Bull. Acad. Pol. Sci; Serie Math. XXX no.7-8, 385-393, 1982**.

For a bounded operator Q on Banach space we consider equation

$$\forall (t>0)\,\exists\,(rvX_t)\,(X_t \text{ independent of } X) \qquad X \stackrel{d}{=} e^{-tQ}X + X_t. \tag{5}$$

cf. the equation p. 390. Hence as in [1], we get that

$$X_{t+s} \stackrel{d}{=} X_t + e^{-tQ}X_s, \quad \text{for all} \quad s,t>0.$$

All solutions to above equation are of the form:

$$X_t \stackrel{d}{=} \int_{(0,t]} e^{-sQ}dY(s), \quad \text{for each } t \geq 0. \tag{6}$$

where again Y is a (unique) Lévy process.

[3] *The classes $L_m(Q)$ of probability measures on Banach spaces*, by ZJJ, **Bull. Pol. Acad. Sc.: Math. 31, no.1-2, pp. 51-62; 1983.**

Here one studies inductively classes L_m, $m = 0, 1, 2, ...$ of laws satisfying the equation (5) with additional requirement that the remainder X_t is from previous class $L_{m-1}(Q)$ with $L_{-1}(Q) := ID$ (all infiniteley divisible laws.) and $L_0(Q)$ is the class of all limiting distribution of X_t, as $t \to \infty$. In other words:

$$L_0(Q) = \{\mathcal{L}(\int_0^\infty e^{-tQ}dY(t)) : \ Y \text{ Lévy proc. } E[\log(1+||Y(1)||)] < \infty\}; \tag{7}$$

(the logarithmic moment guarantees that the improper integral is finite). And similarly,

$$L_m(Q) = \{\mathcal{L}(\int_0^\infty e^{-tQ}dY(\frac{t^{m+1}}{(m+1)!})) : \ E[\log^{m+1}(1+||Y(1)||)] < \infty\}, \tag{8}$$

for $m = 0, 1, 2 ....$ Cf. Corollary 2.11., on page 61.

[4] *Random integral representations for classes of limit distributions similar to Lévy class L_0*, by ZJJ, **Probab. Th. Rel. Fields, vol.78, 473-490, 1988.**

Again in that paper our original goal was the description of some limit laws by random integrals. But the most crucial step was the solution to the equation:

$$\forall(0 < c < 1)\, \exists(\mu_c - \text{probab.meas.}) \;\; \mu = c^Q\, \mu^{*c^\beta} * \mu_c, \tag{9}$$

cf. formula (1.8) on p. 476, or equivalently the equation

$$\mu_{t+s} = e^{-tQ}\, \mu_s^{*e^{-\beta t}} * \mu_t, \tag{10}$$

cf. formula in the first line on p. 483. The solution has the form

$$\mu_t = \mathcal{L}(-\int_{(0,t]} e^{-sQ} dY(e^{-\beta s})), \text{ for a (unique) Lévy proces } Y, \tag{11}$$

cf. page 484; (in the Lévy process Y we run "deterministic" inner clock).

It seems that equations as those in (9) or (10) were not studied in the Mehler theory.

[5] *Some analytic semigroups occurring in probability theory*, K.H. Hofmann and ZJJ, **J. Theoretical Probab. 9, No. 3, 745-763, 1996**

Here among others we investigate a semigroup that is a semidirect product of probability measures (on a Banach space X) and algebra of bounded linear operators with the *operator norm*. Thus one gets equation (5). More explicitly,

$$\mathbf{S} := \mathcal{P}(\mathbf{X}) \times \mathbb{E}\mathbf{nd}(\mathbf{X}), \quad (\mu, \mathbf{A})\Box(\nu, \mathbf{B}) := (\mu * \mathbf{A}\nu, \mathbf{AB}),$$

i.e., $\mathbf{S}$ is a semigroups. All one parameter sub-semigroups T_t are of the form

$$T_t = \Big(\mathcal{L}(\int_{(0,t]} e^{-sQ} dY(s)), e^{-tQ}\Big).$$

Cf. p. 753, Theorem 3.2.

Another novelty in that article was the use of diagrams and the exponential mapping connecting Lie semigroups and Lie groups. For instance see p. 755 or 761.

B.) <u>`Point-wise continuous semigroups of operators`</u>.

Of course, from the very beginning, it was quite obvious and intriguing how to extend the random integral representations to more general semigroups of operators and the equations analogous to (2). At that time we were unaware of the relation to Mehler semigroups and Markov processes.

[6] *Limit distributions and one-parameter groups of linear operators on Banach spaces*, by ZJJ, **J. Multivariate Analysis, 13, no. 4,578-604, <u>1983</u>.**

In that paper one has *a <u>multiplicative</u> strongly continuous one parameter group* $\mathbb{U} = \{U_t : t > 0\}$ of bounded linear operators on a Banach space X, satisfying the initial condition:

$$\lim_{t\to 0} U_t x = 0 \quad \text{for each } x \in X,$$

cf. (1.4) on p. 583. The measure valued equation this time has a form

$$\forall (0 < \alpha < 1)\, \exists\, (\nu_\alpha \text{ probab.meas.}) \;\; \nu = U_\alpha \nu * \nu_\alpha. \tag{12}$$

cf. the formula (3.5), on p. 587. However, the goal at that time was the Lévy-Khintchine type characterization,; cf. Theorems 4.2, 5.1, 6.1.

Also, please make a note of the comments on p. 602.

Of course, as in papers [1] or [2] or [3] or [4] mentioned above, from equation (12) we get the associated cocycle equation

$$\nu_{\alpha\cdot\beta} = \nu_\alpha * U_\alpha \nu_\beta, \text{ for all } 0 < \alpha, \beta < 1. \tag{13}$$

and we could proceeds as in the case of operator norm continuous semigroups. However, at that time (the 80's) my "technical" problem was with the definition of integration of operator-valued function with respect to Lévy processes.

[7] *Polar coordinates in Banach spaces*, by ZJJ, **Bull. Pol. Acad. Sci.; Math. 32, no. 1-2, 61-66, 1983.**

This paper is not directly related to the measure valued cocycles but it was prompted by [6], where we needed so called "polar coordinates" in Banach spaces.

For an one-parameter *group* $\mathbb{U}$ as in [6] there exists an equivalent norm $||\cdot||_{\mathbb{U}}$ and a Borel set $T_{\mathbb{U}}$ such that
(i) functions $t \to ||U_t x||_{\mathbb{U}}$ are strictly increasing, for all $x \neq 0$.
(ii) the mapping

$$\mathbf{\Phi} : \mathbf{T}_{\mathbb{U}} \times (\mathbf{0}, \infty) \to \mathbf{X} \setminus \{\mathbf{0}\} \text{ given by } \mathbf{\Phi}(\mathbf{x}, \mathbf{t}) := \mathbf{U_t x},$$

are Borel isomorphisms. Cf. Thm. 1, p. 62.

In some instances $T_{\mathbb{U}}$ may chosen as a unit sphere in a new norm.

C. `Comments on arbitrary semigroups and cocycle equations.`

(i) At the present moment, needed random integrals (or stochastic integrals) are available . So we can continue investigations from [6], for *one parameter group* or strongly continuous semigroup T_t such that $(T_t)^{-1}$ exists and the mappings $t \to (T_t)^{-1}$ is measurable (we need its integrability below).

Let us change the parameter as follows

$$T_t\, x := U_{e^{-t}} x, \;\; t \in \mathbb{R}, \;\; T_0 = I \text{ is one-parameter group.} \tag{14}$$

Similarly, let us put $\mu_t := \nu_{e^{-t}}$. Then the formula (13) gives

$$\mu_{t+s} = \mu_t \, * \, T_t \mu_s \;\; \text{for all } s, t \geq 0. \tag{15}$$

From the arguments from [1] and [2], i.e., Steps 1 and 2 from p. 2 of this notes, we get that there exists a process with independent increments $Z(t), t \geq 0$, $Z(0) = 0$ and cadlag paths such that

$$Z(t+s) - Z(t) \stackrel{d}{=} T_t \mu_s \;\; \text{for all} \;\; s, t \geq 0.$$

Hence $\mu_t = \mathcal{L}(Z(t))$. But the independence of increments and continuity imply $\mu_t \in ID$!! (infinite divisible). Furthermore,

$$Y(t) := \int_{(0,t]} (T_s)^{-1} \, dZ(s), \quad \text{is a Lévy process.} \tag{16}$$

(note $(T_t)^{-1}$ exists !). Consequently, all solutions to (15) are of the from

$$\mu_t = \mathcal{L}(\int_{(0,t]} T_s \, dY(s), \tag{17}$$

for a unique Lévy process Y. In terms of Fourier transforms we have that

$$\widehat{\mu_t}(y) = \exp \int_{(0,t]} \log(\mathcal{L}(Y(1)))\widehat{\;}(T_s^{\star} y) ds, \quad y \in X^{\star} \;\; \text{(dual space).} \tag{18}$$

Hence, in particular, we have that functions $t \to \widehat{\mu_t}(y)$ are differentiable.

(ii) Let (15) holds for an arbitrary semigroup and the mapping $t \to \mu_t$ be weakly continues. Since $\mu_t \in ID$, the functions $g_t(y) := \log \widehat{\mu_t}(y)$ are well defined and (15) means that

$$g_{t+s}(y) - g_t(y) = g_s(T_t^\star y).$$

If one assumes the differentiability of the functions $t \to g_t(y), y \in X^\star$, is fixed, we conclude that

$$g_t(y) = \int_{(0,t]} \psi(T_s^\star y) ds, \quad \text{where} \quad \psi(y) := \lim_{s \to 0} s^{-1} g_s(y) \text{ is continuous.} \tag{19}$$

Hence,

$$\widehat{\mu_t}(y) = \exp \int_0^t \psi(T_s^\star y)(ds), \quad y \in X^\star, \tag{20}$$

where ψ is the Lévy exponent of a characteristic function of $Y(1)$ from (17).

Equivalently, if one assumes that $lim_{s \to 0}\, s^{-1}\, g_s(y)$ exists then we get the differentiability of $t \to g_t(y)$ and the above holds as well. Or show that $\lim_{s \to 0} \mu_s^{*(1/s)}$ exists in weak topology.

My guess: Except possibly some pathological cases, all solutions to Mehler semigroups are of the form (17), or equivalently of the form (20)??

[Simple calculations show that measures given by (17) satisfy the cocycle (Mehler) equation (15).]

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