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REMARKS ON RESTRICTED NEVANLINNA TRANSFORMS*

Dedicated to Professor Agnieszka Plucińska
on the occasion of her 80th birthday

Abstract. The Nevalinna transform $K_{a,\rho}(z)$ of a positive measure ρ and a constant a, plays an important role in complex analysis and – more recently – in the context of the boolean convolution. We show here that its restriction to the imaginary axis, $k_{a,\,\rho}(it)$, can be expressed as the Laplace transform of the Fourier transform (a characteristic function) of ρ. Consequently, $k_{a\,\rho}$ is sufficient for the unique identification of the measure ρ and the constant a. Finally, we identify a relation between the free additive Voiculescu $\boxplus$ and boolean $\uplus$ convolutions.

The Cauchy $G(z)$ and the Nevanlinna $K(z)$ transforms play an important role in complex analysis and free probability. They are given as follows:

$$(*)\quad G_m\,(z) := \int_{\mathbb{R}} \frac{1}{z-x}\, m(dx), \quad K_{a,\,\rho}(z) := a + \int_{\mathbb{R}} \frac{1+zx}{z-x}\,\rho(dx), \quad z \in \mathbb{C}\setminus\mathbb{R},$$

for some finite measures m and ρ and constants a. In order to retrieve the measure m from G_m one uses the classical inversion formula

$$m([a,b]) = -\lim_{y\to 0}\frac{1}{\pi}\int_a^b \Im\, G_m(x+iy)dx, \text{ provided } \ m(\{a,b\})=0;$$

cf. Akhiezer (1965), p. 125 or Lang (1975), p. 380, Bondesson (1992). Thus, G_m uniquely determines m. It is important to stress that the above in-

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version requires one to know the Cauchy transform in strips $\{x + iy : x \in \mathbb{R}, 0 < y < \epsilon\}$ for some $\epsilon > 0$. Jurek (2006) demonstrates that the values of $G_m(it), t \neq 0$, are sufficient to identify m, using a simple argument of exponentiation of measures; also cf. Proposition 1 below. Of course, as holomorphic functions G_m and $K_{a,\rho}$ are determined by their values on sets having a condensation point, but the proof in Jurek (2006) is notable for avoiding the use of structural theorems from complex analysis.

This paper is an application of the general idea (conjecture) that many transforms in complex analysis and, in particular, in the area of the free probability, are some functionals of the standard Laplace and Fourier transforms when suitably restricted to the imaginary line.

In particular, we will show that the measure ρ, in the Nevalinna transform, can be retrieved from values $K_{a,\rho}(it), t \neq 0$, using the classical (standard) Fourier and Laplace transforms, after restricting $K_{a,\rho}$ to the imaginary axis without the origin; cf. Theorem 1 (The inversion formula). Then we illustrate the inversion formula by an example. Finally we derive a relation between the so-called *boolean convolution* $\uplus$, introduced by Speicher and Woroudi (1997), and the Voiculescu convolution $\boxplus$ (Proposition 2); cf. Acknowledgement below. Finally, Remark 2 identifies a challenging open problem.

1. Notations, results and an example

For a real constant a and a finite Borel measure ρ on the real line, *the restricted Nevanlinna transform* is defined by

(1) $$k_{a,\rho}(it) := a + \int_{\mathbb{R}} \frac{1 + itx}{it - x}\, \rho(dx), \quad \text{for} \quad t \neq 0,$$

and similarly, *the restricted Cauchy transform*, by

(2) $$g_\rho(it) := \int_{\mathbb{R}} \frac{1}{it - x}\, \rho(dx), \quad \text{for} \quad t \neq 0;$$

comp. the equation $(*)$ above. Let us recall also that the Fourier transform (the characteristic function) $\hat{\mu}$ of a measure μ is given by

(3) $$\hat{\mu}(t) := \int_{\mathbb{R}} e^{itx} \mu(dx), \quad t \in \mathbb{R},$$

and the Laplace transform of a function $h : (0, \infty) \to \mathbb{C}$, or of a measure m is given by

(4) $$\mathfrak{L}[h; \lambda] := \int_0^\infty h(x) e^{-\lambda x}\, dx, \quad \mathfrak{L}[m\,; \lambda] := \int_0^\infty e^{-\lambda x}\, m(dx), \quad \lambda > 0$$

where λ is such that those integrals exist; cf. Gradshteyn and Ryzhik (1994), Chapter 17, for examples of those transforms and their inverses.

We begin with our main result that shows how to obtain explicitly the measures ρ knowing only their restricted Nevanlinna transforms. Below, $\Re z$, $\Im z$, $\overline{z}$ denote the real part, the imaginary part and the conjugate of a complex $z \in \mathbb{C}$, respectively.

Theorem 1. (The inversion formula) *For the restricted Nevalinna transform $k_{a,\rho}$ we have that: $a = \Re k_{a,\ \rho}(i)$, $\rho(\mathbb{R}) = -\Im k_{a,\ \rho}(i)$; and the identity*

$$\mathfrak{L}[\hat{\rho};\ w] = \int_0^\infty \hat{\rho}(r) e^{-w\,r} dr = \frac{ik_{a,\rho}(-iw) - i\Re k_{a,\rho}(i) - w\Im k_{a,\rho}(i)}{w^2-1}$$

holds for $w > 0$ and $w \neq 1$. In particular, the constant a and the measure ρ are uniquely determined by the functional $k_{a,\rho}$.

Since part of the above right-hand side formula can be viewed as Laplace transform of some exponential functions we get

Corollary 1. *For the restricted Nevanlinna functional $k_{a,\rho}$ and $w > 1$ we have*

$$\int_0^\infty \left[\hat{\rho}(r) - \frac{1}{2}\big(i\, k_{a,\rho}(i)\, e^{-r} + \overline{i\, k_{a,\rho}(i)}\, e^{r}\big)\right] e^{-w\,r} dr = \frac{ik_{a,\rho}(-iw)}{w^2-1}.$$

In particular, if $a = 0$ and ν is a probability measure then for $k_{0,\nu}$ we get

$$\int_0^\infty (\hat{\nu}(r) - \cosh r)\, e^{-w\,r} dr = \frac{ik_{0,\,\nu}(-iw)}{w^2-1}, \quad w > 1.$$

Proposition 1. *For a finite measure ρ and its restricted Cauchy transform g_ρ we have*

$$\mathfrak{L}[\hat{\rho};\ w] = \overline{i\, g_\rho(iw)},\ w \neq 0,$$

that is, to retrieve ρ one needs to invert Laplace transform of $\hat{\rho}$ and then invert the Fourier transform.

Hence we conclude that the values of restricted Cauchy transform $g_\rho(iw)$, $w \neq 0$, uniquely determine the measure ρ. That fact was already established in Jurek (2006) but not explicitly as it is in the above Proposition 1.

In the following example we show explicitly that shifted reciprocals of restricted Cauchy transforms of discrete measures correspond to restricted Nevannlina transforms; see the formula (5) below.

Example. For a set $\mathbf{b} = \{b_1, b_2, \ldots, b_m\}$ of distinct real numbers let us define a discrete probability measure $\mu_{\mathbf{b}} := \frac{1}{m}\sum_{j=1}^m \delta_{b_j}$ and the canonical polynomial $P_{\mathbf{b}}(z) = \prod_{j=1}^m (z - b_j)$. If $\{\xi_1, \xi_2, \ldots, \xi_{m-1}\}$ is the set of zeros of the polynomial $P'_{\mathbf{b}}(z)$ (the derivative of P) then we have

$$(5)\qquad it-\frac{1}{G_{\mu_{\mathbf{b}}}(it)}=it-\frac{m}{\sum_{j=1}^{m}\frac{1}{it-b_j}}=a_{\mathbf{b}}+\int_{\mathbb{R}}\frac{1+itx}{it-x}\rho_{\mathbf{b}}(dx),\quad t\neq 0,$$

where

$$(6)\qquad \alpha_k:=-m\frac{P(\xi_k)}{P''(\xi_k)}=m\left[\sum_{j=1}^{m}\frac{1}{(\xi_k-b_j)^2}-\left(\sum_{j=1}^{m}\frac{1}{\xi_k-b_j}\right)^2\right]^{-1}>0$$

$$a_{\mathbf{b}}:=\frac{b_1+b_2+\cdots+b_m}{m}-\sum_{j=1}^{m-1}\frac{\alpha_j\,\xi_j}{1+\xi_j^2},\quad \rho_{\mathbf{b}}(dx):=\sum_{j=1}^{m-1}\frac{\alpha_j}{1+\xi_j^2}\delta_{\xi_j}(dx).$$

Note that the procedure described in the Example can be iterated. Namely, in the second step we may start with the probability measure concentrated on the roots $\xi_j,\ j=1,2,\ldots,m-1$, and so on.

Recall that *the self-energy functional* E_μ of the probability measure μ is defined as follows

$$(7)\qquad E_\mu(z)=z-\frac{1}{G_\mu(z)},\quad z\in\mathbb{C}\setminus\mathbb{R}.$$

Similarly to the above relation, we refer to $e_\mu(it):=E_\mu(it), t\neq 0$, as *a restricted self-energy functional.*

To express a and ρ in terms of μ using only the restricted functionals we make use of the following corollary:

Corollary 2. *For a probability measure μ let*

$$(8)\qquad z_\mu:=-g_\mu(i)=c_\mu+i\,d_\mu\equiv\int_{\mathbb{R}}\frac{x}{1+x^2}\mu(dx)+i\int_{\mathbb{R}}\frac{1}{1+x^2}\mu(dx)\in\mathbb{C}.$$

If $e_\mu(it)=k_{a,\rho}(it)$, for all $t\neq 0$, then the constants a and $\rho(\mathbb{R})$ are given by formulae

$$(9)\qquad a=\frac{c_\mu}{|z_\mu|^2}\quad\textit{and}\quad \rho(\mathbb{R})=\frac{d_\mu}{|z_\mu|^2}-1>0\,,$$

and the Fourier transform $\hat\rho$ satisfies the equation

$$(10)\qquad \mathfrak{L}[|z_\mu|^2\hat\rho(x)-\frac{1}{2}(\overline{z_\mu}\,e^{x}+z_\mu\,e^{-x});\ w]=\frac{1}{(w^2-1)\,i\,g_\mu(-iw)},\quad w>1.$$

Since for any probability measures μ and ν there exists a unique probability measure γ such that

$$(11)\qquad E_\mu(z)+E_\nu(z)=E_\gamma(z),$$

we call it *the boolean convolution* and denote it by $\gamma=\mu\uplus\nu$; for more details cf. Speicher–Woroudi (1997) and references therein.

Remark 1. Boolean convolution has the property that *all* probability measures are $\uplus$-infinitely divisible. The *max*-convolution also has that feature because *for each distribution function F, $F^{1/n}$ (the n-th root) is also a distribution function and taking independent identically distributed (as $F^{1/n}$) r.v. $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$, we see that* $\max\{X_{n,1}, \ldots, X_{n,n}\}$ *has the distribution function F.*

For a probability measure μ, let

$$(12)\quad F_\mu(z) := \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad \text{and} \quad V_\mu(z) := F_\mu^{-1}(z) - z, \quad z \in \mathcal{D} \subset \mathbb{C},$$

where $\mathcal{D}$ is the so called Stolz angle in which the inverse F_μ^{-1} exists; cf. Bercovici–Voiculescu (1993) and references therein. Since for any probability measures μ and ν there exists a unique probability measure γ such that

$$V_\mu(z) + V_\nu(z) = V_\gamma(z), \tag{13}$$

we call it *the Voiculescu convolution* and denote it by $\gamma = \mu \boxplus \nu$; cf. Bercovici–Voiculescu (1993) and references therein. A relation between $\boxplus$-infinite divisibility and some random integrals with respect to classical Lévy processes is given in Jurek (2007), Corollary 6.

Here are some unexpected relations between the Voiculescu $\boxplus$ and the boolean $\uplus$ operations on probability measures; cf. Lenczewski (2007), Proposition 2.1 and the Acknowledgements below.

Proposition 2. *For probability measures μ_1 and μ_2 there exist unique probability measures ν_1, ν_2 such that*

$$F_{\mu_1}(F_{\nu_1}(z)) = F_{\mu_2}(F_{\nu_2}(z)) = F_{\mu_1 \boxplus \mu_2}(z), \ z \in \mathbb{C}^+.$$

Furthermore, the above measures satisfy the equation $\nu_1 \uplus \nu_2 = \mu_1 \boxplus \mu_2$.

Corollary 3. *For $n \geq 2$ and probability measures $\mu_1, \mu_2, \ldots, \mu_n$ there exist unique probability measures $\nu_1, \nu_2, \ldots, \nu_n$ such that $F_{\mu_1}(F_{\nu_1}(z)) = F_{\mu_2}(F_{\nu_2}(z)) = \cdots = F_{\mu_n}(F_{\nu_n}(z)) = F_{\mu_1 \boxplus \mu_2 \boxplus \cdots \boxplus \mu_n}(z)$, $z \in \mathbb{C}^+$. Furthermore, the above measures satisfy the equation $(\nu_1 \uplus \nu_2 \uplus \cdots \uplus \nu_n)^{\uplus 1/(n-1)} = \mu_1 \boxplus \mu_2 \boxplus \cdots \boxplus \mu_n$.*

Remark 2. The two identities below, involving $\uplus$ and $\boxplus$, might be of an interest in themselves. More importantly, finding real analytic proofs of them seems to be very challenging.

(a) For probability measures μ and ν there exists a unique measure $\mu \uplus \nu$ such that

$$\frac{1}{\int_{\mathbb{R}} \frac{1}{1-itx}\,\mu(dx)} - 1 + \frac{1}{\int_{\mathbb{R}} \frac{1}{1-itx}\,\nu(dx)} - 1 = \frac{1}{\int_{\mathbb{R}} \frac{1}{1-itx}\,\mu \uplus \nu(dx)} - 1,$$

for $t \in \mathbb{R}$; cf. Theorem 2 and Remark 1.1.1 in Jurek (2006) for other forms of the above formula and some comments.

(b) For measures μ_1 and μ_2 there exist unique measures ν_1, ν_2 and $\mu_1 \boxplus \mu_2$ such that for their restricted Cauchy transforms we have

$$g_{\nu_1}(it) \int_{\mathbb{R}} \frac{1}{1 - x g_{\nu_1}(it)} \mu_1(dx) = g_{\mu_1 \boxplus \mu_2}(it) = g_{\nu_2}(it) \int_{\mathbb{R}} \frac{1}{1 - x g_{\nu_2}(it)} \mu_2(dx),$$

for all $t \neq 0$; cf. Biane (1998), Chistyakov and Goetze (2005). (Using Proposition 1 we may express the above identity in terms of classical Laplace and Fourier transforms.)

2. Auxiliary results and proofs

Note that

$$\overline{g_m(it)} = g_m(-it), \quad \overline{k_\rho(it)} = k_{a,\,\rho}(-it), \quad \overline{e_\mu(it)} = e_\mu(-it), \; t \neq 0,$$

which allows us to consider those function only on the positive half-line.

Proof of Theorem 1. From (1) we get

$$k_{a,\,\rho}(i) = a - i\rho(\mathbb{R}). \tag{14}$$

Further, since

$$\frac{1 + itx}{it - x} = \frac{1 - t^2}{it - x} - it$$

we infer from (1) and (2) that

$$k_{a,\rho}(it) = a + (1 - t^2) g_\rho(it) - it\rho(\mathbb{R}), \quad g_\rho(it) = \frac{k_{a,\rho}(it) - a + it\rho(\mathbb{R})}{1 - t^2}. \tag{15}$$

On the other hand, in Jurek (2006) on p. 189, it was noticed that

$$\int_0^\infty \hat{\rho}(ts) e^{-s} ds = \frac{1}{it} g_\rho\left(\frac{1}{it}\right), \quad t \neq 0, \quad \text{and} \quad \lim_{t \to 0} \frac{1}{it} g_\rho\left(\frac{1}{it}\right) = \rho(\mathbb{R}).$$

This property, along with (14) and (15), yields

$$\int_0^\infty \hat{\rho}(ts) e^{-s} ds = \frac{1}{it} \frac{(k_{a,\rho}(\frac{1}{it}) - \Re k_{a,\rho}(i) - \frac{1}{it} \Im k_{a,\rho}(i))}{1 + (\frac{1}{it})^2}$$
$$= \frac{k_{a,\rho}(\frac{1}{it}) - \Re k_{a,\rho}(i) - \frac{1}{it} \Im k_{a,\rho}(i)}{it + (\frac{1}{it})}.$$

By letting $t = \frac{1}{w} > 0$ we get

$$\int_0^\infty \hat{\rho}\left(\frac{s}{w}\right) e^{-s} ds = \frac{k_{a,\rho}(-iw) - \Re k_{a,\rho}(i) + iw \Im k_{a,\rho}(i)}{\frac{i}{w} - iw}$$
$$= \frac{iw k_{a,\rho}(-iw) - iw \Re k_{a,\rho\mu}(i) - w^2 \Im k_{a,\rho}(i)}{w^2 - 1}$$

which, after substituting $\frac{s}{w} = r > 0$, is as follows

$$\int_0^\infty \hat{\rho}(r) e^{-wr} dr = \frac{iwk_{a,\rho}(-iw) - iw\Re k_{a,\rho}(i) - w^2 \Im k_{a,\rho}(i)}{w(w^2-1)},$$

and thus giving the formula in Theorem 1. Finally, inverting the Laplace transform of $\hat{\rho}$ and then inverting the Fourier transform, we get uniquely the measure ρ from values $k_{a,\rho}(it), t \neq 0$. This completes the proof. ■

Proof of Corollary 1. Simply note that

$$\mathfrak{L}\left[\frac{1}{2}(e^x - e^{-x}); w\right] = \mathfrak{L}[\sinh x; w] = \frac{1}{w^2-1},$$
$$\mathfrak{L}\left[\frac{1}{2}(e^x + e^{-x}); w\right] = \mathfrak{L}[\cosh x; w] = \frac{w}{w^2-1}, \quad w > 1,$$

which taken together with Theorem 1 give the proof. ■

Proof of Proposition 1. Using the definitions (3) and (4) we have

$$\mathfrak{L}[\hat{\rho}; w] = \int_{\mathbb{R}} \int_0^\infty e^{-r(w-ix)}\, dr\, \rho(dx) = \int_{\mathbb{R}} \frac{1}{w-ix} \rho(dx) = -i\, g_\rho(-iw),$$

which completes the proof. ■

Here is an auxiliary lemma where the part (a) is a very standard fact, recalled for completeness. This lemma simplifies the arguments in the proof of the Example.

Lemma 1. (a) *If* $P(z) := \prod_{j=1}^m (z - b_j),\ z \in \mathbb{C}$, *for some complex numbers* b_j, $j = 1, 2, \dots, m$, *and* $P'(z)$ *is its derivative then*

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^m \frac{1}{z-b_j}; \qquad \frac{P''(z)}{P(z)} = \left(\sum_{j=1}^m \frac{1}{z-b_j}\right)^2 - \sum_{j=1}^m \frac{1}{(z-b_j)^2};$$

(b) *If the* b_j*'s are distinct complex numbers and* $\xi_1, \dots, \xi_{m-1}$ *denote the zeros of the equation* $P'(z) = 0$ *then* ξ_j *are different from* $b_1, b_2, \dots, b_m$. *Furthermore,*

$$W_m(z) := \frac{z\, P'(z) - m\, P(z)}{P'(z)} = \frac{b_1 + b_2 + \cdots + b_m}{m} + \sum_{j=1}^{m-1} \frac{\alpha_j}{z - \xi_j} \tag{16}$$

where

$$\alpha_k := -m \frac{P(\xi_k)}{P''(\xi_k)} = m \left[\sum_{j=1}^m \frac{1}{(\xi_k - b_j)^2} - \left(\sum_{j=1}^m \frac{1}{\xi_k - b_j}\right)^2\right]^{-1},$$

for $k = 1, 2, \dots, m-1$.

(c) *If the b_j's are distinct real numbers, for $j = 1, 2, \ldots, m$, then $\alpha_k > 0$, for $k = 1, 2, \ldots, m-1$.*

Proof. (a) Since $P'(z) = \sum_{j=1}^{m} \prod_{k\neq j,k=1}^{m} (z - b_k)$ we get the first part of (a). Differentiating both sides of the identity $P'(z) = P(z) \sum_{j=1}^{m} \frac{1}{z-b_j}$ we get the second part of (a).

(b) Assume that P and P' have a common root. Without loss of generality, lets say that $\xi_1 = b_1$. Then $P'(b_1) = \prod_{k=2}^{m} (b_1 - b_k) = 0$, which contradicts the assumption that all b_j are distinct.

Suppose that ξ_1 and its complex conjugate $\bar{\xi_1}$ are two complex roots of $P'(z) = 0$. Then from (a) we have

$$P'(\xi_1) = P(\xi_1) \sum_{j=1}^{m-1} \frac{1}{\xi_1 - b_j} = 0 = P(\bar{\xi_1}) \sum_{j=1}^{m-1} \frac{1}{\bar{\xi_1} - b_j}.$$

Since $P(\xi_1) \neq 0$ and $P(\bar{\xi_1}) \neq 0$, we have

$$\sum_{j=1}^{m-1} \left[\frac{1}{\bar{\xi_1} - b_j} - \frac{1}{\xi_1 - b_j}\right] = i\,2(\Im\xi_1) \sum_{j=1}^{m} \frac{1}{|\xi_1 - b_j|^2} = 0,$$

and hence $\Im\xi_1 = 0 = \Im\xi_2 = \Im\xi_3 = \cdots = \Im\xi_{m-1}$, that is, all roots of $P'(z) = 0$ are real.

Let us note that

$P(z) = \prod_{k=1}^{m}(z - b_k) = z^m + (-b_1 - b_2 - \cdots - b_m)z^{m-1} + Q_{m-2}(z)$,

for some polynomial Q_{m-2} of degree $m-2$. Then $z\,P'(z) - mP(z) = (b_1 + \cdots + b_m)z^{m-1} + \tilde{Q}_{m-2}(z)$ is a polynomial of degree $m-1$, (for another polynomial of degree $m-2$). Consequently, $W_m(z)$, given by (16), is a rational function (a ratio of two polynomials of degree $m-1$). Since $\xi_1, \ldots, \xi_{m-1}$ are zeros of $P'(z) = 0$, i.e., simple poles of $W_m(z)$, then invoking the theorem on the decomposition of rational function into a sum of simple fractions

$$\begin{aligned} (17)\quad W_m(z) &= z - \frac{m\,P(z)}{P'(z)} \\ &= \frac{(b_1 + \cdots + b_m)z^{m-1} + \tilde{Q}_{m-2}(z)}{mz^{m-1} + (m-1)(-b_1 - b_2 - \cdots - b_m)z^{m-2} + Q'_{m-2}(z)} \\ &= \frac{b_1 + b_2 + \cdots + b_m}{m} + \sum_{j=1}^{m-1} \frac{\alpha_j}{z - \xi_j}. \end{aligned}$$

Putting $\bar{b} := (b_1 + \cdots + b_m)/m$ and multiplying both sides by $z - \xi_k$, we obtain

$$\alpha_k + (z-\xi_k)\sum_{j\neq k, j=1}^{m-1}\frac{\alpha_j}{z-\xi_j} = (z-\xi_k)(z-\bar{b}) - m\,P(z)\left(\frac{P'(z)-P'(\xi_k)}{z-\xi_k}\right)^{-1},$$

and then letting $z \to \xi_k$ we explicitly get that

$$\alpha_k := -m\frac{P(\xi_k)}{P''(\xi_k)} = m\left[\sum_{j=1}^{m}\frac{1}{(\xi_k-b_j)^2} - \left(\sum_{j=1}^{m}\frac{1}{\xi_k-b_j}\right)^2\right]^{-1}.$$

(c) Since $P(x)$ is a polynomial of m-th degree for $x \in \mathbb{R}$ and $P(b_k) = P(b_{k+1}) = 0$ (for $b_j \in \mathbb{R}$) then, by the Mean Value Theorem, there exists exactly one ξ_j (in that interval) such that $P'(\xi_k) = 0$. If $P(\xi_k) > 0$ then P must be concave on that interval and therefore $P''(\xi_k) < 0$. Consequently, $\alpha_j > 0$. In the opposite case we have convex function that also leads to the positivity of the α_k parameter. This completes the proof of Lemma 1. ∎

Proof of the Example. From Lemma 1 we have that the measure $\rho_{\mathbf{b}}$ is finite and positive. Furthermore, for $a_{\mathbf{b}}$ given by (6), using (16) (in Lemma 1) we get

$$\begin{aligned}\int_{\mathbb{R}}\frac{1+zx}{z-x}d\rho_{\mathbf{b}}(x) &= \sum_{j=1}^{m-1}\frac{1+\xi_j^2+z\xi_j-\xi_j^2}{z-\xi_j}\,\frac{\alpha_j}{1+\xi_j^2}\\ &= \sum_{j=1}^{m-1}\frac{\alpha_j}{z-\xi_j} + \sum_{j=1}^{m-1}\frac{\alpha_j\xi_j}{1+\xi_j^2} = W_m(z) - a_{\mathbf{b}}\\ &= z - \frac{m\,P_{\mathbf{b}}(z)}{P'_{\mathbf{b}}(z)} - a_{\mathbf{b}} = z - \frac{1}{G_{\mu_{\mathbf{b}}(z)}} = E_{\mu_{\mathbf{b}}}(z) - a_{\mathbf{b}}.\end{aligned}$$

Substituting *it* for z in the above expression, we get equality (5) in the Example. ∎

Proof of Corollary 2. Using (2) we obtain the expression (8) for $-g_\mu(i)$. From (14) and (7), $e_\mu(i) = a - i\rho(\mathbb{R})$, we then infer the equalities in (9). [Note that $d_\mu(1-d_\mu) \geq c_\mu^2$].

In view of the assumption, $k_{a,\rho}$ in Corollary 1 may be replaced by e_μ, which combined with (7) and (9) yields

$$ik_{a,\rho}(-iw) - i\Re k_{a,\rho}(i) - w\Im k_{a,\rho}(i) = w - \frac{i}{g_\mu(-iw)} - i\frac{c_\mu}{|z_\mu|^2} - w\left(1 - \frac{d_\mu}{|z_\mu|^2}\right).$$

Consequently the required identity follows from Corollary 1. ∎

Proof of Proposition 2. From Theorem 2.1 in Chistyakov and Goetze (2005), (cf. also Biane (1998)) for measures μ_1 and μ_2 there exist uniquely determined probability measures ν_1, ν_2 and μ such that

$$z = F_{\nu_1}(z)+F_{\nu_2}(z)-F_{\mu_1}(F_{\nu_1}(z)) \quad \text{and} \quad F_{\mu_1}(F_{\nu_1}(z)) = F_{\mu_2}(F_{\nu_2}(z)) = F_{\mu}(z),$$

where $\mu = \mu_1 \boxplus \mu_2$ (Voiculescu convolution). Hence

$$\begin{aligned} E_{\nu_1 \uplus \nu_2}(z) &= E_{\nu_1}(z) + E_{\nu_2}(z) = z - F_{\nu_1}(z) + z - F_{\nu_2}(z) = z - F_{\mu_1 \boxplus \mu_2} \\ &= E_{\mu_1 \boxplus \mu_2}(z). \end{aligned}$$

From the uniqueness of the self-energy functional we get $\mu_1 \boxplus \mu_2 = \nu_1 \uplus \nu_2$, which completes the proof. ■

Proof of Corollary 3. From Corollary 2.2 in Chistyakov and Goetze (2005), for measures $\mu_1, \ldots, \mu_n$ there exist uniquely determined probability measures $\nu_1, \ldots, \nu_n$ and μ such that

$$\begin{aligned} &z = F_{\nu_1}(z) + \cdots + F_{\nu_n}(z) - (n-1)F_{\mu_1}(F_{\nu_1}(z)) \\ &\text{and} \;\; F_{\mu_1}(F_{\nu_1}(z)) = \cdots = F_{\mu_n}(F_{\nu_n}(z)) = F_{\mu}(z), \end{aligned}$$

where $\mu = \mu_1 \boxplus \cdots \boxplus \mu_n$ (the Voiculescu convolution). Thus

$$\begin{aligned} E_{\nu_1 \uplus \cdots \uplus \nu_2}(z) = E_{\nu_1}(z) + \cdots + E_{\nu_n}(z) &= z - F_{\nu_1}(z) + \cdots + z - F_{\nu_n}(z) \\ &= (n-1)(z - F_{\mu_1}(F_{\nu_1}(z)) \\ &= (n-1)(z - F_{\mu_1 \boxplus \cdots \boxplus \mu_n}(z) \\ &= (n-1)E_{\mu_1 \boxplus \cdots \boxplus \mu_n}, \end{aligned}$$

which completes the proof. ■

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