s-SELF-DECOMPOSABLE PROBABILITY MEASURES AS PROBABILITY DISTRIBUTIONS OF SOME RANDOM INTEGRALS

Z. J. JUREK

ABSTRACT. The class of the $s$-self-decomposable measures on a Hilbert space coincides with limit distributions of sequences of random variables deformed by some non-linear transformations. In this note the $s$-self-decomposable measures are characterized as the probability distributions of some random integrals with respect to processes with stationary independent increments.

1. NOTATIONS AND PRELIMINARIES. Let $H$ be a real separable Hilbert space with the norm $\| \cdot \|$ and the scalar product $(\cdot, \cdot)$. For arbitrary positive real number $r$, i.e. $r \in \mathbb{R}^+$, we define transformations $T_r$ and $U_r$ from $H$ into $H$ by means of the formulas:
(1.1) \[ T_I x := r x, \]

and

(1.2) \[ U_I x := \max(0, \|x\| - r)x/\|x\|, \quad U_I 0 = 0. \]

The set \( \{ U_I : r \in \mathbb{R}^+ \} \) forms one-parameter semi-group of non-linear mappings of \( H \) called the shrinking operations (for short: \( s \)-operation). By \( P(H) \) and \( ID(H) \) we denote the semi-groups of all probability measures on \( H \) and all infinitely divisible ones, respectively. It is well-known that \( \mu \in ID(H) \) if and only if its characteristic function \( \hat{\mu} \) is of the form

(1.3) \[ \hat{\mu}(y) = \exp \left\{ i(y, z) - \frac{1}{2} (Ry, y) + \right. \]

\[ + \int_{H \setminus \{0\}} [\exp i(y, x) - 1 - i(y, x)/(1 + \|x\|^2)] M(dx) \right\}, \]

where \( z \) is a fixed vector from \( H \), \( R \) is an \( s \)-operator and \( M \) is a Lévy measure, cf. [5], Chapter VI, Theorem 4.10. Since the representation (1.3) is unique we shall write \( \mu = [z, R, M] \) if \( \hat{\mu} \) is of the form (1.3). The Lévy spectral function, \( L_M \), associated with a Lévy measure \( M \) we define by the formula

(1.4) \[ L_M(\lambda, r) := -M(\{x \in H \setminus \{0\} : x/\|x\| \in \lambda, \|x\| > r\}), \]
where \( r \in \mathbb{R}^+ \) and \( A \) is a Borel subset of the unit sphere \( S \) in \( H \). Note that \( L_m \) uniquely determines \( M \). For \( \mu = [z, R, M] \in ID(H) \) and \( t \in \mathbb{R}^+ \) by \( \mu^t \) we mean the infinitely divisible measure \([\varepsilon z, tR, tM]\). Finally, for a Borel measure \( m \) on \( H \) and a Borel mapping \( f \) from \( H \) into \( H \), \( fm \) is a Borel measure given by the formula

\[
(fm)(A) = m(f^{-1}(A))
\]

for all Borel subsets \( A \) of \( H \), and by \( L(x) \) we denote the probability distribution of \( H \)-valued random variable \( x \).

Let \( D_H[a, b] \) denote the set of \( H \)-valued functions that are right continuous on \([a, b]\) and have left-hand limits on \((a, b)\). For stochastic processes \( Y(t, \omega) \) with sample paths in \( D_H[a, b] \) and real valued functions \( f \) with bounded variation on \([a, b]\) we define the random integral as follows

\[
\int_{[a, b]} f(t) dY(t) := f(b)Y(b) - f(a)Y(a) - \int_{[a, b]} Y(t, \omega) df(t),
\]

where the integral on the right-hand side is meant as a Riemann-Stieltjes integral for fixed \( \omega \). Its existence is ensured by Lemma 14.1 in [1]. For our purposes we also can assume LUKÁCS-PRÉKOPA's definition, cf. [4], Chapter
VI. Namely, for \( H \)-valued stochastic process \( Y \) with independent increments and continuous real-valued function \( f \) on \([a, b]\) we put

\[
\int f \, dY := \lim_{k} \sum_{n} f(t_{nk}^{*}) (Y(t_{nk}) - Y(t_{nk-1})) , \quad (a, b)
\]

where the limit is taken in probability as the width of the partition \( \{t_{nk}\} \) of \((a, b)\) tends to zero and \( t_{nk-1} < t_{nk} < t_{nk+1} \). In order to extend the results of [4], Chapter VI to the infinite dimensional space, especially Lemma 6.2.1, we can apply the following inequality

\[
P\left( \left\| \sum_{k=1}^{n} a_{k} \xi_{k} \right\| > t \right) \leq 2P\left( \max_{k} a_{k} \left\| \sum_{k=1}^{n} \xi_{k} \right\| > t \right)
\]

valid for symmetric independent random variables \( \xi_{1}, \xi_{2}, \ldots, \xi_{n} \) and real numbers \( a_{1}, a_{2}, \ldots, a_{n} \), cf. [6], Theorem 1.2. From both definitions if \( Y \) is a \( D_{H} [a, b] \) -valued random variable with stationary independent increments and \( f \) is continuous with bounded variation on \([a, b]\) then

\[
\hat{L} \left( \int f(t) dY(t) \right)(y) =
\]

\[
= \exp \int_{(a, b)} \log \hat{L}(Y(t))(f(t)y) \, dt , \quad y \in H ,
\]
and for $s \in \mathbb{R}^+$

\[
L\left( \int_{(a,b]} f(t) \, dY(t) \right)^{\ast s} = L\left( \int_{(a,b]} f(t) \, dY(st) \right).
\]

Finally, if $0 < a < b < c < \infty$ and $Y$ has independent increments then the random variables

\[
\int_{(a,b]} f(t) \, dY(t), \quad \int_{(b,c]} f(t) \, dY(t)
\]

are independent as well.

2. $s$-SELF-DECOMPOSABLE PROBABILITY MEASURES. Let $x_1, x_2, \ldots$ be a sequence of independent $H$-valued random variables which are not essentially uniformly bounded.

Let $r_1, r_2, \ldots$ be a non-decreasing sequence of positive real numbers and $b_1, b_2, \ldots$ a sequence of vectors from $H$. The limit distributions of sums

\[
\sum_{j=1}^{n} U_{r_n} x_j + b_n,
\]

where the random variables $U_{r_n} x_j$ ($j=1, 2, \ldots, n$; $n=1, 2, \ldots$) form an infinitesimal triangular array, will be called the $s$-self-decomposable probability measures. The $s$-operations $U_{r_n}$ are defined by (1.2) and $U(H)$ denotes the class of all $s$-self-decomposable measures on $H$. After-mentioned theorem collects all known descriptions of the class $U(H)$.

- 621 -
THEOREM 2.1. The following statements are equivalent:

1. \( \mu \in U(H) \).

2. \( \mu = [z, R, M] \) and for all \( t \in \mathbb{R}^+ \), \( M \geq \mathbb{E}_L \) on \( H \setminus \{0\} \).

3. \( \mu = [z, R, M] \) and its Lévy spectral function \( L_M \), for each Borel subset \( A \) of \( S \), has right and left derivatives with respect to \( r \) such that \( dL_M(A, r)/dr \) is non-increasing on \( \mathbb{R}^+ \).

4. \( \hat{\mu}(y) = \exp \left\{ \frac{1}{2} i(y, z_0)^{-1} (Ry, y) + \int_{H \setminus \{0\}} \left[ \frac{\exp i(y, x) - 1}{i(y, x)} - 1 - \frac{i(y, x) \log (1 + \|x\|^2)}{2\|x\|^2} \right] \frac{1}{\|x\|^2} m(dx) \right\} \),

where \( z_0 \in H \), \( R \) is an \( S \)-operator and \( m \) is a finite Borel measure on \( H \setminus \{0\} \).

5. \( \hat{\mu}(y) = \exp \int_0^1 \log \hat{\nu}(ty) \, dt \) for some \( \nu \in ID(H) \).

6. \( \mu = L \left( \int_0^1 t \nu(\tau) \, d\tau \right) \), where \( \nu \) is \( D_H [0, 1] \) -valued random variable with stationary independent increments and \( \nu(0) = 0 \) a.s.

7. \( \mu \in ID(H) \) and for every \( 0 < c < 1 \) there exists \( \mu_c \in ID(H) \) such that \( \mu = T_c^{\alpha_c} \mu_c \).

8. \( \mu \) belongs to the smallest closed subsemigroup of \( ID(H) \) containing all measures of the form \([z, R, 0]\).
and \([z, 0, N_{\alpha, m}]\) where

\[
N_{\alpha, m}(B) = \int \int_{S \times 0} \alpha_{B(tx)} d\mu(dx)
\]

with \(\alpha \in \mathbb{R}^+\) and finite Borel measure \(m\) on \(S\).

PROOF. (1) \(\Rightarrow\) (2) \(\Rightarrow\) (4) is proved in [2], Theorem 5.1 and Corollary 7.1 respectively. (3) \(\Rightarrow\) (2) \(\Rightarrow\) (8) is given in [3] as Proposition 3.1 and Theorem 3.1 respectively. (5) \(\Rightarrow\) (6) in view of the formula (1.8).

(4) \(\Rightarrow\) (5). Let us note that

\[
\int_{0}^{1} \left[ \exp(i(y, x)t) - 1 - i(y, x)t/(1 + t^2 \|x\|^2) \right] dt = 0
\]

\[
= \exp(i(y, x)) - 1 - \frac{i(y, x) \log(1 + \|x\|^2)}{2 \|x\|^2}
\]

and there exists \(z_1 \in H\) such that for all \(y \in H\)

\[
(y, z_1) = 
\]

\[
= \int_{H \setminus \{0\}} \int_{0}^{1} \left[ (1 + t^2 \|x\|^2)^{-1} - (1 + \|x\|^2)^{-1} \right] (y, x) t dt \|x\|^{-\arctg \|x\|} \ p(dx)
\]

where \(p\) is a finite Borel measure on \(H \setminus \{0\}\) related to the measure \(m\) by the formula

\[
p(A) = \int_{A} \left( \|x\|^{-\arctg \|x\|} \right) (1 + \|x\|^2)/\|x\|^3 \ m(dx)
\]

- 623 -
Applying Fubini's Theorem we get

\[ \hat{\mu}(y) = \exp \int_{0}^{1} \left\{ 2it(y, z_0 + z_1) - t^2/2(3R, y) + \right. \\
+ \left. \int_{H \sim \{0\}} \left\{ \exp it(y, x) - 1 - \frac{it(y, x)}{1 + \|x\|^2} \right\} \|x\|/(\|x\| - \arctg\|x\|) p(dx) \right\} dt, \]

which implies (5) with \( \nu = [2(z_0 + z_1), 3R, M] \) where

\[ M(a) := \int_{A} \|x\|/(\|x\| - \arctg\|x\|) p(dx). \]

(6) \( \Rightarrow \) (7). Of course \( \mu = L \left( \int_{(0, 1]} t \, dY(t) \right) \in ID(H) \) and (1.9) implies for \( 0 < c < 1 \)

\[ \mu = L \left( \int_{(0, c]} t \, dY(t) \right) * L \left( \int_{(c, 1]} t \, dY(t) \right) = \]

\[ = L \left( \int_{(0, 1]} c \, t \, dY(t) \right) * L \left( \int_{(c, 1]} t \, dY(t) \right) = T_c \mu_c * \mu_c \]

where \( \mu_c := L \left( \int_{(c, 1]} t \, dY(t) \right) \) which completes the proof.

(7) \( \Rightarrow \) (3). At first note, in view of Theorem 4.10, Chapter VI in [5] that if \( M \) and \( M_c \) are Lévy measures of \( \mu \) and \( \mu_c \) respectively, then

\[ \forall (0 < c < 1) \quad M_c = M - c(T_c M) \geq 0. \]

Hence for fixed Borel subset \( A \) of \( S \) we obtain
\[ L_M(A, r_2) - L_M(A, r_1) \geq c(L_M(A, r_2/c) - L_M(A, r_1/c)) \]

for all \( 0 < r_1 < r_2 \) and all \( 0 < c < 1 \). Taking \( c := r_1/r_2 \) we have

\[ L_M(A, r_2) \geq r_1/(r_1 + r_2) L_M(A, r_2^2/r_1) + r_2/(r_1 + r_2) L_M(A, r_1), \]

i.e., for each \( A \) the function \( R^+ \ni r \rightarrow L_M(A, r) \) is concave. So, it has non-increasing left and right derivatives, which completes the proof of Theorem 1.

Let \( \nu \in ID(H) \) and \( \gamma \) be a \( D_H[0,1] \)-valued random variable with stationary independent increments such that \( L(\gamma(1)) = \nu \). We have established the mapping

\[ J: ID(H) \ni L(\gamma(1)) \rightarrow L \left( \int_0^1 t \circ \gamma(t) \right) \in U(H) \]

that we want to investigate further. Let us note that if \( \nu = [z, R, M] \) and \( J\nu = [z', R', M'] \) then

\[ R' = \frac{1}{3} R, \]

\[ M'(A) = \int_0^1 T_{t'} M(A) \, dt \text{ for Borel subsets } A \text{ of } H \setminus \{0\}, \]

\[ z' = \frac{1}{2} z + \int_{H \setminus \{0\}} x/(1 + \|x\|^2) (M'(dx) - \frac{1}{2} M(dx)). \]
Using the above notations we have the following theorem:

THEOREM 2.2. The mapping $J$ is an isomorphism between the semi-groups $\mathcal{ID}(H)$ and $\mathcal{U}(H)$ with the convolution operation. Moreover we have

(a) $J(\mu \overset{\text{\#}}{\ast} t) = (J\mu) \overset{\text{\#}}{\ast} t$ for $t \in \mathbb{R}^+$

(b) $J(T\mu) = T(J\mu)$ for bounded linear operator $T$ on $H$.

PROOF. Theorem 2.1 ((1)\iff (6)) implies that $J$ is mapping onto $\mathcal{U}(H)$. The formulas $J(\mu \overset{\text{\#}}{\ast} \nu) = (J\mu) \overset{\text{\#}}{\ast} (J\nu)$ and (a) follow from (2.3)-(2.5). The equality

$$T(J\mu) = TL \left[ \int_{(0,1]} t dY(t) \right] = L \left[ \int_{(0,1]} t d(TY(t)) \right] = J(T\mu),$$

where $\mu = L(Y(1))$, implies (b). It remains to show that $J$ is one-to-one. To see this it is enough to prove that the Lévy spectral function $L_M'$ uniquely determines $L_M$. From (2.4) we have

$$L_M'(A,r) = r \int_{r}^{\infty} L_M(A,s)/s^2 ds,$$

and hence

$$L_M(A,r) = L_M'(A,r) - rdL_M'(A,r)/dr$$

for almost all $r \in \mathbb{R}^+$ i.e., $L_M'$ determines $L_M$ uniquely.
which completes the proof.

Remark 2.3. The fact that \( J \) is one-to-one mapping one can prove by the formula (1.8). Putting for fixed

\[ y \in H \]

\[ g_y(s) := \log \int_{(0,1]} (sy) (t) \]  

we obtain

\[ g_y(s) = s^{-1} \int_0^s \log \hat{y}(y(r))(y) dr , \]

and hence

\[ \log \hat{y}(y(r)) = g_y(s) + s \frac{d g_y(s)}{ds} . \]

Consequently,

\[ \hat{y}(y(r))(y) = \exp g_y(1) \exp \left( \frac{d g_y(s)}{ds} \bigg|_{s=1} \right) . \]

Remark 2.4. The formula (2.3) implies that the class of all Gaussian measures is invariant under the mapping \( J \). Further, the class of finite convolutions of the measures of the form \( [z,0,c \delta_x] , c \in \mathbb{R}_+ \) and \( x \in H \setminus \{0\} \) \( (\delta_x \) denotes the probability measure concentrated at \( x) \) forms a dense subset of \( ID(H) \), cf. [5], Theorems 4.7 and 4.10 in Chapter VI. From (2.5) we get
\[(c \delta_x)'(B) = c/\|x\| \int_0^1 B(tx/\|x\|) dt = \]
\[
\int \int_S 1_B(tz) \frac{dt}{\|x\|} \delta_x(\frac{x}{\|x\|}(dz)) ,
\]

for Borel subsets \( B \) of \( H \setminus \{0\} \), i.e. \( J([z,0,c \delta_x]) \)
\((c \in \mathbb{R}^+, \ x \in H \setminus \{0\})\) with Gaussian measures generate the class \( \mathcal{U}(H) \), cf. Theorem 2.1 ((1)→(8)).

REFERENCES


Zbigniew J. Jurek
Institute of Mathematics,
Wrocław University,
Pl. Grunwaldzki 2/4,
50-384 WROCŁAW,
Poland