SELFDECOMPOSABLE LAWS ASSOCIATED WITH HYPERBOLIC FUNCTIONS

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ABSTRACT. It is shown that the hyperbolic functions can be associated with selfdecomposable distributions (in short: SD probability distributions or Lévy class L probability laws). Consequently, they admit associated background driving Lévy processes Y (BDLP Y). We interpret the distributions of $Y(1)$ via Bessel squared processes, Bessel bridges and local times.

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1. Introduction and terminology. The aim of this note is to provide a new way of looking at the hyperbolic functions: *cosh, sinh* and *tanh*, or their modifications, as the members of the class SD of selfdecomposable characteristic functions (often called class L, after Paul Lévy). Analytically one says that *a characteristic function ϕ is selfdecomposable*, we simply write $\phi \in SD$,if

$$\forall(0 < c < 1)\, \exists\, \rho_c\, \forall(t \in \mathbb{R}) \quad \phi(t) = \phi(ct)\rho_c(t), \tag{1}$$

where ρ_c is also a characteristic function. Let us recall here that class SD is a proper subset of ID, the class of all *infinitely divisible characteristic* *functions*, and that the factors ρ_c in (1) are in ID as well ; cf. Jurek and Mason (1993), Section 3.9., or Loéve (1963), Section 23 (there this class is denoted by $\mathfrak{N}$). We also will use the convention that a random variable X (in short: r.v. X) or its probability distribution μ_X or its probability density f_X is selfdecomposable if the corresponding characteristic function is in the class SD. Furthermore, the equation (1) describing the selfdecomposability property, in terms of a r.v. X means that

$$X \in SD \quad \text{iff} \quad \forall (0 < c < 1) \exists (r.v.\ X_c) \quad X \stackrel{d}{=} cX + X_c,$$

where the r.v. X and X_c are independent and $\stackrel{d}{=}$ means equality in distribution.

For further references let us recall the main properties of the selfdecomposable distributions (or characteristic functions or r.v's):

(a) *SD with the convolution and the weak convergence forms a closed convolution subsemigroup of ID;*

(b) *SD is closed under affine mappings, i.e., for all reals a and b one has:* $\phi \in SD$ iff $e^{ibt}\phi(at) \in SD$.

(c) *$X \in SD$ iff there exists a (unique) Lévy process $Y(\cdot)$ such that* $X \stackrel{d}{=} \int_0^\infty e^{-s} dY(s)$, *where Y is called the BDLP (background driving Lévy process) of the X. Moreover, one has that* $\mathbb{E}[\log(1+|Y(1)|)] < \infty$.

[$ID_{\log}$ will stand for the class of all infinitely divisible laws with finite logarithmic moments.]

(d) *Let ϕ and ψ denote the characteristic function of X and $Y(1)$, respectively, in part (c). Then one has* $\log \phi(t) = \int_0^t \log \psi(v) \frac{dv}{v}$, i.e., $\psi(t) = \exp[t(\log \phi(t))'], t \neq 0, \psi(0) = 1$.

(e) *Let M be the Lévy spectral measure in the Lévy-Khintchine formula of $\phi \in SD$. Then M has a density $h(x)$ such that $xh(x)$ is non-increasing on the positive and negative half-lines.*
Furthermore, if h is differentiable almost everywhere then $dN(x) = -(xh(x)'dx$ is the Lévy spectral measure of ψ in (d).
Finally, one has also the following logarithmic moment condition $\int_{\{|x| \geq \epsilon\}} \log(1 + |x|) dN(x) < \infty$, *for all positive ϵ.*

Parts **(a)** and **(b)** follow directly from (1). For **(c)** and **(d)** cf. Jurek & Mason (1993), Theorem 3.6.8 and Remark 3.6.9(4). Part **(e)** is Corollary 1.1 from Jurek (1997).

In this note we will characterize the BDLP's (or the characteristic functions ψ in **d**) for the hyperbolic characteristic functions. The main result shows how to interpret these distributions in terms of squaredd Bessel bridges (Corollary 2) and squared Bessel processes (Corollary 3).

2. Selfdecomposability of the hyperbolic characteristic functions. For this presentation the most crucial example of SD r.v. is that of the *Laplace (or double exponential) random variable* η. So, η has the probability density $\frac{1}{2}e^{-|x|}, x \in \mathbb{R}$, and its characteristic function is equal

$$\phi_\eta(t) = \frac{1}{1+t^2} = \exp\Big[\int_{-\infty}^{\infty}(e^{itx}-1)\frac{e^{-|x|}}{|x|}\,dx\Big] = \exp\int_0^t\Big[\int_{-\infty}^{\infty}(e^{ivx}-1)e^{-|x|}dx\Big]\frac{dv}{v} \in SD. \tag{2}$$

To see its selfdecomposability property simply note that

$$\frac{\phi_\eta(t)}{\phi_\eta(ct)} = \frac{1+c^2t^2}{1+t^2} = c^2 1 + (1-c^2)\frac{1}{1+t^2} \quad \text{is the characteristic function of } \rho_c,$$

in the formula (1). The rest follows from appropriate integrations; cf. Jurek (1996).

Another, more "stochastic" argument for selfdecomposability of Laplace rv η, as a counterpart to the above analytic one, is as follows.

Firstly, notice that for three independent rvs $\mathcal{E}(1)$, $\tilde{\mathcal{E}}(1)$ and b_c, where the first two have exponential distribution with parameter 1 and the third one has Bernoulli distribution ($P(b_c = 1) = 1 - c$ and $P(b_c = 0) = c$), one has equality

$$\mathcal{E}(1) \stackrel{d}{=} c\mathcal{E}(1) + b_c\tilde{\mathcal{E}}(1),$$

which means that $\mathcal{E}(1)$ is a selfdecomposable rv. (The above distributional equality is easily checked by using the Laplace or Fourier transform.)

Secondly, taking two independent Brownian motions B_t, $\tilde{B}_t$, $t \geq 0$ and independently of them an exponential rv $\mathcal{E}(1)$ satisfying the above decomposition, we infer that

$$B_{\mathcal{E}(1)} \stackrel{d}{=} \sqrt{c}B_{\mathcal{E}(1)} + \tilde{B}_{b_c\tilde{\mathcal{E}}(1)},$$

and thus proving that stopped Brownian motion $B_{\mathcal{E}(1)}$ is selfdecomposable as well.

Thirdly, let us note that $B_{\mathcal{E}(1)}$ has the double exponential distribution. More explicitly we have

$$\mathbb{E}[e^{it(\sqrt{2}B_{\mathcal{E}(1)})}] = \mathbb{E}[e^{-t^2\mathcal{E}(1)}] = \frac{1}{1+t^2} = \phi_\eta(t).$$

For more details and a generalization of this approach cf. Jurek (2001), Proposition 1 and Bondesson (1992), p. 19.

PROPOSITION 1. *The following three hyperbolic functions:* $\frac{1}{\cosh t}$, $\frac{t}{\sinh t}$, $\frac{\tanh t}{t}$, $t \in \mathbb{R}$, *are characteristic functions of selfdecomposable probability distributions, i.e., they are in the class SD.*

Proof. From the following product representations :

$$\cosh z = \prod_{k=1}^{\infty}(1 + \frac{4z^2}{(2k-1)^2\pi^2}), \quad \sinh z = z\prod_{k=1}^{\infty}(1+\frac{z^2}{k^2\pi^2}), \tag{3}$$

for all complex z, and from (2) with **(a)** we conclude that the first two hyperbolic functions are characteristic functions from SD. Moreover, these are characteristic functions of series of independent Laplace r.v.; cf. Jurek (1996).

Note that for $0 < a < b$ the fraction

$$\frac{1+a^2t^2}{1+b^2t^2} = \frac{a^2}{b^2}1 + (1-\frac{a^2}{b^2})\frac{1}{1+b^2t^2} \quad \text{is a characteristic function,}$$

$$\text{and so is} \quad \frac{\tanh t}{t} = \prod_{k=1}^{\infty}\frac{1+(k\pi)^{-2}t^2}{1+((k-\frac{1}{2})\pi)^{-2}t^2},$$

as a converging infinite series of characteristic functions of the above form. Its selfdecomposbility follows from Yor (1997), p. 133, or Jurek (2001), Example 1(b). □

REMARK 1. The selfdecomposability of $\frac{\tanh t}{t}$, i.e., the formula (1), would follow in an elementary manner if for all $0 < c < 1$ and all $0 < w < u$, the functions

$$\frac{1+(c^2u+w)t^2+c^2uwt^4}{1+(u+c^2w)t^2+c^2uwt^4} = \frac{1+wt^2}{1+ut^2}\frac{1+c^2ut^2}{1+c^2wt^2} \quad \text{were characteristic functions.}$$

[The above is a ratio of *two* fractions of the form as in the product representation of $\tanh t/t$, with the fraction in the denominator computed at ct.] However, they *can not be* characteristic functions ! Affirmative answer would mean that Laplace rv is in L_1 (these are those SD rv for which $BDLP\ Y(1)$ in **(c)**, is SD). Equivalently, the characteristic function ρ_c in (1) is in SD).) But from (2) we see that $Y(1)$ for Laplace rv η has compound Poisson distribution with Lévy spectral measure $dM(x) = e^{-|x|}dx$ which does not satisfy the criterium **(e)**. Cf. also Jurek (1997).

3. The BDLP's of the hyperbolic characteristic functions. Since the three hyperbolic characteristic functions are infinitely divisible one can insert them into Lévy processes: $\hat{C}_s$, $\hat{S}_s$, $\hat{T}_s$, for $s \geq 0$, corresponding to *cosh, sinh* and *tanh* characteristic functions. Those processes were studied from the ID class point of view in the recent paper Pitman-Yor (2003). Here we are looking at them from the SD class point of view, i.e., via the corresponding BDLP's.

Below ϕ with subscript $\hat{C}, \hat{S}$ or $\hat{T}$ denotes one of the three hyperbolic characteristic functions, M with similar subscripts denotes the Lévy spectral measure in the appropriate Lévy-Khintchine formula, furthermore ψ with the above subscripts is the corresponding characteristic function in the random integral representation (properties **(c)** and **(d)** of class SD) and finally N with one of the above subscripts is the Lévy spectral measure of ψ (as in **(e)**). Thus we have the equalities :

$$\phi_{\hat{C}}(t) = \phi_{\hat{S}}(t) \cdot \phi_{\hat{T}}(t), \quad \text{i.e.,} \quad \frac{1}{\cosh t} = \frac{t}{\sinh t} \cdot \frac{\tanh t}{t}, \tag{4}$$

$$M_{\hat{C}}(\cdot) = M_{\hat{S}}(\cdot) + M_{\hat{T}}(\cdot), \quad \text{where} \quad \frac{dM_{\hat{C}}(x)}{dx} = \frac{1}{2x\sinh(\pi x/2)};$$
$$\frac{dM_{\hat{S}}(x)}{dx} = \frac{e^{-\pi|x|/2}}{2x\sinh(\pi x/2} = \frac{1}{2|x|}(\coth(\frac{\pi|x|}{2}) - 1);$$
$$\frac{dM_{\hat{T}}(x)}{dx} = \frac{1}{2|x|}\frac{e^{-\pi|x|/4}}{\cosh(\pi|x|/4)} = \frac{1}{2|x|}[1 - \tanh(\pi|x|/4)]. \tag{5}$$

These are consequences of the appropriate Lévy-Khintchine formulas for the hyperbolic characteristic functions or see Jurek (1996) or Pitman-Yor (2003) or use (2) and the product formulas for $\cosh z, \sinh z$; (for $\tanh z$ use the ratio of the two previous formulas).

COROLLARY 1. *For the SD hyperbolic characteristic functions $\phi_{\hat{C}}$, $\phi_{\hat{S}}$ and $\phi_{\hat{T}}$, their background driving characteristic functions are $\psi_{\hat{C}}$, $\psi_{\hat{S}}$ and $\psi_{\hat{T}}$, where*

$$\psi_{\hat{C}}(t) = \psi_{\hat{S}}(t)\cdot\psi_{\hat{T}}(t); \quad \psi_{\hat{C}}(t) = \exp[-t\tanh t],\ \psi_{\hat{S}}(t) = \exp[1-t\coth t],$$
$$\psi_{\hat{T}}(t) = \exp\Big[\frac{1}{\cosh t}\cdot\frac{t}{\sinh t} - 1\Big] = \exp\Big[\frac{2t}{\sinh(2t)} - 1\Big]. \tag{6}$$

Probability distributions corresponding to $\psi_{\hat{C}}$, $\psi_{\hat{S}}$ and $\psi_{\hat{T}}$ are infinitely divisible with finite logarithmic moments.

Proofs follow from (4) and the properties **(c)** and **(d)** of the selfdecomposable distributions.

Let us note that $\psi_{\hat{T}}$ is the characteristic function of the compound Poisson distribution with summand being the sum of independent rv's with the *cosh* and *sinh* characteristic functions.

Finally, on the level of the Lévy measures N of $Y(1)$, from the $BDLP's$ in the property **(e)**, we have the following :

$$N_{\hat{C}}(\cdot) = N_{\hat{S}}(\cdot) + N_{\hat{T}}(\cdot), \quad \text{where} \quad \frac{dN_{\hat{C}}(x)}{dx} = \frac{\pi}{4}\frac{\cosh(\frac{\pi x}{2})}{\sinh^2(\frac{\pi x}{2})},$$
$$\frac{dN_{\hat{S}}(x)}{dx} = \frac{\pi}{4}\frac{1}{\sinh^2(\frac{\pi x}{2})}; \quad \frac{dN_{\hat{T}}(x)}{dx} = \frac{\pi}{8}\frac{1}{\cosh^2(\frac{\pi x}{4})}. \tag{7}$$

Explicitly, as in (4), on the level of the BDLP one has factorization

$$\exp[-t\tanh t] = \exp[1 - t\coth t]\cdot\exp\Big[\frac{t}{\cosh t\sinh t} - 1\Big]. \tag{8}$$

Taking into account all the above and the property **(e)** we arrive at the identities :

$$\int_{R\setminus\{0\}}(1-\cos tx)\,\frac{\pi}{4}\frac{\cosh(\pi x/2)}{\sinh^2(\pi x/2)}\,dx = t\tanh t; \tag{9}$$

$$\int_{R\setminus\{0\}}(1-\cos tx)\,\frac{\pi}{4}\frac{1}{\sinh^2(\pi x/2)}\,dx = t\coth t - 1; \tag{10}$$

$$\int_{R\setminus\{0\}}(1-\cos tx)\,\frac{\pi}{8}\frac{1}{\cosh^2(\pi x/4)}\,dx = 1 - \frac{2t}{\sinh 2t}. \tag{11}$$

Furthermore, since $\psi_{\hat{T}}$ corresponds to a compound Poisson distribution, the last equality implies that

$$\int_{R\setminus\{0\}} \cos tx\,[\frac{\pi}{8}\frac{1}{\cosh^2(\pi x/4)}]\,dx = \frac{2t}{\sinh 2t}, \tag{12}$$

where we recover the known relation between $(\cosh u)^{-2}$ being the probability density corresponding to the characteristic function $\frac{at}{\sinh at}$ and vice versa by the inversion formula; cf. P. Lévy (1950) or Pitman and Yor (2003), Table 6.

4. Stochastic interpretation of BDLP's for hyperbolic functions. The functions $\psi_{\hat{C}}(t)$ and $\psi_{\hat{S}}(t)$ were identified as characteristic functions of the *background driving random variable* $Y(1)$ (in short: BDRV) for *cosh* and *sinh* SD rv in Jurek (1996), p. 182. [By the way, the question raised there has an affirmative answer. More precisely: (23) implies (24). To see that note that $D_1 \stackrel{d}{=} \frac{1}{2}\tilde{D}_1 + C$, where $\tilde{D}_1$ is a copy of D_1 and independent of C . The notations here are from the paper in question.]

More recently, in Jurek (2001) it was noticed that the conditional characteristic function of the Lévy's stochastic area integral is a product of the *sinh* characteristic function and its BDLP $\psi_{\hat{S}}$. Similar factorization one has in Wenocur formula; Wenocur (1986). Cf. also Yor (1992a), p. 19.

In this section we give some *"stochastic"* interpretation of the characteristic functions $\psi_{\hat{C}}(t)$ and $\psi_{\hat{S}}(t)$ in terms of Bessel processes.

Let us recall here some basic facts and notations from Pitman and Yor (1982) and Yor (1992a, 1997). Also cf. Revuz and Yor (1999).

For δ-dimensional Brownian motion $(B_t, t \geq 0)$, starting from a vector a, we define the process $X_t = |B_t|^2, t \geq 0$, which in turn defines the probability distribution (law) $Q^{\delta}_x, x := |a|^2$, on the canonical space $\Omega := C([0,\infty); [0,\infty))$ of non-negative functions defined on the half-line $[0,\infty)$, equipped with the σ-field $\mathcal{F}$ such that mappings $\{\omega \to X_s(\omega)\}$ are measurable. In fact, $(X_t, t \geq 0)$ is the unique strong solution of a stochastic integral equation

$$X_t = x + 2\int_0^t \sqrt{X_s}\,d\beta_s + \delta\, t, \quad t \geq 0,$$

where $(\beta_t, t \geq 0)$ is 1-dimensional Brownian motion.

The laws Q^{δ}_x satisfy the following convolution equation due to Shiga-Watanabe:

$$Q^{\delta}_x \star Q^{\delta'}_{x'} = Q^{\delta+\delta'}_{x+x'} \quad \text{for all} \quad \delta, \delta', x, x' \geq 0, \tag{13}$$

where, for P and Q two probabilities on $(\Omega, \mathcal{F})$, $P\star Q$ denotes the distribution of $(X_t+Y_t, t \geq 0)$, with $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ two independent processes, respectively P and Q distributed; cf. Revuz and Yor (1999), Chapter XI, Theorem 1.2.

Similarly, let $Q^\delta_{x\to y}$ be δ-dimensional squared Bessel bridge of $(X_s, 0 \leq s \leq 1)$, given $X_1 = y$, viewed as a probability on $C([0,1],[0,\infty))$.

Below we use integrals of functionals F with respect to measures Q over function spaces. To simplify our notation, as in Revuz and Yor (1999), we use $Q(F)$ to denote such integrals. From Yor (1992a, Chapter 2), the Lévy's stochastic area formula is given in the form

$$Q^\delta_{x\to 0}\Big[\exp(-\frac{\lambda^2}{2}\int_0^1 dsX_s)\Big] = \Big(\frac{\lambda}{\sinh\lambda}\Big)^{\delta/2}\exp\Big(-\frac{x}{2}(\lambda\coth\lambda - 1)\Big) \quad (14)$$

However, since $Q^\delta_{x\to 0} = Q^\delta_{0\to 0}\star Q^0_{x\to 0}$, (cf. Yor (1992), Pitman and Yor (1982)) we have in fact that

$$Q^0_{x\to 0}\Big(\exp(-\frac{\lambda^2}{2}\int_0^1 dsX_s)\Big) = \exp\Big(-\frac{x}{2}(\lambda\coth\lambda - 1)\Big)$$

Thus we may conclude the following

COROLLARY 2. *The BDLP Y, for the SD characteristic function $\phi_{\hat{S}}(t) = \frac{t}{\sinh t}$, is such that $Y(1)$ has the characteristic function*

$$\psi_{\hat{S}}(t) = \exp(1 - t\coth t) = Q^0_{2\to 0}\Big(\exp(it\gamma_{(\int_0^1 dsX_s)})\Big) \in ID_{\log}, \quad (15)$$

where $(\gamma_s, s \geq 0)$ is a Brownian motion independent of the Bessel squared process X.

[Here it may be necessary to enlarge the probability space to support independent γ and X.]

In a similar way, in view of Yor (1992a), Chapter 2, we have

$$Q^\delta_x\Big(\exp(-\frac{\lambda^2}{2}\int_0^1 dsX_s)\Big) = \Big(\frac{1}{\cosh\lambda}\Big)^{\delta/2}\exp(-\frac{x}{2}\lambda\tanh\lambda),$$

so, in particular,

$$Q^0_x\Big(\exp(-\frac{\lambda^2}{2}\int_0^1 dsX_s)\Big) = \exp\Big(-\frac{x}{2}\lambda\tanh\lambda\Big).$$

Thus, as above, we conclude the following:

COROLLARY 3. *The BDLP Y, for the SD characteristic function $\phi_{\hat{C}}(t) = \frac{1}{\cosh t}$, is such that $Y(1)$ has characteristic function*

$$\psi_{\hat{C}}(t) = \exp(-t \tanh t) = Q_2^0\Big(\exp(it\gamma_{(\int_0^1 ds X_s)})\Big) \in ID_{\log}, \tag{16}$$

where a process $(\gamma_s, s \geq 0)$ is a Brownian motion independent of the Bessel squared process X.

Let us return again to functions $\psi_{\hat{C}}(t)$ and $\psi_{\hat{S}}(t)$, given in (6), but viewed this time as Laplace transforms in $t^2/2$. From Yor (1997), p. 132, we have

$$\frac{1}{\cosh t} = \mathbb{E}\Big[\exp(-\frac{t^2}{2}\, T_1^{(1)})\Big], \qquad \frac{t}{\sinh t} = \mathbb{E}\Big[\exp(-\frac{t^2}{2}\, T_1^{(3)})\Big], \tag{17}$$

where $T_1^{(\delta)} := \inf\{t : \mathcal{R}_t^{(\delta)} = 1\}$ denotes the hitting time of 1 by δ-dimensional Bessel process $\mathcal{R}_t^{(\delta)}$, $t \geq 0$, starting from zero. Jeanblanc-Pitman-Yor (2002), Theorem 3, found that the corresponding BDLP's Y are of the form

$$Y(h) = \int_0^{\tau_h^{(\delta)}} du\, 1_{(\mathcal{R}_u^{(\delta)} \leq 1)}, \qquad h \geq 0, \tag{18}$$

where $(\tau_h^r, h \geq 0)$ is the inverse of the local time of $\mathcal{R}_u^{(\delta)}$ at r; cf. Revuz and Yor (1999), Chapter VI, for all needed notion and definitions. From the above we also recover the formulae

$$\mathbb{E}\Big[\exp(-\frac{\lambda^2}{2} \int_0^{\tau_1^{(1)}} du 1_{(|B_u| \leq 1)})\Big] = \exp(-\lambda \tanh \lambda),$$

$$\mathbb{E}\Big[\exp(-\frac{\lambda^2}{2} \int_0^{\tau_1^{(3)}} du 1_{(\mathcal{R}_u^{(3)} \leq 1)})\Big] = \exp(-\lambda(\coth \lambda - 1)). \tag{19}$$

These as well provide another "stochastic view" of the analytic formulae for the *BDRV* of two *SD* hyperbolic characteristic functions in (4), i.e., $1/\cosh t$ and $t/\sinh t$.

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