Remarks on the factorization property of some random integrals*

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Abstract. Two families of improper random integrals and the two corresponding convolution semigroups of infinitely divisible laws are studied. A relation (a factorization property) between those random integrals is established. For the proof we use the method of *the random integral mappings* $I^{h,r}_{(a,b]}$ that is also valid for infinitely divisible measures on Banach spaces. Furthermore, using that technique we established new relations between those two families of random integrals.

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In the last few decades there have appeared many papers on random integral representations of convolution subsemigroups of the (master) semigroup, ID, of all infinitely probability distributions. Jurek-Vervaat(1983) on the class, L, of selfdecomposable measures seems to be one of the first in that area. For more references we refer to the survey article of Jurek (2011); also cf. Kumar and Schreiber (1997), Wolfe (1982), Sato (2006), Grigelionis (2007) and Maejima, Perez-Abreu and Sato (2012).

Some of the subsemigroups of infinitely divisible distributions were introduced via the random integrals (cf. (∗) below) while the others were described by transformations defined on the Lévy (spectral) measures in the Lévy-Khintchine representation. The latter approach was presented already in Jurek (1990) and the resulting measures were called there as λ-*mixtures*. Most of the research was done for measures on Euclidean spaces but our techniques and proofs are applicable in any infinite dimensional separable Banach space.

In this note, using the random integral technique, in Proposition 1 we provided another and simpler proof of the factorization property between two families of random integrals proved in Sato (2006). Moreover, our method allows us to establish new relations between those integrals; cf. Proposition 2 and Corollary 2. It seems that the general random integral method is more useful than the considerations of some specific cases.

1. For an interval $(a, b]$ in the positive half-line, two deterministic functions h (*space change*) and r (*inner clock time change*), and a Lévy process $Y_\nu(t), t \geq 0$ on a real separable Banach space E, where $\nu \in ID$ is the law of random variable $Y_\nu(1)$, we consider the following mapping (or the operator):

$$ID \ni \nu \longmapsto I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\big(\int_{(a,b]} h(t)\, dY_\nu(r(t))\big) \in ID, \qquad (\star)$$

and $\mathcal{L}$ denotes the probability distribution of the random (stochastic) integral. Random integrals (∗) are defined by formal integration by parts formula, i.e.,

$$\int_{(a,b]} h(t) dY_\nu(r(t)) :=$$
$$h(b)Y_\nu(r(b)) - h(a)Y_\nu(r(a)) - \int_{(a,b]} Y_\nu(r(t)-)dh(t) \in E,$$

cf. Jurek and Vervaat (1983) or Jurek and Mason (1993). Note that the above definition is sufficient for our purposes. Improper mappings $I^{h,r}_{(a,\infty)}$ are defined as weak limits as $b \to \infty$; similarly, as weak limits, are defined the improper random integrals $I^{h,r}_{(a,b)}$; cf. Jurek (2011) (invited Lecture at 10^{th} Vilnius Conference on Probability in 2010) or Jurek (2012).

Families of probability distributions (∗) form a convolution semigroup of the semigroup, ID, of all infinitely divisible probability measures. In fact, whole semigroup ID can be written as a weak closure a sum of an increasing family of the semigroups $I^{t,t^{\beta_n}}_{(0,1]}(ID)$, as $\beta_n \to \infty$.

Recall here that the integral $I^{h,r}_{(a,b]}$ commute with each other, that is,

$$I^{h_1,r_1}_{(a_1,b_1]}\big(I^{h_2,r_2}_{(a_2,b_2]}(\mu)\big) = I^{h_2,r_2}_{(a_2,b_2]}\big(I^{h_1,r_1}_{(a_1,b_1]}(\mu)\big),$$

provided μ is in appropriate domains. It follows from the Lévy-Khintchine formula for characteristic functions of infinitely divisible distributions; cf. for details in Jurek (2012).

2. For $-\infty < \beta < \alpha < \infty$, let us define the following two families of time change clocks:

$$r_\alpha(t) := \int_t^\infty u^{-\alpha-1} e^{-u} du, \qquad \text{for} \quad 0 < t < \infty; \qquad \text{and}$$

$$r_{\beta,\alpha}(t) := (\Gamma(\alpha-\beta))^{-1} \int_s^1 (1-u)^{\alpha-\beta-1}\, u^{-\alpha-1} du, \qquad \text{for} \quad 0 < t < 1\,. \quad (1)$$

Sato (2006) used the implicitly given inverse functions r_α^{-1} and $r_{\alpha,\beta}^{-1}$ to define two improper random integrals. In our terminology and notations these were random integral mappings $I^{t,\,r_\alpha(t)}_{(0,\infty)}$ and $I^{s,\,r_{\alpha,\beta}(s)}_{(0,1)}$. One of the results is the following factorizations of the above two mappings:

PROPOSITION 1. *For $-\infty < \beta < \alpha < \infty$ and infinitely divisible ν, on a real separable Banach space, such that the following integrals are well defined we have that*

$$I^{t,r_\beta(t)}_{(0,\infty)}\big(I^{s,r_{\beta,\alpha}(s)}_{(0,1)}(\nu)\big) = I^{t,\,\int_t^\infty u^{-\beta-1}e^{-u}du}_{(0,\infty)}\big(I^{s,\,(\Gamma(\alpha-\beta))^{-1}\int_s^1 (1-u)^{\alpha-\beta-1}\,u^{-\alpha-1}du}_{(0,1)}(\nu)\big)$$
$$= I^{t,\,\int_t^\infty u^{-\alpha-1}e^{-u}du}_{(0,\infty)}(\nu) = I^{t,r_\alpha(t)}_{(0,\infty)}(\nu). \quad (2)$$

Remark 1. (i) Above we keep the explicit form the inner clock time for an ease of a reference and a comparison.

(ii) For general questions related to domains of the above random integrals we refer to Sato (2006) and Jurek (2012). However, from Jurek (2012),Corollary 10, we infer that in (2) for ν we can take stable measures with the exponent $p > \alpha$.

(iii) Also, the proof below is valid for any real separable infinite dimensional Banach space – not only for Euclidean space $\mathbb{R}^d$ as it's in Sato (2006).

Proof of Proposition 1. As in Theorem 2, Section 4.2 in Jurek (2012), let us define Borel measures ρ_i using the inner clock time change from (1). Namely, let

$$\rho_1((c,d]) := \int_{(c,d]} u^{-\beta-1}e^{-u}du, \quad (c,d] \subset (0,\infty) \tag{3}$$

and

$$\rho_2((c,d]) := (\Gamma(\alpha-\beta))^{-1}\int_{(c,d]} (1-u)^{\alpha-\beta-1}\, u^{-\alpha-1}du, \quad (c,d] \subset (0,1) \tag{4}$$

Furthermore, let define the space change functions as follows

$$h_1(t) := t,\ t \in (0,\infty) \qquad \text{and} \qquad h_2(s) := s,\ s \in (0,1) \tag{5}$$

Finally, let

$$\boldsymbol{\rho} := \rho_1 \times \rho_2 \text{ and } \quad \mathbf{h}(t,s) := h_1 \otimes h_2(t,s) = h_1(t)h_2(s) \text{ (tensor product)} \tag{6}$$

Now observe that for the image measure $\mathbf{h}\boldsymbol{\rho}$ and $u > 0$ we have

$$(\mathbf{h}\boldsymbol{\rho})(x : x > u) = \int_0^\infty 1_{(x:x>u)}(v)\mathbf{h}\boldsymbol{\rho}(dv) = \int_0^\infty \int_0^1 1_{(x:x>u)}(s{\cdot}t)\rho_1(ds)\rho_2(dt)$$
$$= (\Gamma(\alpha{-}\beta))^{-1}\int_0^\infty \Big(\int_0^1 1_{(x:x>u)}(s{\cdot}t)(1{-}s)^{\alpha-\beta-1}\, s^{-\alpha-1}ds\Big)t^{-\beta-1}e^{-t}dt (\text{w:=st})$$
$$= (\Gamma(\alpha-\beta))^{-1}\int_0^\infty \Big(\int_0^t 1_{(x:x>u)}(w)(1-\frac{w}{t})^{\alpha-\beta-1}\,(\frac{w}{t})^{-\alpha-1}\frac{dw}{t}\Big)t^{-\beta-1}e^{-t}dt$$
$$= (\Gamma(\alpha{-}\beta))^{-1}\int_0^\infty \Big(\int_0^t 1_{(x:x>u)}(w)(t{-}w)^{\alpha-\beta-1}\, w^{-\alpha-1}dw\Big)e^{-t}\, dt \text{ (changing order)}$$
$$= (\Gamma(\alpha-\beta))^{-1}\int_0^\infty 1_{(x:x>u)}(w)w^{-\alpha-1}\Big(\int_w^\infty (t-w)^{\alpha-\beta-1}e^{-t}dt\Big)dw$$
$$= (\Gamma(\alpha-\beta))^{-1}\int_0^\infty 1_{(x:x>u)}(w)w^{-\alpha-1}e^{-w}\Big(\int_w^\infty (t-w)^{\alpha-\beta-1}e^{-(t-w)}dt\Big)dw$$
$$= \int_u^\infty w^{-\alpha-1}e^{-w}dw.$$

Hence and from Theorem 2 in Jurek (2012) we get the equality (2) which completes the proof.

COROLLARY 1. *(a) For $-\infty < \beta < \alpha < \infty$ and the inner clock changes r_α and $r_{\beta,\alpha}$ given in (1) we have a factorization*

$$I_{(0,\infty)}^{t,\, r_\beta(t)} \circ I_{(0,1)}^{s,\, r_{\beta,\alpha}(s)} = I_{(0,\infty)}^{t,\, r_\alpha(t)}$$

(b) For $-\infty<\alpha_k<\alpha_{k-1}<\alpha_{k-2}<...<\alpha_2<\alpha_1<\infty$ we have

$$I^{t,\,r_{\alpha_k}(t)}_{(0,\infty)}\circ I^{s,\,r_{\alpha_k,\alpha_{k-1}}(s)}_{(0,1)}\circ I^{s,\,r_{\alpha_{k-1},\alpha_{k-2}}(s)}_{(0,1)}\circ ...\circ I^{s,\,r_{\alpha_2,\alpha_1}(s)}_{(0,1)}=I^{t,\,r_{\alpha_1}(t)}_{(0,\infty)}\,,$$

where $\circ$ denotes the composition of the random integral mappings.

Proofs follows from Proposition 1 by mathematical induction argument.

3. The following factorization was predicted but not proved in Sato (2006) in Comment 2 on p. 86. Here it is phrased in the terms of our integral mappings $I^{h,r}_{(a,b]}$.

PROPOSITION 2. *For $-\infty<\gamma<\beta<\alpha<\infty$ and an infinitely divisible ν, on a real separable Banach space, such that the following integral are well defined, we have that*

$$I^{t,\,r_{\beta,\alpha}(t)}_{(0,1)}\big(I^{s,\,r_{\gamma,\beta}(s)}_{(0,1)}(\nu)\big)=I^{s,\,r_{\gamma,\beta}(s)}_{(0,1)}\big(I^{t,\,r_{\beta,\alpha}(t)}_{(0,1)}(\nu)\big)=I^{u,\,r_{\gamma,\alpha}(u)}_{(0,1)}(\nu) \qquad (7)$$

Proof of Proposition 2. For later use let recall the relation between the special functions beta and gamma. Namely, for $a>0,\ b>0$

$$B(a,b):=\int_0^1(1-u)^{a-1}u^{b-1}du, \quad \text{and} \quad B(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}.$$

As in proof of Proposition 1 we use Theorem 2 from Jurek (2012).

For the Lévy exponent of the ID measure on the right hand side in (7) we have

$$\Gamma(\alpha-\beta)\,\Gamma(\beta-\gamma)\int_0^1\big(\int_0^1\Phi(s\,t\,y)|dr_{\alpha,\beta}(t)|\big)|dr_{\beta,\gamma}(t)|$$

$$=\int_0^1\big(\int_0^1\Phi(s\,t\,y)(1-t)^{\alpha-\beta-1}t^{-\alpha-1}dt\big)(1-s)^{\beta-\gamma-1}s^{-\beta-1}ds \ (\text{put } st=:w)$$

$$=\int_0^1\big(\int_0^s\Phi(w\,y)(1-\frac{w}{s})^{\alpha-\beta-1}(\frac{w}{s})^{-\alpha-1}\frac{dw}{s}\big)(1-s)^{\beta-\gamma-1}s^{-\beta-1}ds$$

$$=\int_0^1\Phi(wy)w^{-\alpha-1}\big(\int_w^1(s-w)^{\alpha-\beta-1}(1-s)^{\beta-\gamma-1}ds\big)dw \quad (\text{put } \ 1-s=:z)$$

$$=\int_0^1\Phi(wy)w^{-\alpha-1}\big(\int_0^{1-w}(1-w-z)^{\alpha-\beta-1}z^{\beta-\gamma-1}dz\big)dw \ (\text{put } (1-w)^{-1}z=:x)$$

$$=\int_0^1\Phi(wy)w^{-\alpha-1}(1-w)^{\alpha-\gamma-1}dw\int_0^1(1-x)^{\alpha-\beta-1}x^{\beta-\gamma-1}dx$$

$$=B(\alpha-\beta,\beta-\gamma)\,\Gamma(\alpha-\gamma)\int_0^1\Phi(wy)|dr_{\alpha,\gamma}(w)|$$

$$=\Gamma(\alpha-\beta)\ \Gamma(\beta-\gamma)\int_0^1\Phi(wy)|dr_{\alpha,\gamma}(w)|,$$

which proves identity (7) and Proposition 2.

COROLLARY 2. *For positive integer $k \geq 2$ and reals α_i, $i = 1, 2, ..., k$ such that $-\infty < \alpha_k < \alpha_{k-1} < ... < \alpha_2 < \alpha_1 < \infty$ we have*

$$I_{(0,1)}^{t,\, r_{\alpha_2,\alpha_1}(t)} \circ I_{(0,1)}^{t,\, r_{\alpha_3,\alpha_2}(t)} \circ I_{(0,1)}^{t,\, r_{\alpha_4,\alpha_3}(t)} \circ ... \circ I_{(0,1)}^{t,\, r_{\alpha_k,\alpha_{k-1}}(t)} = I_{(0,1)}^{t,\, r_{\alpha_k,\alpha_1}(t)}$$

where $\circ$ denotes the composition of the random integral mappings.

It's proof follows from Proposition 2 via the induction argument.

Last but not least, from the few instances showed in this note, one may expect that the images of measures through tensor product will find more applications and may provide new simpler proofs as well; cf. Jurek (2012) for more examples.

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