A calculus on Lévy exponents and new properties of selfdecomposable probability measures*

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ABSTRACT. In a recent paper of Iksanov-Jurek-Schreiber in the Annals of Probability **32**, 2004, it was proved that in some cases (e.g. the for Lévy stochastic area integrals) a convolution of a selfdecomposable measure with its background driving probability measure leads to a new selfdecomposable measures (so called *factorization property*. Here we have proved a complementing result that each selfdecomposable measure can by factored as another selfdecomposable measure and its background driving measure. To this end we have introduced *a calculus on Lévy exponents* of infinitely divisible probability measures, which maybe of an interest in itself.

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Abbreviated title: *A calculus on Lévy exponents*

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1. An introduction. The importance of the class of *selfdecomposable probability distributions*, (denoted by L and also known as *Lévy class L*), follows from the fact that it is a natural extension of the class of *stable laws* (and in particular, the central limit theorem). Explicitly, these are weak limit distributions in the following scheme

$$a_n(X_1 + X_2 + ... + X_n) + x_n \to Z, \quad \text{as} \quad n \to \infty, \tag{1}$$

where random variables $X_1, X_2, ...$ are independent and the summands $\{a_n X_j : j = 1, 2, .., n; n = 1, 2, ...\}$ are uniformly infinitesimal; cf. Loeve (1963), Section 23, p. 319. If one assumes that in (1) laws of X_j are in class L then we say that laws of Z is 2-times selfdecomposable or that they belong to the class L_2, and so on by an induction; cf. Jurek (1983) and references therein; or Nguyen van Thu (1986); or for a similar concept see in Maejima and Rosiński (2001). The class L is quite large and it includes many well known distributions in probability and mathematical statistics: Student t-distributions, log t, Fisher F, log-normal, gamma, log-gamma and many others; cf. also Shanbhag and Sreehari (1977); Jurek (2001) and Jurek and Yor (2004).
Equivalently, if μ is a probability distribution of Z from (1) and $\mathcal{P}$ stands for the convolution semigroup of all probability measures on E (a Banach space), then we have the following characterization of the class L:

$$\mu \in L \ \text{ iff } \ \forall(t > 0)\exists(\nu_t \in \mathcal{P}) \ \ \mu = T_{e^{-t}}\mu \star \nu_t, \tag{2}$$

where $T_c\mu(A) := \mu(c^{-1}A)$ for all Borel sets A; cf. Loéve (1963), Section 23, p. 319, Jurek and Mason (1993), Section 3.9, p. 177 or Sato (1999), Section 15, p. 90. In fact, the factorization property (2) holds also for $t = 0$ and $t = \infty$ with $\nu_\infty = \mu, \nu_0 = \delta_0$. Furthermore, the convolution equation (2) also justifies the term *selfdecomposability.* Of course, well known and extensively studied stable laws (i.e., limit laws in the above scheme for identically distributed $X_i's$) satisfy the convolution equation (2). In fact, for there exists $0 < p \leq 2$ (called *an exponent*) such that for all $a, b > 0$ there exists x such that $T_a\mu * T_b\mu = T_{(a^p+b^p)^{1/p}}\mu * \delta_x$; cf. Samorodnitsky and Taqqu (1994) for the theory of stable processes and measures.

As in Jurek (1985) and in Iksanov-Jurek-Schrieber (2004) we will work in a generality of a real separable Banach space E with the norm $||.||$ and the conjugate Banach space E', i.e., in (1) random variables X_j are E-valued and measures μ's in (2) are Borel probability measures on E with their Fourier transforms $\hat{\mu}(y)$ and $y \in E'$. However, as in the two previous papers our results are new for distributions on the real line as well.

All selfdecomposable probability measures μ and their convolution factors ν_t in (2) are *infinitely divisible* (a such class will be denoted below by ID). Hence their Fourier transforms (the Lévy-Khintchine formula) can be written as follows

$$\hat{\mu}(y) = e^{\Phi(y)}, \;\; \hat{\nu}_t(y) = e^{\Phi_t(y)} \;\; \text{and the exponents are of the form}$$

$$\Phi(y) = i < y, a > -2^{-1} < y, Ry > + \int_{E\setminus\{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M(dx), \quad (3)$$

where E is a Banach or a Euclidean space, $< ., . >$ is an appropriate bilinear form between E' and E, a is *a shift vector*, S is *a covariance operator* corresponding to the Gaussian part of μ and M is *a Lévy spectral measure.* There is one to one corresponds between $\mu \in ID$ and the triples $[a, R, M]$ in its Lévy-Khintchine formula (2); cf. Araujo-Giné (1980), Chapter 3, Section 6, p. 136.

The function $\Phi(y)$ from (3) is called *the Lévy exponent* of μ. If E is a Hilbert space then Lévy spectral measures M are completely characterized by the integrability condition $\int_E (1 \wedge ||x||^2) M(dx) < \infty$ and Gaussian covariance operators S coincide with the class of trace operators ; cf. Parthasarathy (1967), Chapter VI, Theorem 4.10. Consequently, formula (2) gives the following description

$$\hat{\mu}(y) = e^{\Phi(y)} \in L \;\; \text{iff} \;\; \Phi_t(y) := \Phi(y) - \Phi(e^{-t}y), y \in E' \quad \text{is a Lévy exponent for all} \;\; t > 0. \quad (4)$$

Of course, $\Phi_\infty(y) = \Phi(y)$ and $\Phi_0(y) = 0$. Recall that in a case when E is an Euclidean space then Lévy exponents are characterized as a continuous negative-definite functions; cf. Cuppens (1975) and Schoenberg's Theorem on p. 80.

Finally, let us recall also that *a Lévy process* $Y(t), t \geq 0$, is a process with stationary and independent increments and $Y(0) = 0$. Without loss of generality we may and do assume that it has paths in Skorochod space $D_E[0, \infty)$ of E-valued *cadlag functions* (i.e., right continuous with left hand limits.) There is one to one correspondence between the class ID and the class of Lévy processes. Namely, for $\nu \in ID$ there is unique, in distribution, Lévy process $Y_\nu(t)$ such that $\mathcal{L}(Y_\nu(1)) = \nu$. Conversely, the distribution of Lévy process is uniquely determined by $\mathcal{L}(Y(1))$ from the class ID.

The cadlag paths of a process Y allows us to define *random integrals* of the form $\int_{(a,b]} h(s)Y(r(ds))$ via the formal formula of integration by parts. Namely,

$$\int_{(a,b]} h(s)Y(r(ds)) := \\ h(b)Y(r(b)) - h(a)Y(r(a)) - \int_{(a,b]} Y(r(s))dh(s), \quad (5)$$

where h is a real valued function of bounded variation and $r(.)$ is a monotone and right-continuous function. Cf. Jurek& Mason (1993), Section 3.6, p. 116, or Jurek-Vervaat (1983).
Furthermore, using Riemann-Stieltjes approximating sums for (5) we have the following formula for the characteristic function of the above integrals:

$$\mathcal{L}\Big(\widehat{\int_{(a,b]} h(s)Y(r(ds))}\Big)(y) = \exp \int_{(a,b]} \log \widehat{\mathcal{L}(Y(1))}(h(s)y)dr(s), \qquad (6)$$

where $\mathcal{L}(.)$ denotes the probability distribution and $\hat{\mu}(.)$ denotes the Fourier transform of a measure μ; cf. Jurek-Vervaat (1983) or Jurek (1985) or Jurek-Mason (1993). The usefulness of the random integral representations can be seen in the following:

$$\mu \in L \ \text{ iff } \ \mu = \mathcal{L}(\int_{(0,\infty)} e^{-s}Y(ds)), \qquad (7)$$

for a unique (in distribution) Lévy process Y such that $\mathbf{E}[\log(1+||Y(1)||)] < \infty$; we refer to (7) as *the random integral representation* of distributions from the class L. (Integrals over half-line are defined as a limit in probability (almost surly, or in distribution) of integrals (6) as $b \to \infty$.) The above let us introduce *a random integral mapping*

$$\mathcal{I} : ID_{\log} \ni \mathcal{L}(Y(1)) \to \mathcal{L}(\int_{(0,\infty)} e^{-s}Y(ds)) \in L.$$

In terms of Lévy exponents, characterization (7) means that if Φ and Ψ are Lévy exponents of μ and $Y(1)$, respectively, then

$$\Phi \in L \ \text{ iff } \ \Phi(y) = \int_0^\infty \Psi(e^{-s}y)ds = \int_0^1 \Psi(sy)\frac{ds}{s}, \quad \text{for all } \ y \in E',$$

which follows from (6) with appropriately chosen parameters and integration over positive half-line. Above and in what follows, a phrase "$\Phi \in L$" or "$M \in L$" will mean that a characteristic function $\exp[\Phi(y)]$, $y \in E'$ corresponds to a class L probability measure. And similarly in the second instance we mean that $[a, R, M]$ is a class L probability measure.

To $\mathcal{L}(Y(1))$ we refer to as the *background driving probability distribution* for μ; in short: $BDPD$. Similarly to $Y(t), t \geq 0$, we refer as the *background driving Lévy process*; in short: $BDLP$. Since $Y(1)$ has a characteristic function $\exp \Psi(y)$, $y \in E'$, we call it *the background driving characteristic function* of a class L characteristic function $\exp \Phi(y)$; in short $BDCF$.

Similarly to the formula (7) we introduce a class $\mathcal{U}$ as follows:

$$\mu \in \mathcal{U} \quad \text{iff} \quad \mu = \mathcal{L}(\int_{(0,1)} s\, Y(ds)), \tag{8}$$

and the following random integral mapping

$$\mathcal{J} : ID \ni \mathcal{L}(Y(1)) \to \mathcal{L}(\int_{(0,1)} s\, Y(ds)) \in \mathcal{U},$$

where Y is an arbitrary Lévy process. Measures from the class $\mathcal{U}$ are called *s-selfdecomposable* and they were orignally introduced using some *non-linear shrinking transforms*, in short: s-operations; cf. Jurek (1985) and references therein and Iksanov-Jurek-Schreiber (2004).

2. A calculus on Lévy exponents. Let $\mathcal{E}xp$ denotes the totality of all functions $\Phi : E' \to \mathbb{C}$ appearing as the exponent in the Lévy-Khintchine formula (2). Hence we have that

$$\mathcal{E}xp + \mathcal{E}xp \subset \mathcal{E}xp, \quad \lambda \cdot \mathcal{E}xp \subset \mathcal{E}xp, \quad \text{for all postive} \ \ \lambda, \tag{9}$$

which means that $\mathcal{E}xp$ forms a cone in the space of all complex valued functions defined in E'. These follows from the fact that infinite divisibility is preserved under convolution and under convolution powers to positive real numbers.

Here we consider two integral operators acting on $\mathcal{E}xp$. Namely,

$$\mathcal{J} : \mathcal{E}xp \to \mathcal{E}xp, \quad (\mathcal{J}\Phi)(y) := \int_0^1 \Phi(sy)ds, \quad y \in E';$$

$$\mathcal{I} : \mathcal{E}xp_{\log} \to \mathcal{E}xp, \qquad (\mathcal{I}\Phi)(y) := \int_0^1 \Phi(sy)s^{-1}ds, \ \ y \in E'. \tag{10}$$

Indeed, $\mathcal{J}$ is well defined on whole $\mathcal{E}xp$ and $\mathcal{J}\Phi$ is the Lévy exponent of the integral (8). However, $\mathcal{I}$ is only defined on $\mathcal{E}xp_{\log}$ which corresponds to infinitely divisible measures with finite logarithmic moments. In fact, $\mathcal{I}\Phi$ and $\mathcal{J}\Phi$ are the Lévy exponents corresponding to the random integrals (8) and (7), respectively.

Here are the main properties of $\mathcal{J}$ and $\mathcal{I}$ mappings.

LEMMA 1. *The operators $\mathcal{I}$ and $\mathcal{J}$ acting on Lévy exponents and defined by (10) have the following basic properties:*

(*a*) $\mathcal{I}, \mathcal{J}$ *are additive and positive homogeneous operators on $\mathcal{E}xp$;*

(*b*) $\mathcal{I}, \mathcal{J}$ *commute under the composition and $\mathcal{J}(\mathcal{I}(\Phi)) = (\mathcal{I} - \mathcal{J})\Phi$;*

(*c*) $\mathcal{J}(I + \mathcal{I}) = \mathcal{I}$;

(*d*) $\mathcal{I}(I - \mathcal{J}) = \mathcal{J}$;

(*e*) $(I - \mathcal{J})(I + \mathcal{I}) = I.$

Proof. Part (a) follows from the fact that $\mathcal{E}xp$ forms a cone. For part (b) note that

$$(\mathcal{J}(\mathcal{I}(\Phi)))(y) = \int_0^1 (\mathcal{I}(\Phi))(ty)\,dt = \int_0^1 \int_0^1 \Phi(sty)s^{-1}dsdt =$$
$$\int_0^1 \int_0^t \Phi(ry)r^{-1}drdt = \int_0^1 \int_r^1 \Phi(ry)dt\,r^{-1}dr =$$
$$\int_0^1 \Phi(ry)r^{-1}dr - \int_0^1 \Phi(ry)dr = \mathcal{I}\Phi(y) - \mathcal{J}\Phi(y) = (\mathcal{I} - \mathcal{J})\Phi(y),$$

which proves equality in (b). Note that from the above (first line) we also infer that that operators $\mathcal{I}$ and $\mathcal{J}$ commute. All the remaining parts are straightforward consequences of the equality in (b). □

LEMMA 2. *The operators $\mathcal{I}$ and $\mathcal{J}$, defined by (10), have the following additional properties:*

(*a*) $\mathcal{J} : \mathcal{E}xp_{\log} \to \mathcal{E}xp_{\log}$ *and* $\mathcal{I} : \mathcal{E}xp_{(\log)^2} \to \mathcal{E}xp_{\log}$

(*b*) *If $(I - \mathcal{J})\Phi \in \mathcal{E}xp$ then the corresponding infinitely divisible measure $\tilde{\mu}$ with the Lévy exponent $(I - \mathcal{J})\Phi(y)$, $y \in E'$, has finite logarithmic moment.*

(*c*) $(I-\mathcal{J})\Phi+\mathcal{I}(I-\mathcal{J})\Phi = (I-\mathcal{J})\Phi+\mathcal{J}\Phi = \Phi$ *for all* $\Phi \in \mathcal{E}xp$.

Proof. (a) Since the function $E \ni x \to \log(1 + ||x||)$ is subadditive therefore for an infinitely divisible probability measure $\mu = [a, R, M]$ we have

$$\int_E \log(1 + ||x||)\mu(dx) < \infty \ \text{ iff } \int_{\{||x||>1\}} \log(1 + ||x||)M(dx) < \infty$$
$$\text{iff} \int_{\{||x||>1\}} \log ||x|| M(dx) < \infty; \qquad (11)$$

cf. Jurek and Mason (1993), Proposition 1.8.13 and references therein. Furthermore, if M is the spectral Lévy measure appearing in the Lévy exponent Φ then $\mathcal{J}\Phi$ has a Lévy spectral measure $\mathcal{J}M$, where

$$(\mathcal{J}M)(A) := \int_{(0,1)} M(t^{-1}A)dt = \int_{(0,1)} \int_E 1_A(tx)M(dx)dt, \qquad (12)$$

for all Borel subsets A of $E \setminus \{0\}$. Hence

$$\int_{||x||>1} \log ||x|| (\mathcal{J}M)(dx) = \int_{(0,1)} \int_E 1_{\{||x||>1\}}(tx) \log(t||x||)M(dx)dt$$
$$= \int_{(0,1)} \int_{\{||x||>t^{-1}\}} \log(t||x||)M(dx)dt = \int_{\{||x||>1\}} \int_{||x||^{-1}}^1 \log(t||x||)dtM(dx)$$
$$= \int_{\{||x||>1\}} ||x||^{-1} \int_1^{||x||} \log w dw M(dx)$$
$$= \int_{\{||x||>1\}} ||x||^{-1}[||x|| \log ||x|| - ||x|| + 1]M(dx)$$
$$= \int_{\{||x||>1\}} \log ||x|| M(dx) - \int_{\{||x||>1\}} [1 - ||x||^{-1}]M(dx).$$

Since the last integral is always finite as we integrate a bounded function with respect to a finite measure, we get the first part of (a). For the second one, let us note that

$$\int_{||x||>1} \log ||x|| (\mathcal{I}M)(dx) = 1/2 \int_{||x||>1} \log^2 ||x|| M(dx),$$

where $\mathcal{I}M$ is the Lévy spectral measure corresponding to the Lévy exponent $\mathcal{I}\Phi$.

For the part (b), note that the assumption made there implies that the measure

$$\widetilde{M}(A) := M(A) - \int_{(0,1)} M(t^{-1}A)dt \geq 0, \text{ for all Borel sets } A \subset E\backslash\{0\}, \quad (13)$$

is the Lévy spectral measure of $\tilde{\mu}$. [Note that there is no restriction on Gaussian part.] In fact, if $\widetilde{M}$ is nonnegative measure then it is necessarily Lévy spectral measure because $0 \leq \widetilde{M} \leq M$ and M is Lévy spectral measure; comp. Arujo-Giné (1980), Chapter 3, Theorem 4.7 , p. 119.
To establish the logarithmic moment of $\tilde{\mu}$ we argue as follows. Observe that for any constant $k > 1$ we have

$$\int_{(1<||x||\leq k)} \log ||x|| \widetilde{M}(dx) =$$
$$\int_{(1<||x||\leq k)} \log ||x|| M(dx) - \int_{(0,1)} \int_{(1<||x||\leq k)} \log ||x|| M(t^{-1}dx)dt =$$
$$\int_{(1<||x||\leq k)} \log ||x|| M(dx) - \int_{(0,1)} \int_{\{t^{-1}<||x||\leq kt^{-1}\}} \log(t||x||) dM(dx)dt =$$
$$\int_{(1<||x||\leq k)} \log ||x|| M(dx) - \int_{(1<||x||\leq k)} \int_{||x||^{-1}}^{1} \log(t||x||)dtM(dx)$$
$$- \int_{(k<||x||)} \int_{||x||^{-1}}^{k||x||^{-1}} \log(t||x||)dtM(dx) =$$
$$\int_{(1<||x||\leq k)} \log ||x|| M(dx) - \int_{(1<||x||\leq k)} ||x||^{-1} \int_{1}^{||x||} \log(w)dwM(dx)$$
$$- \int_{(k<||x||)} ||x||^{-1} \int_{1}^{k} \log(w)dwM(dx) =$$
$$\int_{(1<||x||\leq k)} \log ||x|| M(dx) - \int_{(1<||x||\leq k)} ||x||^{-1}(||x|| \log ||x|| - ||x|| + 1)M(dx)$$
$$- (k \log k - k + 1) \int_{(||x||>k)} ||x||^{-1} M(dx) =$$
$$\int_{(1<||x||\leq k)} (1 - ||x||^{-1})M(dx) - (k \log k - k + 1) \int_{(||x||>k)} ||x||^{-1} M(dx)$$
$$\leq M(||x|| > 1) < \infty,$$

and consequently $\int_{(||x||>1)} \log ||x|| \widetilde{M}(dx < \infty$. This with property (11), completes the proof of the part (b).

Finally, since $(I - \mathcal{J})\Phi$ is in a domain of definition of the operator $\mathcal{I}$ thus part (c) is a consequence of Lemma 1(e) and (d). Thus the proof is complete. $\square$

3. New factorizations of selfdecomposable distributions. Here we will apply the operators $\mathcal{I}$ and $\mathcal{J}$ to Lévy exponents of selfdecomposable probability measures.

LEMMA 3. *If μ is a selfdecomposable probability measure on a Banach space E with a characteristic function $\hat{\mu}(y) = \exp \Phi(y), y \in E'$ then*

$$\widetilde{\Phi}(y) := \Phi(y) - \int_{(0,1)} \Phi(sy)ds = (I - \mathcal{J})\Phi(y), \ y \in E',$$

is a Lévy exponent corresponding to an infinitely divisible probability measure with finite logarithmic moment.

Equivalently, if M is a Lévy spectral measure of a selfdecomposable μ then the measure $\widetilde{M}$ given by

$$\widetilde{M}(A) := M(A) - \int_0^1 M(t^{-1}A)dt, \quad A \subset E \setminus \{0\},$$

is also a Lévy spectral measure on E, that additionally integrates logarithmic function on any complement of a neighborhood of zero.

Proof. If $\mu = [a, R, M]$ is selfdecomposable, i.e., it satisfies the condition (2), for probability measures, that in turn is equivalent to the claim (4), for Lévy exponents. Hence we infer that the following inequalities

$$M(A) - M(e^t A) \geq 0, \quad \text{for all } \ t > 0 \text{ and Borell } \ A \subset E \setminus \{0\},$$

hold true and that there is no restriction on the remaining two parameters (a shift vector and a Gaussian covariance operator) in the Lévy-Khintchine formula (3). Multiplying both sides by e^{-t} and then integrating over positive half-line we conclude that $\widetilde{M}$ is non-negative Borel measure. Since $\widetilde{M} \leq M$ and M is a Lévy spectral measure then so is $\widetilde{M}$; comp. Theorem 4.7 in Chapter 3 of Araujo-Giné (1980). Finally, Lemma 2b) gives the finiteness of the logarithmic moment. Thus the proof is complete. $\square$

THEOREM 1. *For each selfdecomposable probability measure μ, on a Banach space E, there exists a unique s-selfdecomposable probability measure $\tilde{\mu}$ with finite logarithmic moment such that*

$$\mu = \tilde{\mu} * \mathcal{I}(\tilde{\mu}) \quad \textit{and} \quad \mathcal{J}(\mu) = \mathcal{I}(\tilde{\mu}) \tag{14}$$

In fact, if $\hat{\mu}(y) = \exp \Phi(y)$ then $(\tilde{\mu})\hat{}(y) = \exp[\Phi(y) - \int_{(0,1)} \Phi(ty)dt]$, $y \in E'$.

In other words, if Φ is a Lévy exponent of a selfdecomposable probability measure then $(I - \mathcal{J})\Phi$ is a Lévy exponent of a s-selfdecomposable measure with a finite logarithmic moment and

$$\Phi = (I - \mathcal{J})\Phi + \mathcal{I}(I - \mathcal{J})\Phi = (I - \mathcal{J})\Phi + \mathcal{J}\Phi. \tag{15}$$

Proof. Let $\hat{\mu}(y) = \exp \Phi(y) \in L$. From (4), $\Phi_t(y) := \Phi(y) - \Phi(e^{-t}y)$ are Lévy exponents. Hence,

$$\widetilde{\Phi}(y) := \int_{(0,\infty)} \Phi_t(ty)e^{-t}dt = \Phi(y) - \int_{(0,\infty)} \Phi(e^{-t}y)e^{-t}dt = ((I - \mathcal{J})\Phi)(y)$$

is a Lévy exponent as well, because of Lemma 3. Again by Lemma 3 (or Lemma 2 b)), a probability measure $\tilde{\mu}$ defined by the Fourier transform $(\tilde{\mu})\hat{}(y) = \exp(I - \mathcal{J})\Phi(y)$ has logarithmic moment. Consequently, $\mathcal{I}(\tilde{\mu})$ is well defined probability measure whose Lévy exponent is equal to $\mathcal{I}(I - \mathcal{J})\Phi$. Finally, Lemmas 1(d) and 2(c) give the factorization (15).

Since $\mathcal{I}(\tilde{\mu}) \in L$ has the property that $\tilde{\mu} * \mathcal{I}(\tilde{\mu})$ is again in L, therefore Theorem 1 from Iksanov-Jurek-Schreiber(2004) gives that $\tilde{\mu} \in \mathcal{U}$, i.e., it is a s-selfdecomposable probability distribution.

To see the second equality in (14) one should observe that it is equivalent to equality $\mathcal{J}\Phi = \mathcal{I}(I - \mathcal{J})\Phi$ that indeed holds true in view of Lemma 1(d).

Suppose there exists another factorization of the form $\mu = \rho * \mathcal{I}(\rho)$ and let $\Xi(y)$ be the Lévy exponent of ρ. Then we get that $\Phi(y) = \Xi(y) + (\mathcal{I}\,\Xi)(y) = (I + \mathcal{I})\,\Xi(y)$. Hence, applying to both sides $\mathcal{I} - \mathcal{J}$ we conclude that

$$(I - \mathcal{J})\Phi = ((I - \mathcal{J})(I + \mathcal{I}))\,\Xi = \Xi,$$

where the last equality is from Lemma 1(e). This proves the uniqueness in representation (14) and thus the proof of Theorem 1 is complete. □

REMARK 1. In a case of Euclidean space $\mathbb{R}^d$, using Schoenberg's Theorem, one gets immediately that $\widetilde{\Phi}$ is a Lévy exponent; cf. Cuppens (1975), pp. 80-82.

Following Iksanov, Jurek and Schreiber (2004),p. 1360, we will say that a selfdecomposable probability measure μ has *the factorization property* if $\mu * \mathcal{I}^{-1}(\mu)$ is selfdecomposable as well. In other words, a class L probability measure convoluted with its background driving probability distribution is again class L distribution. As in Iksanov-Jurek-Schreiber (2004), Proposition 1, if L^f denotes the set of all class L distribution with factorization property then

$$L^f = \mathcal{I}(\mathcal{J}(ID_{\log})) = \mathcal{J}(\mathcal{I}(ID_{\log})) = \mathcal{J}(L) \text{ and } L^f \subset L \subset \mathcal{U}, \qquad (16)$$

REMARK 2. All the three sets of probability measures form closed topological subsemigroups of the semigroup ID of infinitely divisible probability measures.

COROLLARY 1. *Each selfdecomposable μ admits factorization $\mu = \nu_1 * \nu_2$, where ν_1 is an s-selfdecomposable measure (i.e., $\nu_1 \in \mathcal{U}$) and ν_2 is a selfdecomposable one with the factorization property (i.e., $\nu_2 \in L^f$). Moreover, we have inclusions $L^f \subset L \subset \mathcal{U}$ and $L \subset L^f * \mathcal{U}$.*

Proof. Because of (14) we infer that $\nu_1 := \tilde{\mu}$ is s-selfdecomposable measure. In view of (14) and (16), $\nu_2 := \mathcal{I}(\tilde{\mu})$ has the factorization property, i.e., $\nu_2 \in L^f$, which completes the proof. □

EXAMPLES. 1) Let Σ_p be a symmetric stable distribution on a Banach space E, with the exponent p. Then its Lévy exponent, Φ_p, is equal $\Phi_p(y) = -\int_S |< y, x >|^p \, m(dx)$, where m is a finite Borel measure on the unit sphere S of E; cf. Samorodnitsky and Taqqu (1994). Hence $(I - \mathcal{J})\Phi_p(y) = p/(p+1)\Phi_p(y)$, which means that in Corollary 1, both ν_1 and ν_2 are stable with the exponent p and measures $m_1 := (p/(p+1))m$ and $m_2 := (1/(p+1))m$, respectively.

2) Let η denotes the Laplace (double exponential) distribution on real line $\mathbb{R}$. Then its Lévy exponent Φ_η is equal $\Phi_\eta(t) := -\log(1+t^2)$, $t \in \mathbb{R}$. Consequently, $(I - \mathcal{J})\Phi_\eta(t) = 2(\arctan t - t)t^{-1}$ is the Lévy exponent of the class $\mathcal{U}$ probability measure ν_1 from Corollary 1, and $(2t - \arctan t - t \log(1+t^2))t^{-1}$ is the Lévy exponent of the class L^f measure ν_2 from Corollary 1.

Before we formulate next result we need to recall that, by (8), the class $\mathcal{U}$ is defined here as $\mathcal{U} = \mathcal{J}(ID)$. (For other description cf. Jurek (1985) and references therein.) Consequently, by an iterative argument we can define

$$\mathcal{U}^{<1>} := \mathcal{U}, \;\; \mathcal{U}^{<k+1>} := \mathcal{J}(\mathcal{U}^{<k>}) = \mathcal{J}^{k+1}(ID), \;\; k = 1, 2, ...; \qquad (17)$$

cf. Jurek (2004) for other characterization of classes $\mathcal{U}^{<k>}$. Elements from semigropus $\mathcal{U}^{<k>}$ are called *k-times s-selfdecomposable probability measures.*

THEOREM 2. *Let n be any natural number and μ be a selfdecopmosable probability measure. Then there exist k-times s-selfdecomposable probability measures $\tilde{\mu}_k$, $k = 1, 2, ..., n$, such that*

$$\mu = \tilde{\mu}_1 * \tilde{\mu}_2 * ... * \tilde{\mu}_n * \mathcal{I}(\tilde{\mu}_n), \quad \mathcal{J}^k(\mu) = \mathcal{I}(\tilde{\mu}_k), \quad k = 1, 2, ..., n. \tag{18}$$

In fact, if Φ is the exponent of μ then $\tilde{\mu}_k$ has the exponent $\mathcal{I}^{k-1}(I - \mathcal{J})^k\Phi = (I - \mathcal{J})\mathcal{J}^{k-1}\Phi$ and

$$\Phi = (I-\mathcal{J})\Phi + (I-\mathcal{J})\mathcal{J}\Phi + ... + (I-\mathcal{J})\mathcal{J}^{k-1}\Phi + ... + (I-\mathcal{J})\mathcal{J}^{n-1}\Phi + \mathcal{J}^n\Phi$$
$$= (I - \mathcal{J}^n)\Phi + \mathcal{J}^n\Phi. \tag{19}$$

Proof. For $n = 1$ the factorization (18) and the formula (19) are true by Theorem 1, with $\tilde{\mu}_1 := \tilde{\mu}$. Suppose our claim (18) is true for n. Since $\rho := \mathcal{I}(\tilde{\mu}_n)$ is selfdecomposable, therefore applying to it Theorem 1, we have that $\rho = \tilde{\rho} * \mathcal{I}(\tilde{\rho})$, where $\tilde{\rho}$ has the Lévy exponent $(I-\mathcal{J})\mathcal{J}^n\Phi = \mathcal{J}^n(I-\mathcal{J})\Phi$ and thus it corresponds to $n+1$-times s-selfdecomposable probability because, by Theorem 1, $(I - \mathcal{J})\Phi$ is already s-selfdecomposable and then we apply n-times the operator $\mathcal{J}$; compare the definition (17). Thus the factorization (18) holds for $n+1$, which completes the proof of the first part of the theorem. Similarly, applying inductively decomposition (15), from Theorem 1, and part (d) of Lemma 1 we will get the formula (19). Thus the proof is complete. □

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