SELFDECOMPOSABILITY PERPETUITY LAWS AND STOPPING TIMES*

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In the probability theory limit distributions (or probability measures) are often characterized by some convolution equations (factorization properties) rather than by Fourier transforms (the characteristic functionals). In fact, usually the later follows the first one. Equations, in question, involve the multiplication by the positive scalars c or an action of the corresponding dilation T_c on measures. In such a setting, it seems that there is no way for stopping times (or in general, for the stochastic analysis) to come into the "picture". However, if one accepts the view that the primary objective, in the classical limit distributions theory, is to describe the limiting distributions (or random variables) by the tools of random integrals/functionals then one can use the stopping times. In this paper we illustrate such a possibility in the case of selfdecomposability random variables (i.e. Lévy class L) with values in a real separable Banach space. Also some applications of our approach to perpetuity laws are presented; cf. [2], [3], [4]. In fact, we show that all selfdecomposable distributions are perpetuity laws.

1. Let E be a real separable Banach space. An E-valued random variable (rv) X, defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, is said to be *selfdecomposable* (or *a Lévy class* L) if for each $t > 0$ there exists a rv X_t independent of X such that

$$X \stackrel{d}{=} X_t + e^{-t}X. \tag{1}$$

where " $\stackrel{d}{=}$ " means equality in distribution.

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Remark 1. (i) The class of selfdecomposabile rv's (distributions) coincides with the class of limiting rv's of the following infinitesimal triangular arrays:

$$a_n(Z_1 + Z_2 + \ldots + Z_n) + b_n,$$

where $a_n > 0$, $b_n \in \mathrm{E}$ and $Z_1, Z_2, \ldots$ are independent E-valued rv's; cf. for instance [8], Chapter 3.
(ii) In case of i.i.d. Z_n's one gets in the above scheme the class of all stable distributions.
(iii) In terms of probability distributions equation (1) reads that for each $0 < c < 1$ there exists a probability measure μ_c such that

$$\mu = \mu_c * T_c\mu. \tag{2}$$

where " $*$ " denotes the convolution of measures and $(T_c\mu)(\,\cdot\,) = \mu(c^{-1}\,\cdot\,)$. In other words, $T_c\mu$ is the image of a measure μ under the linear mapping $T_c\colon \mathrm{E} \to \mathrm{E}$ given by $T_c x = cx$, $x \in \mathrm{E}$.

Let us also recall that a stochastic base is an increasing and right continuous family of σ-fields $\mathcal{F}_t \subset \mathcal{F}$; (i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$).

Furthermore, any mapping $\tau\colon \Omega \to [0, \infty)$ such that

$$\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t,\ t \ge 0. \tag{3}$$

is called *a stopping time.*

A family $Y(t)$, $t \ge 0$, of E-valued random variables is called *a Lévy process* provided

- $Y(0) = 0$ $\mathcal{P}$-a.s., $Y(t+s) - Y(s) \stackrel{d}{=} Y(t)$ for all $s, t \ge 0$,
- $Y(t_k) - Y(t_{k-1})$, $k = 1, 2, \ldots, n$, *are independent* for all $0 \le t_0 < \ldots < t_n$, $n \ge 1$,
- $t \mapsto Y(t, \omega)$ *are cadlag functions, for* $\mathcal{P}$*-a.a.* $\omega \in \Omega$.

Of course, for any $c > 0$ and a Lévy process Y, one has that $Y_c(t) := Y(t+c) - Y(t)$, $t \ge 0$, is a new Lévy process with $Y_c \stackrel{d}{=} Y$ in the Skorohod space $D_E[0, \infty)$ of all cadlag functions. Moreover, Y_c is independent of the σ-field $\sigma\{Y(t) : 0 \le t \le c\} = \mathcal{F}_c^Y$ In fact, for any stopping τ with respect to $\mathcal{F}_t^Y$ and has that

$$Y_\tau(t) := Y(t+\tau) - Y(\tau),\ t \ge 0, \tag{4}$$

is a Lévy process such that $Y_\tau \stackrel{d}{=} Y$ and Y_τ is independent of the σ-field $\mathcal{F}_\tau$ defined as follows

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap [\tau \le t] \in \mathcal{F}_t \text{ for each } t \ge 0\}; \tag{5}$$

cf. for instance [1], Theorem 32.5 to derive the above statements for any Lévy process.

Here is the main result which extends (1) for some stopping times.

THEOREM 1. *Suppose X is a selfdecomposable* E*-valued rv. Then there exists a stochastic base $(\mathcal{F}_t)_{t\ge0}$ such that for each $\mathcal{F}_t$-stopping time τ there are independent* E*-valued rv's X_τ and X' satisfying*

(i) $X \stackrel{d}{=} X_\tau + e^{-\tau}X'$;

(ii) *X' is independent of X and $X' \stackrel{d}{=} X$;*

(iii) *vector $(X_\tau, e^{-\tau})$ is independent of X'.*

Proof. From [5] or [8] p.124 we conclude that rv X is selfdecomposable (i.e. (1) holds) if and only if there exists a unique, in distribution, Lévy process Y such that $\mathbb{E}[\log(1 + \|Y(1)\|)] < \infty$ and

$$X \stackrel{d}{=} \int_{(0,\infty)} e^{-s} dY(s). \tag{6}$$

(We refer to Y as the **b**ackground **d**riving **L**évy **p**rocess of X; in short: Y is BDLP for X, cf. [6]). Taking $\mathcal{F}_t = \sigma(Y(s) : s \le t)$ and defining

$$Z(t) := \int_{(0,t]} e^{-s} dY(s) \equiv e^{-t}Y(t) + \int_{(0,t]} Y(s-)e^{-s} ds, \tag{7}$$

we have that $Z(\tau(\omega))$ is $\mathcal{F}_\tau$-measurable for a stopping time τ; cf. [9], p. 18–20. Finally using (4) and (7) one gets

$$X \stackrel{d}{=} \int_{(0,\tau]} e^{-s} dY(s) + \int_{(\tau,\infty)} e^{-s} dY(s) =$$

$$= Z(\tau) + e^{-\tau} \int_{(0,\infty)} e^{-s} dY_\tau(s) = X_\tau + e^{-\tau}X', \tag{8}$$

with $X_\tau = Z(\tau)$ independent of $X' = \int_{(0,\infty)} e^{-s} dY(s) \stackrel{d}{=} X$ because Y_τ is independent of $\mathcal{F}_\tau$. This completes the proof of Theorem 1. □

Remark 2. The integral in (6) is defined as a limit of $Z(t)$, given by (7), as $t \to \infty$. Existence of the limit (in probability, a.s., or in distribution) is equivalent to the condition $\mathbb{E}[\log(1 + \|Y(1)\|)] < \infty$; cf. [5] or [8] p.122.

COROLLARY 1. *Let Y be a BDLP of a selfdecomposable rv X and let $\mathcal{F}_t = \sigma(Y(s) : s \le t)$, $t \ge 0$ be the stochastic base given by Y. Then for each $\mathcal{F}_t$-stopping time τ there exists a rv X_τ independent of X such that*

$$X \stackrel{d}{=} X_\tau + e^{-\tau} X. \tag{9}$$

The equality in distribution in (9) can be strengthen as follows.

COROLLARY 2. *Let Y be* E*-valued Lévy process such that $\mathbb{E}[\log(1 + \|Y(1)\|)] < \infty$ and $(\mathcal{F}_t)_{t\ge 0}$ be the natural filtration given by Y. Then for any $\mathcal{F}_t$-stopping time τ one has*

$$\int_{(0,\infty)} e^{-s} dY(s,\omega) = \int_{(0,\tau(\omega)]} e^{-s} dY(s,\omega) + \\ + e^{-\tau(\omega)} \int_{(0,\infty)} e^{-s} dY(s + \tau(\omega), \omega) \tag{10}$$

for $\mathcal{P}$-a.a. $\omega \in \Omega$.

Proof. This is a consequence of Remark 2 and the equality in the formula (8). □

It is well-know that any Lévy process Y can be written as a sum of two independent Lévy processes Y^c and Y^d, i.e., $Y = Y^c + Y^d$, where Y^c is purely continuous (Gaussian) while Y^d is purely discontinuous (cadlag) process. Furthermore, for a Borel subset A separated from zero (i.e., $A \subset \{x \in E : \|x\| \ge \epsilon\}$ for some $\epsilon > 0$) we define

$$Y^d(t; A) := \sum_{0<s\le t} \Delta Y^d(s) 1_A(\Delta Y^d(s)),$$

where the jumps $\Delta Y^d(s) := Y^d(s) - Y^d(s-)$ are in the set A, which a Lévy process independent of the process $Y^d(t) - Y^d(t; A)$. All the above allows us to have the following:

COROLLARY 3. **(i)** *Let Y^d be a purely discontinuous Lévy process with finite logarithmic moment and*

$$\tau_0 = \inf\{t > 0 : Y^d(t) \neq 0\}$$

be the stopping time of the first non-zero value. Then

$$\int_{(0,\infty)} e^{-s} dY^d(s) = e^{-\tau_0} Y^d(\tau_0) + \\ e^{-\tau_0} \int_{(0,\infty)} e^{-s} dY^d(s + \tau_0), \ \mathcal{P}\text{-}a.s. \quad (11)$$

(ii) *Let $\tau_A = \inf\{t > 0 : Y^d(t; A) \neq 0\}$ be the stopping time of the first jump whose values is in A. Then*

$$\int_{(0,\infty)} e^{-s} dY^d(s; A) = e^{-\tau_A} Y^d(\tau_A; A) + \\ + e^{-\tau_A} \int_{(0,\infty)} e^{-s} dY^d(s + \tau_A; A), \ \mathcal{P}\text{-}a.s. \quad (12)$$

Proof. Apply the above stopping times in the equation (10). □

Remark 3. (a) Random integrals appearing in (12) are independent, identically distributed and selfdecomposable. Similarly holds for integrals in (11) and the outmost integrals in (10).
(b) If $\tau_1 = \tau_A$ and τ_k, $k \geq 1$, are the consecutive random times of the jumps of the process $Y^d(t; A)$ with $\tau_k \uparrow +\infty$, a.e., then one gets factorization

$$\int_{(0,\infty)} e^{-s} dY^d(t; A) = \sum_{k=1}^{\infty} e^{-\tau_k} \Delta Y(\tau_k; A), \ a.e., \quad (13)$$

where $\Delta Y(\tau_k; A)$ are independent of $\tau_k - \tau_{k-1}$ for $k \geq 1$.

2. In this section we consider only *real* valued random variables. Let (A, B), (A_1, B_1), $(A_2, B_2), \ldots$ be a sequence of i.i.d. random vectors in $\mathbb{R}^2$ which define the stochastic difference equation

$$Z_{n+1} = A_n Z_n + B_n, \ n \geq 1. \quad (14)$$

Equation (14) appear in modelling many real situations including economics, finanse or insurance; cf. for instance [3], [4] and the reference there. One many look at (14) as an iteration of the affine random mapping $x \mapsto Ax + B$, So, starting with Z_0 and $(A_0, B_0) = (A, B)$ we get

$$Z_{n+1} = A_n A_{n-1} \dots A_0 Z_0 + \sum_{k=0}^{n} B_k A_{k+1} A_{k+2} \dots A_n.$$

Putting $Z_0 = 0$ and assuming (Z_n) converges to Z we get

$$Z \stackrel{d}{=} \sum_{k=1}^{\infty} B_k \prod_{l=1}^{k-1} A_l.$$

In insurance mathematics distributions of Z are called *perpetuities*. Note that by (14) perpetuities are the solution to

$$Z \stackrel{d}{=} AZ + B, \tag{15}$$

i.e., Z is *a distributional fixed-point* of the random affine mapping $x \mapsto Ax + B$, $x \in \mathbb{R}$.

What triplets A, B, X satisfy (15) with (A, B) independent of X? Or are there independent rv's A, C, Z such that

$$Z \stackrel{d}{=} A(Z + C) \tag{16}$$

It seems that there are not to many explicite examples of (15) or (16); cf. [2], p.288. Results from previous section can now be phrased as follows:

COROLLARY 4. (i) *All selfdecomposable distributions are perpetuities, i.e, satisfy (15) with non-trivial* $0 \le A \le 1$ *a.s.*

(ii) *All sefldecomposable distributions whose BDLP* Y *have non-zero purely discontinuous part, have convolution factors that satisfy the equation (16).*

Let $\gamma_{\alpha,\lambda}$ denotes a gamma rv with parameters $\alpha > 0$, $\lambda > 0$, i.e., it has the probability density

$$f_{\alpha,\lambda} = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} 1_{(0,\infty)}(x).$$

It is known, cf. [6], [7], that $\gamma_{\alpha,\lambda}$ is selfdecomposable and its BDLP is given by $\lambda^{-1}Y_0(\alpha t)$, where

$$Y_0(t) = \sum_{j=1}^{N(t)} \gamma_{1,1}^{(j)}, \tag{17}$$

$\gamma_{1,1}^{(1)}, \gamma_{1,1}^{(2)} \ldots$ are i.i.d. copies of $\gamma_{1,1}$ and $N(t)$ is a standard Poisson process, i.e., it has stationary, independent increments, $N(0) = 0$ a.e. and for $t > s > 0$

$$\mathcal{P}[N(t) - N(s) = k] = e^{-(t-s)}\frac{(t-s)^k}{k!}, \ k = 0, 1, 2, \ldots \ .$$

If $0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ are the consecutive random times (arrival times) of the jumps of N then $\tau_n - \tau_{n-1} \stackrel{d}{=} \gamma_{1,1}$ for $n \geq 1$ are independent and $\tau_n \stackrel{d}{=} \gamma_{n,1}$ for $n \geq 1$.

PROPOSITION 1. *For gamma rv* $\gamma_{\alpha,\lambda}$ *one has*

(i)
$$\gamma_{\alpha,\lambda} \stackrel{d}{=} e^{-\gamma_{\alpha,1}}(\gamma_{1,\lambda} + \gamma_{\alpha,\lambda}) \stackrel{d}{=} U^{1/\alpha}\gamma_{\alpha+1,\lambda}$$

where U *is uniformly distributed on* $[0, 1]$ *independent of* $\gamma_{\alpha+1,\lambda}$ *and the three middle rv's are independent too.*

(ii)
$$\gamma_{\alpha,\lambda} \stackrel{d}{=} \sum_{n=1}^{\infty} U_1^{1/\alpha}U_2^{1/\alpha}\ldots U_n^{1/\alpha}\gamma_{1,\lambda}^{(n)}$$

where $\gamma_{1,\lambda}^{(1)}, \gamma_{1,\lambda}^{(2)} \ldots$ *are i.i.d. copies of* $\gamma_{1,\lambda}$, $U_1, U_2 \ldots$ *are i.i.d. copies of* U *and both sequences are independent too.*

Proof. (i) Since $\gamma_{\alpha,\lambda}$ has BDLP $Y(t) = \lambda^{-1}Y_0(\alpha t)$, therefore by Corollary 3(i) and Remark 3(a) we have

$$\begin{aligned}\gamma_{\alpha,\lambda} &\stackrel{d}{=} e^{-\gamma_{\alpha,\lambda}}\gamma_{1,\lambda}^{(1)} + e^{-\gamma_{\alpha,1}}\int_{(0,\infty)} e^{-s}dY(s+\tau_1) = \\ &= U^{1/\alpha}(\gamma_{1,\lambda}^{(1)} + \tilde{\gamma}_{\alpha,\lambda} \stackrel{d}{=} U^{1/\alpha}\gamma_{1+\alpha,\lambda}.\end{aligned}$$

(Equality of the two outmost terms in (i) can be also easily checked by comparing the corresponding characteristic functions.)

(ii) Repeating the middle equality in (i) and using facts that $\tau_n \uparrow +\infty$ a.s., and $Y(\cdot)$ is independent of $Y(\cdot + \tau_1 + \tau_2 + \ldots + \tau_k) - Y(\tau_1 + \tau_2 + \ldots + \tau_k)$ one arrives at (ii). □

3. The metod of random integral representation is also applicable to operator-selfdecomposable distributions; cf. [8] Chapter 3, or [5]. Recall that a Banach space E-valued rv X is Q-selfdecomposable if for each $t > 0$ there exists a rv X_t independent of X such that

$$X \stackrel{d}{=} X_t + e^{-tQ}X; \tag{18}$$

Q is a bounded linear operator on E and e^{-tQ} is the operator given by a power series.

THEOREM 2. *Suppose that X is Q-decomposable* E*-valued rv and $e^{-tQ} \to 0$, as $t \to \infty$, in the norm topology. Then there is a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that for each stopping times τ there exist independent* E*-valued rv's X_τ and X' satisfying*

(i) $X \stackrel{d}{=} X_\tau + e^{-\tau Q}X'$*;*

(ii) *X' is independent copy of X;*

(iii) *the random vector $(X_\tau, e^{-\tau Q})$ is independent of X'.*

Proof. From [5] or [8] Chapter III we have that X is Q-selfdecomposable if and only if

$$X \stackrel{d}{=} \int_{(0,\infty)} e^{-tQ} dY(t) \tag{19}$$

for a uniquely defined Lévy process Y such that $\mathbb{E}[\log(1 + \|Y(t)\|] < \infty$. So, (19) allows us to proceed as in the proof of Theorem 1. □

Remark 4. (a) Corollaries from Section 1 have their "operator" counterparts.
(b) For a given E-valued rv B and a random bounded linear operator A on a Banach space E, consider the affine random mapping $x \mapsto Ax + B$. The question of finding all distributial fix-points, i.e., all E-valued rv's X such that

$$X \stackrel{d}{=} AX + B, \tag{20}$$

seems, to be more difficult as the composition of operators is not commutative. However, random integrals of the form

$$\int_{(a,b]} f(t) dY(r(t)), \tag{21}$$

where Y is a Lévy process, $r(t)$ is a change of time, f is a process or deterministic funtions, *might provide a tool* of constructing X satisfying the equation (20) or its variants (like (16)). The present paper illustrates this approach in a case of the selfdecomposable distributions and their random integral representations.

Added in proof. Corollary 1 is also true when the stopping time τ is replaced by a non-negative random varaible T independent of the BDLP Y.

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