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s-SELF-DECOMPOSABLE PROBABILITY MEASURES AS PROBABILITY DISTRIBUTIONS OF SOME RANDOM INTEGRALS

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ABSTRACT. The class of the s-self-decomposable measures on a Hilbert space coincides with limit distributions of sequences of random variables deformed by some non-linear transformations. In this note the s-self-decomposable measures are characterized as the probability distributions of some random integrals with respect to processes with stationary independent increments.

1. NOTATIONS AND PRELIMINARIES. Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $(\cdot,\cdot)$. For arbitrary positive real number r , i.e. $r\in\mathbb{R}^+$, we define transformations T_r and U_r from H into H by means of the formulas:

(1.1) $$T_r x := rx ,$$

and

(1.2) $$U_r x := \max(0, \|x\| - r) x / \|x\| , \quad U_r 0 = 0 .$$

The set $\{U_r : r \in \mathbb{R}^+\}$ forms one-parameter semi-group of non-linear mappings of H called *the shrinking operations* (for short: *s -operation*). By $P(H)$ and $ID(H)$ we denote the semi-groups of all probability measures on H and all infinitely divisible ones, respectively. It is well-known that $\mu \in ID(H)$ if and only if its characteristic function $\hat{\mu}$ is of the form

(1.3) $$\hat{\mu}(y) = \exp\Big\{ i(y,z) - \frac{1}{2}(Ry,y) + \int_{H \setminus \{0\}} [\exp i(y,x) - 1 - i(y,x)/(1+\|x\|^2)] M(dx) \Big\} ,$$

where z is a fixed vector from H, R is an S-operator and M is a Lévy measure, cf. [5], Chapter VI, Theorem 4.10. Since the representation (1.3) is unique we shall write $\mu = [z, R, M]$ if $\hat{\mu}$ is of the form (1.3). *The Lévy spectral function*, L_M, associated with a Lévy measure M we define by the formula

(1.4) $$L_M(A, r) := -M(\{x \in H \setminus \{0\} : x/\|x\| \in A, \|x\| > r\}),$$

where $r \in \mathbb{R}^+$ and A is a Borel subset of the unit sphere S in H. Note that L_M uniquely determines M. For $\mu = [z,R,M] \in ID(H)$ and $t \in \mathbb{R}^+$ by μ^{*t} we mean the infinitely divisible measure $[tz,tR,tM]$. Finally, for a Borel measure m on H and a Borel mapping f from H into H, fm is a Borel measure given by the formula

$$(1.5) \qquad (fm)(A) := m(f^{-1}(A))$$

for all Borel subsets A of H, and by $L(X)$ we denote *the probability distribution* of H-valued random variable X.

Let $D_H[a,b]$ denote the set of H-valued functions that are right continuous on $[a,b)$ and have left-hand limits on $(a,b]$. For stochastic processes $Y(t,\omega)$ with sample paths in $D_H[a,b]$ and real valued functions f with bounded variation on $[a,b]$ we define the random integral as follows

$$(1.6) \quad \int_{(a,b]} f(t)dY(t) := f(b)Y(b) - f(a)Y(a) - \int_{(a,b]} Y(t,\omega)df(t) ,$$

where the integral on the right-hand side is meant as a Riemann-Stieltjes integral for fixed ω. Its existence is ensured by Lemma 14.1 in [1]. For our purposes we also can assume LUKÁCS-PRÉKOPA's definition, cf. [4], Chapter

VI. Namely, for H -valued stochastic process Y with independent increments and continuous real-valued function f on $[a,b]$ we put

$$(1.7) \qquad \int_{(a,b]} f dY := \lim \sum_k f(t^*_{nk})(Y(t_{nk})-Y(t_{nk-1})) ,$$

where the limit is taken in probability as the width of the partition $\{t_{nk}\}$ of $(a,b]$ tends to zero and $t_{nk-1} < t^*_{nk} \le t_{nk}$. In order to extend the results of [4], Chapter VI to the infinite dimensional space, especially Lemma 6.2.1, we can apply the following inequality

$$P\left(\left\| \sum_{k=1}^{n} a_k \xi_k \right\| > t\right) \le 2P\left(\max_k |a_k| \left\| \sum_{k=1}^{n} \xi_k \right\| > t\right)$$

valid for symmetric independent random variables ξ_1, $\xi_2, \dots, \xi_n$ and real numbers $a_1, a_2, \dots, a_n$, cf. [6], Theorem 1.2. From both definitions if Y is a $D_H[a,b]$ -valued random variable with stationary independent increments and f is continuous with bounded variation on $[a,b]$ then

$$(1.8) \qquad \hat{L}\left(\int_{(a,b]} f(t)dY(t)\right)(y) =$$

$$= \exp \int_{(a,b]} \log \hat{L}(Y(1))(f(t)y)dt \ , \quad y \in H \ ,$$

and for $s \in \mathbb{R}^+$

$$(1.9) \qquad L\left(\int_{(a,b]} f(t)\,dY(t)\right)^{*s} = L\left(\int_{(a,b]} f(t)\,dY(st)\right) .$$

Finally, if $0<a<b<c<\infty$ and Y has independent increments then the random variables

$$\int_{(a,b]} f(t)\,dY(t) , \qquad \int_{(b,c]} f(t)\,dY(t)$$

are independent as well.

2. s-SELF-DECOMPOSABLE PROBABILITY MEASURES. Let $X_1, X_2, \ldots$ be a sequence of independent H-valued random variables which are not essentially uniformly bounded. Let $r_1, r_2, \ldots$ be a non-decreasing sequence of positive real numbers and $b_1, b_2, \ldots$ a sequence of vectors from H. The limit distributions of sums

$$(2.1) \qquad \sum_{j=1}^{n} U_{r_n} X_j + b_n ,$$

where the random variables $U_{r_n} X_j$ $(j=1,2,\ldots,n \,;\, n=1,2,\ldots)$ form an infinitesimal triangular array, will be called the *s-self-decomposable probability measures*. The s-operations U_{r_n} are defined by (1.2) and $\mathcal{U}(H)$ denotes the class of all s-self-decomposable measures on H. After-mentioned theorem collects all known descriptions of the class $\mathcal{U}(H)$.

THEOREM 2.1. *The following statements are equivalent:*

(1) $\mu \in \mathcal{U}(H)$.

(2) $\mu = [z,R,M]$ *and for all* $t \in \mathbb{R}^+$, $M \geq U_t M$ *on* $H \setminus \{0\}$.

(3) $\mu = [z,R,M]$ *and its Lévy spectral function* L_M , *for each Borel subset* A *of* S , *has right and left derivatives with respect to* r *such that* $dL_M(A,r)/dr$ *is non-increasing on* $\mathbb{R}^+$.

(4) $$\hat{\mu}(y) = \exp\left\{ i(y,z_0) - \frac{1}{2}(Ry,y) + \int_{H\setminus\{0\}} \left[\frac{\exp i(y,x) - 1}{i(y,x)} - 1 - \frac{i(y,x)\log(1+\|x\|^2)}{2\|x\|^2} \right] \frac{1+\|x\|^2}{\|x\|^2}\, m(dx) \right\} ,$$

where $z_0 \in H$, R *is an S-operator and* m *is a finite Borel measure on* $H \setminus \{0\}$.

(5) $\hat{\mu}(y) = \exp \int_0^1 \log \hat{\nu}(ty)\,dt$ *for some* $\nu \in ID(H)$.

(6) $\mu = L\left(\int_{(0,1]} t\,dY(t) \right)$, *where* Y *is* $D_H[0,1]$ *-valued random variable with stationary independent increments and* $Y(0)=0$ *a.s.*

(7) $\mu \in ID(H)$ *and for every* $0<c<1$ *there exists* $\mu_c \in ID(H)$ *such that* $\mu = T_c \mu^{*c} * \mu_c$.

(8) μ *belongs to the smallest closed subsemigroup of* $ID(H)$ *containing all measures of the form* $[z,R,0]$

and $[z,0,N_{\alpha,m}]$ *where*

$$N_{\alpha,m}(B) = \int_S \int_0^{\alpha} 1_B(tx)\,dt\,m(dx)$$

with $\alpha \in \mathbb{R}^+$ *and finite Borel measure* m *on* S .

PROOF. (1)⇔(2)⇔(4) is proved in [2], Theorem 5.1 and Corollary 7.1 respectively. (3)⇔(2)⇔(8) is given in [3] as Proposition 3.1 and Theorem 3.1 respectively. (5)⇔(6) in view of the formula (1.8).

(4)⇒(5). Let us note that

$$\int_0^1 [\exp i(y,x)t - 1 - i(y,x)t/(1+t^2\|x\|^2)]\,dt =$$

$$= \frac{\exp i(y,x)-1}{i(y,x)} - 1 - \frac{i(y,x)\log(1+\|x\|^2)}{2\|x\|^2}$$

and there exists $z_1 \in H$ such that for all $y \in H$

$$(y,z_1) =$$

$$= \int_{H\setminus\{0\}} \int_0^1 \left[(1+t^2\|x\|^2)^{-1} - (1+\|x\|^2)^{-1}\right](y,x)\,t\,dt\,\frac{\|x\|}{\|x\|-\operatorname{arctg}\|x\|}\,p(dx) ,$$

where p is a finite Borel measure on $H\setminus\{0\}$ related to the measure m by the formula

$$p(A) = \int_A (\|x\|-\operatorname{arctg}\|x\|)(1+\|x\|^2)/\|x\|^3\,m(dx) .$$

Applying Fubini's Theorem we get

$$\hat{\mu}(y) = \exp \int_0^1 \Big\{ 2it(y,z_0+z_1) - t^2/2(3Ry,y) +$$

$$+ \int_{H\setminus\{0\}} [\exp it(y,x)-1-it(y,x)/(1+\|x\|^2)]\|x\|/(\|x\|-\mathrm{arctg}\|x\|)p(dx)\Big\}\, dt ,$$

which implies (5) with $\nu = [2(z_0+z_1), 3R, M]$ where

$$M(A) := \int_A \|x\|/(\|x\|-\mathrm{arctg}\|x\|)p(dx) .$$

(6)⇒(7). Of course $\mu = L\Big(\int_{(0,1]} t\,dY(t)\Big) \in ID(H)$ and (1.9) implies for $0<c<1$

$$\mu = L\Big(\int_{(0,c]} t\,dY(t)\Big) * L\Big(\int_{(c,1]} t\,dY(t)\Big) =$$

$$= L\Big(c\int_{(0,1]} t\,dY(t)\Big)^{*c} * L\Big(\int_{(c,1]} t\,dY(t)\Big) = T_c\mu^{*c} * \mu_c$$

where $\mu_c := L\Big(\int_{(c,1]} t\,dY(t)\Big)$ which completes the proof.

(7)⇒(3). At first note, in view of Theorem 4.10, Chapter VI in [5] that if M and M_c are Lévy measures of μ and μ_c respectively, then

$$\forall(0<c<1) \quad M_c = M - c(T_cM) \geq 0 .$$

Hence for fixed Borel subset A of S we obtain

$$L_M(A,r_2) - L_M(A,r_1) \geq c(L_M(A,r_2/c) - L_M(A,r_1/c))$$

for all $0 < r_1 < r_2$ and all $0<c<1$. Taking $c := r_1/r_2$ we have

$$L_M(A,r_2) \geq r_1/(r_1+r_2)L_M(A,r_2^2/r_1)+r_2/(r_1+r_2)L_M(A,r_1) ,$$

i.e., for each A the function $\mathbb{R}^+ \ni r \to L_M(A,r)$ is concave. So, it has non-increasing left and right derivatives, which completes the proof of Theorem 1.

Let $\nu \in ID(H)$ and Y be a $D_H[0,1]$ -valued random variable with stationary independent increments such that $L(Y(1)) = \nu$. We have established the mapping

$$(2.2) \qquad J\colon ID(H) \ni L(Y(1)) \to L\left(\int_{(0,1]} t\,dY(t)\right) \in U(H)$$

that we want to investigate further. Let us note that if $\nu = [z,R,M]$ and $J\nu = [z',R',M']$ then

$$(2.3) \qquad R' = \frac{1}{3} R ,$$

$$(2.4) \quad M'(A) = \int_0^1 (T_t M)(A)\,dt \quad \text{for Borel subsets } A \text{ of } H\setminus\{0\} ,$$

$$(2.5) \quad z' = \frac{1}{2} z + \int_{H\setminus\{0\}} x/(1+\|x\|^2)(M'(dx)-\tfrac{1}{2}M(dx)) .$$

Using the above notations we have the following theorem:

THEOREM 2.2. *The mapping* J *is an isomorphism between the semi-groups* $ID(H)$ *and* $U(H)$ *with the convolution operation. Moreover we have*

(a) $J(\mu^{*t}) = (J\mu)^{*t}$ *for* $t \in \mathbb{R}^+$

(b) $J(T\mu) = T(J\mu)$ *for bounded linear operator* T *on* H.

PROOF. Theorem 2.1 ((1)⇔(6)) implies that J is mapping onto $U(H)$. The formulas $J(\mu * \nu) = (J\mu) * (J\nu)$ and (a) follow from (2.3)-(2.5). The equality

$$T(J\mu) = TL\left(\int_{(0,1]} t\,dY(t)\right) = L\left(\int_{(0,1]} t\,d(TY(t))\right) = J(T\mu) ,$$

where $\mu = L(Y(1))$, implies (b). It remains to show that J is one-to-one. To see this it is enough to prove that the Lévy spectral function $L_{M'}$ uniquely determines L_M. From (2.4) we have

$$L_{M'}(A,r) = r \int_r^\infty L_M(A,s)/s^2 ds ,$$

and hence

$$L_M(A,r) = L_{M'}(A,r) - r\,dL_{M'}(A,r)/dr$$

for almost all $r \in \mathbb{R}^+$ i.e., $L_{M'}$ determines L_M uniquely, which completes the proof.

Remark 2.3. The fact that J is one-to-one mapping one can prove by the formula (1.8). Putting for fixed $y \in H$

$$g_y(s) := \log \hat{L}\left(\int_{(0,1]} t dY(t) \right)(sy) , \quad s \in \mathbb{R} ,$$

we obtain

$$g_y(s) = s^{-1} \int_0^s \log \hat{L}(Y(1))(ry)dr ,$$

and hence

$$\log \hat{L}(Y(1))(sy) = g_y(s) + s(dg_y(s)/ds) .$$

Consequently,

$$\hat{L}(Y(1))(y) = \exp g_y(1) \exp(dg_y(s)/ds|_{s=1}) .$$

Remark 2.4. The formula (2.3) implies that the class of all Gaussian measures is invariant under the mapping J . Further, the class of finite convolutions of the measures of the form $[z,0,c\delta_x]$, $c \in \mathbb{R}^+$ and $x \in H \setminus \{0\}$ (δ_x denotes the probability measure concentrated at x) forms a dense subset of $ID(H)$, cf. [5], Theorems 4.7 and 4.10 in Chapter VI. From (2.5) we get

$$(c\delta_x)'(B) = c/\|x\| \int_0^{\|x\|} 1_B(tx/\|x\|)dt =$$

$$= \int_S \int_0^{\|x\|} 1_B(tz)dt(c/\|x\| \delta_{x/\|x\|}(dz)) ,$$

for Borel subsets B of $H\setminus\{0\}$, i.e. $J([z,0,c\delta_x])$ $(c\in\mathbb{R}^+,\ x\in H\setminus\{0\})$ with Gaussian measures generate the class $U(H)$, cf. Theorem 2.1 ((1)⇔(8)).

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