A note on the composition of two random integral mappings $\mathcal{J}^{\beta}$ and some examples

Agnieszka Czyżewska-Jankowska and Zbigniew J. Jurek*

Stochastic Anal. Appl., vol. 27, 2009, pp. 1212-1222.

Abstract. A method of *random integral representation*, that is, a method of representing a given probability measure as the probability distribution of some random integral, was quite successful in the past few decades. In this note we show that a composition of two random integral mappings $\mathcal{J}^{\beta}$ is again a random integral mapping. We illustrate our results on some examples.

Mathematics Subject Classifications(2000): Primary 60F05 , 60E07, 60B11; Secondary 60H05, 60B10.

Key words and phrases: Class $\mathcal{U}_{\beta}$ distributions; s-selfdecomposable distributions; infinite divisibility; Lévy-Khintchine formula; Euclidean space; Lévy process; random integral; Banach space.

Abbreviated title: A Composition of random mappings $\mathcal{J}^{\beta}$

We say that a probability distribution (measure) μ admits *a random integral representation* if we have

$$\mu = \mathcal{L}\big(\int_I h(t)dY(r(t))\big),$$
$$\textit{where } I=(a,b]\subset\mathbb{R}^+,\ h:\mathbb{R}^+\to\mathbb{R},\quad Y(\cdot) \textit{ is a Lévy process and,}$$
$$r:\mathbb{R}^+\to\mathbb{R}^+ \textit{is a monotone function (deterministic time change in } Y) \ . \tag{1}$$

In fact, in the past it was proved that many classes of limit laws can be described as probability distributions of random integrals of the form (1).

*This research has been in part supported by a Maria Curie Transfer of Knowledge Fellowship of the European Community's Sixth Framework under contract number MTKD-CT-2004-013389.

Moreover, it was conjectured that *all classes of limit laws derived for sequences of independent random variables* should admit a random integral representation; cf. Jurek (1985; 1988) and see the Conjecture on *www.math.uni.wroc.pl/*∼zjjurek . The random integral approach was also successfully used by others; see for instance Aoyama-Maejima (2007). Last but not least, one should emphasis that from the integral representations (1) very easily follow formulae for the characteristic functions and the Lévy-Khintchine representation.

In this note we examine how some random integral mappings of the form (1) behave under a composition of such mappings.

1. Introduction and main results. Throughout the paper $\mathcal{L}(X)$ will denote the probability distribution of a $\mathbb{R}^d$-valued[1] random vector X. Similarly, by $Y_\nu(t), t \geq 0$, we will denote an $\mathbb{R}^d$- valued Lévy process such that $\mathcal{L}_\nu(Y(1)) = \nu$. By a Lévy process we mean process starting from zero, with stationary and independent increments and with paths that are right continuous and have finite left hand limits. Of course, we always have that $\nu \in ID$, where ID stands for the set of all *infinitely divisible* measures on $\mathbb{R}^d$, so in particular ν^{*c}, $c > 0$, is well-defined infinitely divisible probability measure.

For $\beta > 0$ and a Lévy process $Y_\nu(t), t \geq 0$, we define an integral mapping $\mathcal{J}^\beta : ID \to ID$ and a class $\mathcal{U}_\beta$ as follows

$$\mathcal{J}^\beta(\nu) := \mathcal{L}\big(\int_0^1 t^{1/\beta}\, dY_\nu(t)\big) = \mathcal{L}\big(\int_0^1 t\, dY_\nu(t^\beta)\big), \quad \text{and} \quad \mathcal{U}_\beta := \mathcal{J}^\beta(ID). \quad (2)$$

For another equivalent characterizations of classes $\mathcal{U}_\beta$, even in a greater generality than we are interested in the present note, we refer to Jurek (1988). To the distributions from the class $\mathcal{U}_\beta$ we refer to as *generalized s-selfdecomposable distributions.*

Recall here that the importance and the interest in the increasing family $\mathcal{U}_\beta$ of convolution semigroups comes from the fact that

$$\overline{(\cup_n \mathcal{U}_{\beta_n})} = ID, \quad \text{for any incresing seqeunce} \quad \beta_n \to \infty\,,$$

where the bar means a closure in the weak topology; cf. Jurek () and the references therein. More explicitly, let us note that

$$\mathcal{J}^\beta(\nu) \Rightarrow \mathcal{L}(\int_0^1 dY_\nu(t)) = \nu, \quad \text{as } \beta \to 0.$$

[1]Our proofs are a such that they hold true for a real separable Banach space valued random elements as well. Interested Readers in probability on Banach spaces, e.g., those who prefer to see a stochastic process with continuous paths on $[0,1]$ interval as an $C[0,1]$-valued random element, we refer to the monograph by Araujo and Gine (1980).

Since we assume that the paths of Y are almost surely cadlag (i.e., right continuous with finite left hand limits) and the random integral in (2) we define by a formal integration by parts formula, therefore the random integral in question exists; for details cf. Jurek-Vervaat (1983), Lemma or Jurek-Mason (1993), Section 3.6, p. 116.

Here are the results:

Proposition 1. *For $\nu \in ID$ and for positive $\alpha \neq \beta$ we have*

$$\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\nu)) = \mathcal{L}\big(\int_0^1 u\,dY_\nu(r_{(\alpha,\beta)}(u))\big) = \mathcal{L}\big(\int_0^1 r^{-1}_{(\alpha,\beta)}(u)\,dY_\nu(u)\big) \tag{3}$$

where $r_{(\alpha,\beta)} : [0,1] \to [0,1]$ is a continuous strictly increasing time change given by the formula $r_{(\alpha,\beta)}(u) := \frac{\beta}{\beta-\alpha}u^{\alpha} - \frac{\alpha}{\beta-\alpha}u^{\beta}$, which is symmetric in α and β, that is, $r_{(\alpha,\beta)}(u) = r_{(\beta,\alpha)}(u)$.

Corollary 1. *For $0 < \alpha < \beta$ and $\nu \in ID$ we have the identity:*

$$\mathcal{J}^{\alpha}\big(\mathcal{J}^{\beta}(\nu^{*(\beta-\alpha)})\big) * \mathcal{J}^{\beta}(\nu^{*\alpha}) = \mathcal{J}^{\alpha}\big(\nu^{*\beta}\big) \tag{4}$$

Equivalently, in terms of characteristic functions we have

$$(\beta-\alpha)\log(\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\nu))\widehat{)}(y) = \beta\log(\mathcal{J}^{\alpha}(\nu)\widehat{)}(y)) - \alpha\log(\mathcal{J}^{\beta}(\nu)\widehat{)}(y) \tag{5}$$

Proposition 2. *For $\beta > 0$ and $\nu \in ID$ we have*

$$\mathcal{J}^{\beta}(\mathcal{J}^{\beta}(\nu)) = \mathcal{L}\big(\int_0^1 t\,dY_\nu(r_{(\beta,\beta)}(t))\big), \tag{6}$$

where $r_{(\beta,\beta)}(u) := u^{\beta}(1-\beta\log u)$ is an increasing time change in a Lévy process Y.

Remark 1. *Let us note that for the functions $r_{(\alpha,\beta)}$, $\alpha \neq \beta$, and $r_{(\beta,\beta)}$, from Propositions 1 and 2, we have*

$$\lim_{\alpha\to\beta} r_{(\alpha,\beta)}(u) = r_{(\beta,\beta)}(u), \quad \textit{for } 0 \le u \le 1.$$

Remark 2. *For $\beta = 1$, probability measures from $\mathcal{U}^{<2>} := \mathcal{J}^1(\mathcal{J}^1(ID))$ were called* 2-times s-selfdecomposable distributions *in Jurek (2004). In fact,* m-times s-selfdecomposability *was defined there inductively, and the corresponding classes $\mathcal{U}^{<m>}$, for $m = 1, 2, \dots$ were described in many ways; cf. Propositions 3 and 4.*

Each infinitely divisible probability distribution μ is uniquely determined by a triplet: a shift vector a, a Gaussian covariance operator R and a Lévy spectral measure M that appear in the Lévy-Khintchine formula, as it is recalled at beginning of Section 4. Therefore, following the notation from Parthasarathy (1967), Chapter VI, we will write that $\mu = [a, R, M]$.

Directly from Jurek (1988), or from Lemma 1 below, if $\mu = [a, R, M]$ and $\mathcal{J}^{\beta}(\mu) = [a^{(\beta)}, R^{(\beta)}, M^{(\beta)}]$ and

$$b_{M,\beta} := \int_{\{||x||>1\}} x\,||x||^{-1-\beta}\,M(dx) \in \mathbb{R}^d \text{ (or } E) \tag{7}$$

then we have

$$a^{(\beta)} := \beta(\beta+1)^{-1}(a + b_{M,\beta}), \quad R^{(\beta)} := \beta(2+\beta)^{-1}R$$
$$M^{(\beta)}(A) := \int_0^1 T_{t^{1/\beta}}M(A)dt = \int_0^1 \int_{\mathbb{R}^d} 1_A(t^{1/\beta}\,x)M(dx)dt, \quad \text{for } A \in \mathcal{B}_0. \tag{8}$$

The above $\mathcal{B}_0$ stands for all Borel subsets of $\mathbb{R}^d \setminus \{0\}$ (or $E \setminus \{0\}$ if one consider results on Banach space E). Note that one needs to change the order of integration in the formula (1.10) in Jurek (1988) to get the above form of $a^{(\beta)}$.

Proposition 3. *For positive $\alpha \neq \beta$, if $\mu = [a, R, M]$ and $\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\mu)) = [a^{(\alpha,\beta)}, R^{(\alpha,\beta)}, M^{(\alpha,\beta)}]$ then*

(i) $a^{(\alpha,\beta)} = \alpha\beta(1+\alpha)^{-1}(1+\beta)^{-1}\; a + \alpha\beta(\beta-\alpha)^{-1}$
$\Big[(\alpha{+}1)^{-1} \int_{\{||x||>1\}} x||x||^{-1-\alpha}M(dx) - (\beta{+}1)^{-1} \int_{\{||x||>1\}} x||x||^{-1-\beta}M(dx)\Big]$
$= \frac{\beta}{\beta-\alpha}\, a^{(\alpha)} - \frac{\alpha}{\beta-\alpha}\, a^{(\beta)}\,.$

(ii) $R^{(\alpha,\beta)} = \frac{\alpha}{2+\alpha} \cdot \frac{\beta}{2+\beta} R = \frac{\beta}{\beta-\alpha} R^{(\alpha)} - \frac{\alpha}{\beta-\alpha} R^{(\beta)}\,.$

(iii) $M^{(\alpha,\beta)}(A) = \int_0^1 \int_0^1 T_{t^{1/\alpha}s^{1/\beta}} M(A)\; ds\; dt = \frac{\beta}{\beta-\alpha}\, M^{(\alpha)}(A) - \frac{\alpha}{\beta-\alpha}\, M^{(\beta)}(A)$

for all Borel sets A in $\mathcal{B}_0$.

Remark 3. *Let us remark here that the parameters in the triple corresponding to the composition $\mathcal{J}^{\alpha} \circ \mathcal{J}^{\beta}$ are linear combinations of parameters corresponding to the mappings $\mathcal{J}^{\alpha}$ and $\mathcal{J}^{\beta}$ in an identical way as they appear in the formula of $r_{(\alpha,\beta)}$ in Proposition 1.*

From Jurek(1988), Corollary 1.1, if $0 < \alpha < \beta$ then we have $\mathcal{U}_{\alpha} \subseteq \mathcal{U}_{\beta}$. The converse inclusion is determined as follows.

Corollary 2. *Let $0 < \alpha < \beta$. In order that $\mathcal{J}^{\beta}(\rho) \in \mathcal{U}_{\alpha}$, for some $\rho \in ID$, it is necessary and sufficient that ρ admits the following convolution factorization $\rho = \mathcal{J}^{\alpha}\big(\nu^{*(1-\alpha/\beta)}\big) * \nu^{*\alpha/\beta} = \big(\mathcal{J}^{\alpha}(\nu)\big)^{*(1-\alpha/\beta)} * \nu^{*\alpha/\beta}$, for some $\nu \in ID$.*

3. Examples. We will illustrate our results in the following examples.

(a) For $\alpha > 0$ and $\beta := 2\alpha$, Proposition 1 gives that $r_{(\alpha,2\alpha)}(u) = 2u^{\alpha} - u^{2\alpha}$ and $r^{-1}_{(\alpha,2\alpha)}(t) = \big(1 - \sqrt{1-t}\big)^{1/\alpha}$, and we have that:

$$\mathcal{J}^{\alpha}\big(\mathcal{J}^{2\alpha}(\nu)\big) = \mathcal{L}\Big(\int_0^1 u\, dY_{\nu}(2u^{\alpha} - u^{2\alpha})\Big) = \mathcal{L}\,\Big(\int_0^1 \Big(1 - \sqrt{1-t}\Big)^{1/\alpha} dY_{\nu}(t)\Big) =$$

$$\mathcal{L}\,\Big(\int_0^1 \Big(1 - \sqrt{1-t}\Big)^{1/\alpha} d\widetilde{Y}_{\nu}(t)\Big) = \mathcal{L}\Big(\int_0^1 \Big(1 - \sqrt{t}\Big)^{1/\alpha} dY_{\nu}(t)\Big). \quad (9)$$

where $\widetilde{Y}_{\nu}(t) := Y_{\nu}(1) - Y_{\nu}(1-t), 0 \le t \le 1$, and $Y_{\nu}(t), 0 \le t \le 1$, have the same distributions, and $d\widetilde{Y}_{\nu}(t) = d\,Y_{\nu}(t)$.

Remark 4. *From Proposition 3 we may get the formulae for the triple $a^{(\alpha,2\alpha)}$, $R^{(\alpha,2\alpha)}$ and $M^{(\alpha,2\alpha)}$. However, since in our example we have an explicit form for $r^{-1}_{(\alpha,2\alpha)}$ therefore applying Lemma 1 and the formula (13) to the last integral in (9) we have an alternative way of getting the triplet in question. Thus we have*

$$M^{(\alpha,2\alpha)}(A) = \int_0^1 T_{(1-\sqrt{t})^{1/\alpha}} M(A) dt = \int_0^1 \int_{\mathbb{R}^d} 1_A((1-\sqrt{t})^{1/\alpha}\, x) M(dx)\, dt\,,$$

for all Borel sets A in $\mathcal{B}_0$.

(b) Let $\sigma_p := [a, 0, M_p]$ denotes *a stable distribution* with an exponent $0 < p < 2$, that is, for some finite measure γ on the unit sphere $S = \{x : ||x|| = 1\}$ we have

$$M_p(A) := \int_S \int_0^{\infty} 1_A(r\,x) r^{-p-1} dr \gamma(dx), \quad A \in \mathcal{B}_0\,; \qquad (10)$$

cf. Araujo-Gine (1980), Chapter 3, Theorem 6.15. Then

$$b_{M_p,\beta} = (\beta + p)^{-1} \bar{\gamma}, \quad \text{where} \;\; \bar{\gamma} := \int_S u\, \gamma(du); \quad M_p^{(\beta)} = \beta(\beta + p)^{-1}\, M_p,$$

by (7) and (8), respectively. Consequently, from Proposition 3 we get

$$\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\sigma_p))$$
$$= \Big[\frac{\alpha\beta}{(\alpha+1)(\beta+1)} \big(a + \frac{\alpha+\beta+p+1}{(\alpha+p)(\beta+p)}\,\bar{\gamma}\big), \quad 0, \quad \frac{\alpha\beta}{(\alpha+p)(\beta+p)}\, M_p \Big]$$
$$= \sigma_p^{*\,\frac{\alpha\beta}{(\alpha+p)(\beta+p)}} * \delta_{x_0}, \tag{11}$$

where a vector x_0 is given by the formula

$$x_0 := \frac{\alpha\beta\,(\alpha+\beta+p+1)}{(\alpha+1)(\beta+1)(\alpha+p)(\beta+p)}\,\big[(p-1)\,a + \bar{\gamma}\,\big],$$

that is, in (11), up to a shift vector, we get a convolution power of the stable measure σ_p.

(c) Let e_λ denotes the exponential distribution with the parameter λ. Then its Lévy spectral measure M_e has the density $e^{-\lambda\,x}\,x^{-1}1_{(0,\infty)}(x)$ and has the characteristic function

$$\frac{1}{\lambda - i\,y} = \widehat{e_\lambda}(y) = \exp \int_0^\infty (e^{iyx} - 1)\frac{e^{-\lambda\,x}}{x} dx\,, y \in \mathbb{R}\,.$$

Then the Lévy spectral measure $M_e^{(\beta)}$ has the density $\lambda\beta(\lambda\,x)^{\beta-1}\Gamma(-\beta, \lambda\,x)$, $x > 0$, where for $c \in \mathbb{R}$

$$\Gamma(c,x) =: \int_x^\infty u^{c-1}e^{-\lambda\,u}du, x > 0, \quad \text{is the incomplete Euler gamma function;}$$

cf. Gradsteyn and Ryzhik (1994), Section 8.3 for other representations of the gamma function. Then for $A \subset (0,\infty)$, we get the formula

$$M_e^{(\alpha,\beta)}(A) = \frac{\alpha\beta\lambda}{\beta-\alpha}\int_A [(\lambda\,x)^{\alpha-1}\Gamma(-\alpha,\lambda\,x) - (\lambda\,x)^{\beta-1}\Gamma(-\beta,\lambda\,x)]\,dx, \tag{12}$$

i.e., the density is a combinations (with variable coefficients) of two incomplete gamma (Euler) functions.

4. Proofs and auxiliary results. Let us recall that for a probability Borel measures μ on $\mathbb{R}^d$, its *characteristic function* $\hat{\mu}$ is defined as

$$\widehat{\mu}(y) := \int_{\mathbb{R}^d} e^{i<y,x>}\mu(dx), \;\; y \in \mathbb{R}^d,$$

where $< \cdot, \cdot >$ denotes the scalar product (or a bilinear form between the conjugate Banach space E' and the space E). Recall that for infinitely divisible measures μ their characteristic functions admit the following Lévy-Khintchine formula

$$\widehat{\mu}(y) = e^{\Phi(y)}, \ y \in \mathbb{R}^d, \quad \text{and the Lévy exponents} \ \ \Phi(y) = i < y, a > - \frac{1}{2} < y, Ry > + \int_{\mathbb{R}^d \setminus \{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M(dx), \tag{13}$$

where a is a *shift vector*, R is a *covariance operator* corresponding to the Gaussian part of μ and M is a *Lévy spectral measure.* Since there is a one-to-one correspondence between a measure $\mu \in ID$ and the triples a, R and M in its Lévy-Khintchine formula (13) we will write $\mu = [a, R, M]$. Finally, let recall that[2]

$$M \textit{ is Lévy spectral measure on } \mathbb{R}^d \textit{ iff } \int_{\mathbb{R}^d} \min(1, ||x||^2) M(dx) < \infty \tag{14}$$

For this note it is important to recall the following crucial fact.

Lemma 1. *If the random integral $X := \int_I h(t) dY(r(t))$ exists then we have*

$$\log(\mathcal{L}(X))\widehat{\ }(y) = \int_I \log(\mathcal{L}(Y(1))\widehat{\ }(h(s)y) dr(s) = \int_I \Phi(h(s)y) dr(s), \quad y \in \mathbb{R}^d \tag{15}$$

where Φ is the Lévy exponent of $(\mathcal{L}(Y(1)))\widehat{\ }$.

The formula is a straightforward consequence of our definition (integration by parts) of the random integrals (1). The proof is analogous to that in Jurek-Vervaat(1983), Lemma 1.1 or Jurek-Mason (1993), Lemma 3.6.4 or Jurek (1988), Lemma 2.2 (b).

Note that for bounded sets $I \subset \mathbb{R}^+$ and continuous h integrals of the form (1) are well-defined. In particular, we have

$$(\mathcal{J}^\beta(\nu))\widehat{\ }(y) = \exp \int_0^1 \log \hat{\nu}(t^{1/\beta} y) \, dt, \quad y \in \mathbb{R}^d \ \ (\text{or} \ \ y \in E'), \tag{16}$$

Since $\mathcal{J}^\beta$ are mappings from ID into ID, therefore we may consider their compositions. Here we recall their basic properties for further references.

[2]The integrability criterium (14) is true also in real separable Hilbert spaces, cf. Parthasarathy (1967), Chapter VI. But no such a characterization is available for infinite dimensional Banach spaces; cf. Araujo-Gine (1980).

Lemma 2. Relations between the mappings $\mathcal{J}^{\beta}$, for $\beta > 0$.

(a) Each mapping $\mathcal{J}^{\beta}$ is a continuous isomorphism between convolution semigroups ID and $\mathcal{U}_{\beta}$.

(b) For $\beta, c > 0$ and $\nu \in ID$ we have

$$(\mathcal{J}^{\beta}(\nu))^{*c} = \mathcal{J}^{\beta}(\nu^{*c}) \tag{17}$$

(c) The composition of the mappings $\mathcal{J}^{\beta}, \beta > 0$, is commutative, that is, for $\alpha, \beta > 0$ and $\nu \in ID$

$$\mathcal{J}^{\alpha}\big(\mathcal{J}^{\beta}(\nu)\big) = \mathcal{J}^{\beta}\big(\mathcal{J}^{\alpha}(\nu)\big). \tag{18}$$

Proof of Lemma 2. Parts (a) and (b) have proofs along the lines of a proof of Theorem 1.3 (a) , (b) and (c) in Jurek (1988) for the operator $Q = I$. But basically we utilize the formula (16). For the commutativity property note that using the formula (16) we get

$$\begin{aligned}
&\Big(\mathcal{J}^{\alpha}\left(\mathcal{J}^{\beta}\left(\nu\right)\right)\Big)\widehat{\ }(y) = \\
&\quad = \exp\Big(\int_0^1 \log\left(\mathcal{J}^{\beta}\left(\nu\right)\right)\widehat{\ }\left(t^{1/\alpha}y\right) dt\Big) = \exp\Big(\int_0^1\int_0^1 \log\hat{\nu}\left(s^{1/\beta}t^{1/\alpha}y\right) ds\, dt\Big) \\
&\quad = \exp\Big(\int_0^1\int_0^1 \log\hat{\nu}\left(s^{1/\beta}t^{1/\alpha}y\right) dt\, ds\Big) = \mathcal{J}^{\beta}\big(\mathcal{J}^{\alpha}\left(\nu\right)\big)\widehat{\ }(y),
\end{aligned}$$

which indeed proves the commutativity in (c). This completes a proof of Lemma 2.

Proof of Proposition 1. Note that the above can be rewritten as follows

$$\begin{aligned}
\log\big(\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\nu)\big)\widehat{\ }(y) &= \int_0^1\int_0^1 \log\hat{\nu}\left(s^{1/\beta}t^{1/\alpha}y\right) dt\, ds \quad (\text{putting } t := u^{\alpha}s^{-\alpha/\beta}) \\
= \int_0^1\int_0^{s^{1/\beta}} \log\hat{\nu}\left(uy\right)\frac{\alpha u^{\alpha-1}}{s^{\alpha/\beta}}\, du\, ds &= \int_0^1 \log\hat{\nu}(uy)\alpha u^{\alpha-1}\int_{u^{\beta}}^1 s^{-\alpha/\beta}\, ds\, du \\
= \frac{\beta}{\beta-\alpha}\int_0^1 \log\hat{\nu}(uy)\alpha u^{\alpha-1}\, du &- \frac{\alpha}{\beta-\alpha}\int_0^1 \log\hat{\nu}(uy)\beta u^{\beta-1}\, du \\
= \int_0^1 \log\hat{\nu}(uy)d\, r_{(\alpha,\beta)}(u) &= \log\Big(\mathcal{L}\big(\int_0^1 u\, dY_{\nu}(r_{(\alpha,\beta)}(u))\big)\widehat{\ }(y)\Big),
\end{aligned} \tag{19}$$

where the function $r_{(\alpha,\beta)}$ is given in Proposition 1 and the last equality follows from Lemma 1 .

Proof of Corollary 1. From the property (17) and then (19) we get

$$\log\left(\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\nu^{*(\beta-\alpha)}))\widehat{)}(y) = (\beta-\alpha)\log\left(\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\nu))\widehat{)}(y)\right.\right.$$
$$= \beta\int_0^1 \log\hat{\nu}(uy)\alpha u^{\alpha-1}\,du - \alpha\int_0^1 \log\hat{\nu}(uy)\beta u^{\beta-1}\,du$$
$$= \beta\int_0^1 \log\hat{\nu}(t^{1/\alpha}y)\,dt - \alpha\int_0^1 \log\hat{\nu}\left(t^{1/\beta}y\right)dt$$
$$= \int_0^1 \log(\nu^{*\beta}\widehat{)}(t^{1/\alpha}y)\,dt - \int_0^1 \log(\nu^{*\alpha}\widehat{)}(t^{1/\beta}y)dt \quad \text{(by (17))}$$
$$= \log(\mathcal{J}^{\alpha}(\nu^{\beta}\widehat{)}(y) - \log(\mathcal{J}^{\beta}(\nu^{*\alpha}))\widehat{)}(y)). \quad (20)$$

In other words we have

$$(\mathcal{J}^{\alpha}(\mathcal{J}^{\beta}(\nu^{*(\beta-\alpha)})))\widehat{\ }(y)\cdot(\mathcal{J}^{\beta}\left(\nu^{*\alpha}\right))\widehat{)}(y) = (\mathcal{J}^{\alpha}(\nu^{*\beta}))\widehat{)}(y) \quad (21)$$

But (21), in terms of probability measures, coincides with the formula (4) that completes the proof of the Corollary.

Proof of Proposition 2. Similarly as at the beginning of (15) we get

$$\log\left(\mathcal{J}^{\beta}(\mathcal{J}^{\beta}(\nu))\widehat{)}(y) = \int_0^1\int_0^1 \log\hat{\nu}\left(s^{1/\beta}t^{1/\beta}y\right)dt\,ds\right.$$
$$= \int_0^1\int_0^{s^{1/\beta}} \log\hat{\nu}\,(uy)\,\frac{\beta u^{\beta-1}}{s}\,du\,ds = \int_0^1 \log\hat{\nu}\,(uy)\,\beta u^{\beta-1}(-\beta\log u)du$$
$$= \int_0^1 \log\hat{\nu}\,(uy)\,d[u^{\beta}(1-\beta\log u)]du, \quad (22)$$

which, via Lemma 1, is the the statement (6) in Proposition 2.

Proof of Proposition 3. First, note that for $M^{(\beta)}$ given by (8), using (7) we get

$$b_{M^{(\beta)},\alpha} = \int_0^1\int_{\mathbb{R}^d} 1_{B^c}(x)\,x\,||x||^{-1-\alpha}M(t^{-1/\beta}dx)dt$$
$$= \int_0^1\int_{\mathbb{R}^d} 1_{B^c}(t^{1/\beta}x)\,x\,||x||^{-1-\alpha}t^{-\alpha/\beta}M(dx)dt$$
$$= \int_{\{||x||>1\}} x\,||x||^{-1-\alpha}\int_{||x||^{-\beta}}^1 t^{-\alpha/\beta}dt\,M(dx) = \beta(\beta-\alpha)^{-1}(b_{M,\alpha}-b_{M,\beta}).$$

Second, successively using (8) (for the shift vector) and the above one gets

$$
\begin{aligned}
&(a^{(\beta)})^{(\alpha)} = \alpha(\alpha+1)^{-1}[a^{(\beta)} + b_{M^{(\beta)},\alpha}] \\
&= \alpha(\alpha+1)^{-1}[\beta(\beta+1)^{-1}a + \beta(\beta+1)^{-1}b_{M,\beta} + \beta(\beta-\alpha)^{-1}(b_{M,\alpha} - b_{M,\beta})] \\
&= \alpha\beta(1+\alpha)^{-1}(1+\beta)^{-1}\ a + \alpha\beta(\beta-\alpha)^{-1}[(\alpha+1)^{-1}b_{M,\alpha} - (\beta+1)^{-1}b_{M,\beta}] \\
&= \frac{\beta}{\beta-\alpha}\,a^{(\alpha)} - \frac{\alpha}{\beta-\alpha}\,a^{(\beta)},
\end{aligned}
$$

and the last equality one checks by straightforward computation. This proves the formula for the shift vector **(i)**. Part **(ii)** follows easily from (5). Finally we have

$$
\begin{aligned}
M^{(\alpha,\beta)}(A) &= (M^{(\alpha)})^{(\beta)}(A) = \int_0^1 T_{t^{1/\beta}}M^{(\alpha)}(A)dt \\
&= \int_0^1\int_0^1 T_{t^{1/\beta}s^{1/\alpha}}M(A)dtds = \int_0^1\int_0^{s^{\beta/\alpha}} T_{w^{1/\beta}}M(A)s^{-\beta/\alpha}dw\,ds \\
&= \int_0^1\int_{w^{\alpha/\beta}}^1 T_{w^{1/\beta}}M(A)s^{-\beta/\alpha}ds\,dw = \alpha(\alpha-\beta)^{-1}\int_0^1(1-w^{(\alpha-\beta)/\beta})T_{w^{1/\beta}}(A)dw \\
&= \alpha(\alpha-\beta)^{-1}M^{(\beta)}(A) - \alpha(\alpha-\beta)^{-1}\int_0^1 T_{w^{1/\beta}}(A)w^{\alpha/\beta-1}dw \\
&= \alpha(\alpha-\beta)^{-1}M^{(\beta)}(A) - \beta(\alpha-\beta)^{-1}M^{(\alpha)}(A) = \beta(\beta-\alpha)^{-1}M^{(\alpha)} - \alpha(\beta-\alpha)^{-1}M^{(\beta)},
\end{aligned}
$$

which gives the equality **(iii)**. Thus the proof of Proposition 3 is completed.

Proof of Corollary 2. Replacing ν by $\nu^{*1/\beta}$ in Corollary 1 and using the commutativity of the mappings $\mathcal{J}^\alpha$ and $\mathcal{J}^\beta$ (Lemma 2 c)), we get

$$
\mathcal{J}^\beta\big(\mathcal{J}^\alpha\big(\nu^{*(1-\alpha/\beta)}\big) * \nu^{*\alpha/\beta}\big) = \mathcal{J}^\alpha\big(\nu\big), \tag{23}
$$

and hence for $\rho := \mathcal{J}^\alpha\big(\nu^{(1-\alpha/\beta)}\big) * \nu^{*\,\alpha/\beta}$ we get that $\mathcal{J}^\beta(\rho) \in \mathcal{U}_\alpha$. (Note that from the above we may also conclude that $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$).

Conversely, let $\mathcal{J}^\beta\big(\rho\big) = \mathcal{J}^\alpha\big(\nu\big)$, for some $\nu \in ID$. Hence the equality (23) implies that $\rho = \mathcal{J}^\alpha\big(\nu^{*(1-\alpha/\beta)}\big) * \nu^{*\alpha/\beta}$, because $\mathcal{J}^\beta$ is one-to-one mapping (Lemma 2 a)). Thus this completes the proof.

References

[1] A. Araujo i E. Gine (1980). *The central limit theorem for real and Banach valued random variables.* John Wiley & Sons, New York.

[2] T. Aoyama and M. Maejima (2007). Characterizations of subclasses otf type G distributions on $\mathbb{R}^d$ by stochastic random integral representation, *Bernoulli*, vol. 13, pp. 148-160.
[3] A. Czyżewska-Jankowska, Z. J. Jurek (2008). Random integral representation of the class L^f distributions and some related properties; *submitted.*
[4] I. S. Gradshteyn and I. M. Ryzhik (1994). *Table of integrals, series, and products*, 5th Edition, Academic Press.
[5] Z. J. Jurek (2004). The random integral representation hypothesis revisited: new classes of s-selfdecomposable laws. In: Abstract and Applied Analysis; *Proc. International Conf. ICAAA* , Hanoi, August 2002, str. 495-514. World Scientific, Hongkong.
[6] Z. J. Jurek (1988). Random Integral representation for Classes of Limit Distributions Similar to Lavy Class L_0, *Probab. Th. Fields.* 78, pp. 473-490.
[7] Z. J. Jurek and W. Vervaat (1983). An integral representation for selfdecomposable Banach space valued random variables, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 62, pp. 247-262.
[8] K. R. Parthasarathy (1967). *Probability measures on metric spaces.* Academic Press, New York and London.

Institute of Mathematics
University of Wrocław
Pl.Grunwaldzki 2/4
50-384 Wrocław, Poland
e-mail: zjjurek@math.uni.wroc.pl or czyzew@math.uni.wroc.pl
www.math.uni.wroc.pl/~zjjurek