ON A METHOD OF INTRODUCING FREE-INFINITELY DIVISIBLE PROBABILITY MEASURES[1]

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Demonstratio Math. vol. **49**, No 2, (2016), pp. 235-251.

Abstract. Random integral mappings $I^{h,r}_{(a,b]}$ give isomorphism between the sub-semigroups of the classical $(ID, *)$ and the free-infinite divisible $(ID, \boxplus)$ probability measures. This allows us to introduce new examples of such measures, more precisely their corresponding characteristic functionals.

Mathematics Subject Classifications(2010): Primary 60E07, 60H05, 60B11; Secondary 44A05, 60H05, 60B10.

Key words and phrases: Classical infinite divisibility; free infinite divisibility Lévy-Khintchine formula; Nevanlinna-Pick formula; characteristic (Fourier) functional; Voiculescu transform.

Abbreviated title: Free-infinitely divisible measures

In this paper we prove Fourier type characterizations for new classes of subsets of infinitely divisible laws in the free probability theory. One of them (Proposition 1) is the free analog of the class $\mathcal{U}$. The class $\mathcal{U}$ originally was defined as the class of weak limits for sequences of the form:

$$U_{r_n}(X_1) + U_{r_n}(X_2) + ... + U_{r_n}(X_n) + x_n \Rightarrow \mu, \qquad (\star)$$

where random variables (X_n) are stochastically independent, the above summands are infinitesimal, (x_n) are real numbers and *the non-linear shrinking deformations* $U_r, r > 0$, are defined as follows;

$$U_r(0) := 0, \quad U_r(x) := \max\{|x| - r, 0\}\, \frac{x}{|x|}, \text{ for } x \neq 0.$$

Probability measures μ in $(\star)$ are called *s-selfdecomosable* ("s", because of the shrinking operators U_r.) They were introduced for Hilbert space valued random variables in Jurek (1977) with complete proofs given in Jurek (1981); cf. p. 46 (accepted for publication on Nov. 29, 1977). Among others, characteristic functions of s-selfdecomposable distributions were found via Choquet's Theorem on extreme points. Later on, in Jurek (1984) and (1985)

[1]Research funded by Narodowe Centrum Nauki (NCN), grant no Dec2011/01/B/ST1/01257.

s-selfdecomposable distributions were described by random integrals of the form (4) and (5) given below, from which one gets characterizations in terms of characteristic functions.

More recently, in Bradley and Jurek (2015), the Gaussian limit in ($\star$) was proved in a case when the independence was replaced by some strong mixing conditions. On other hand, in Arizmendi and Hasebe (2014), Section 5, free analog of measures from the class $\mathcal{U}$, have defined them via the unimodality property of their Lévy spectral measures; comp. Jurek (1984).

Replacing in ($\star$) the $U_r's$ by the linear dilations $T_r(x) := r\,x$ we get the Lévy class L of so called *selfdecomposable* distributions. In particular, we obtain *stable distributions*, when $X_i's$ are also identically distributed.

For purposes in this paper we need the descriptions of the classes L and $\mathcal{U}$ in terms the random integral representations; cf. Jurek and Vervaat (1983) and Theorem 2.1 in Jurek (1984), respectively.

0. The isomorphism. Traditionally, let ϕ_μ denotes the Fourier transform (the characteristic function) of a probability measure μ and let V_ν denotes the Voiculescu transform of a probability measure ν (the definition is given in the subsection **1.2.** below). Then for a class $\mathcal{C}$ of classical $*$ - infinitely divisible probability measures we define its free $\boxplus$ -infinitely divisible counter part $\mathfrak{C}$ as follows:

$$\mathfrak{C} = \{\nu : V_\nu(it) = it^2 \int_0^\infty \log \phi_\mu(-v) e^{-tv} dv,\ t > 0;\ \text{ for some } \mu \in \mathcal{C}\} \quad (1)$$

Conversely, for a class $\mathfrak{C}$ of $\boxplus$ -infinitely divisible measures we define its classical $*$ - infinitely divisible counterpart $\mathcal{C}$ as follows:

$$\mathcal{C} = \{\mu : it^2 \int_0^\infty \log \phi_\mu(-v) e^{-tv} dv = V_\nu(it),\ t > 0;\ \text{ for some } \nu \in \mathfrak{C}\} \quad (2)$$

It is notably that above, and later on, we consider V_ν (and Cauchy transforms) only on the imaginary axis. Still, it is sufficient to perform the explicit inverse procedures; cf. Section **1.3.** below.

We illustrate the relation between classes $\mathcal{C}$ and $\mathfrak{C}$ (in fact, an isomorphism between their Fourier and Voiculescu transforms, respectively) via examples and will prove among others that :

ν is $\boxplus$ s-selfdecomposable if and only if for t>0

$$V_\nu(it) = \frac{a}{2} + \frac{\sigma^2}{3}\frac{1}{it} + \int_{\mathbb{R}\setminus\{0\}} \Big(\frac{(it)^2\,[\log(it - x) - \log(it)]}{x} - it - \frac{1}{2} x 1_{(|x|\le 1)}(x) \Big) M(dx)$$

for some constants $a \in \mathbb{R}, \sigma^2 \geq 0$ (variance) and a Lévy spectral measure M. The parameters a, σ^2 and the measure M correspond to $\mu = [a, \sigma^2, M]$ in (3) and (3a); for other details cf. Proposition 1.

The method (idea) of inserting the same characteristics (parameters) into different integral kernels can be traced to Jurek and Vervaat (1983), p. 254. There it was used to describe the characteristic functions of the class L of selfdecomposable distributions. Namely, the triple from the characteristic function of class L was identified with the Lévy-Khintchine formula, recalled below; ($[a, R, M]_L \longleftrightarrow [a, R, M]$). For other such examples cf. Jurek (2011), an invited talk at 10th Vilnius Conference.

Similarly, Bercovici and Pata (1999), Theorem 3.1 (note some inaccuracy there) introduced a bijection between the semigroups of classical and free infinite divisible probability measures. Namely, the pair of (b, ρ) (from (3a), in one-to-one way was inserted into the Voiculescu formula (10); see below.

In this paper, the one-to-one identification is done on the level of the Lévy exponents $\log \phi_\mu$ (cumulant transform) and the corresponding Voiculescu transforms V_ν. Thus there is one-to-one and onto correspondence between the classes $\mathcal{C}$ and $\mathfrak{C}$ in (1) and (2). It should be also emphasized that throughout the paper we use only restricted Cauchy and Voiculescu transforms.

Furthermore, we define (identify) classes of measures via transforms (characteristic and Voiculescu functions) but not as classes of limiting distributions. It seems that as of now there is no description of that free class $\mathcal{U}$ (s-selfdecomposable distributions) as a collection of limits in free analog of the scheme $(*)$. It is worthy to mention that in Arizmendi-Hasebe (2014) (already quoted above) the free class $\mathcal{U}$ was defined by using the unimodality property of Lévy spectral measures of distributions in $\mathcal{U}$.

In particular, we shall see the explicit relation between $V_{\mathbf{w}}(z) = 1/z$ and $\phi_{N(0,1)}(t) = \exp(-t^2/2)$, where $\mathbf{w}$ is the Wigner semicircle distribution and $N(0, 1)$ is the standard Gaussian measure.

1. The classical $*$- and the $\boxplus$- free infinite divisibility.

1.1. A probability measure μ is $*$-*infinitely divisible* (ID, $*$) if for each natural $n \geq 2$ there exists probability measure μ_n such that $\mu_n^{*n} = \mu$. Equivalently, its characteristic function ϕ_μ (Fourier transform) admits the following form (Lévy-Khinchine formula)

$$\log \phi_\mu(t) = ita - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{itx} - 1 - itx 1_{(|x|\leq 1)}\big) M(dx), \quad t \in \mathbb{R}, \quad (3)$$

and the triplet $a \in \mathbb{R}, \sigma^2 \geq 0$ (covariance) and a positive Borel measure M are uniquely determined by μ; in short we write $\mu = [a, \sigma^2, M]$. A sigma-finite measure M in (3) is finite on all open complements of zero and integrates $|x|^2$ in every finite neighborhood of zero. It is called *the Lévy spectral measure* of μ. By $ID_{\log}$ we denotes those $\nu \in ID$ that have finite logarithmic moments.

The formula (3) can be equivalently written as

$$\log \phi_\mu(t) = itb + \int_{\mathbb{R}} \big(e^{itx} - 1 - \frac{itx}{1+x^2}\big)\frac{1+x^2}{x^2}\rho(dx), \quad t \in \mathbb{R}, \qquad (3a)$$

where $b \in \mathbb{R}$ and ρ is finite Borel measure such that $\rho(\{0\}\}) = \sigma^2$. Of course, parameters b and ρ are uniquely expressed by the triple from (3).

In recent year has been considerable interest in studying *random integral representations* of infinitely divisible probability measures or their Lévy measures M; cf. Jurek (2011), (2012) and references therein. Namely, for a continuous h and a monotone right continuous r, on an interval $(a,b]$, one defines

$$I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\big(\int_{(a,b]} h(s)dY_\nu(r(s))\big), \qquad (4)$$

where $\mathcal{L}(X)$ denotes the probability distribution of X and Y_ν is a cadlag Lévy process such that $\mathcal{L}(Y_\nu(1)) = \nu$.

In terms of characteristic functions $(\widehat{(...)})$ (4) means that

$$(\widehat{I^{h,r}_{(a,b]}(\nu)})(t) = \exp \int_{(a,b]} \log \phi_\nu(\pm\, h(s)\, t)(\pm) dr(t), \quad t \in \mathbb{R}, \qquad (5)$$

where the minus sign is for decreasing r and plus for increasing r; cf. Jurek and Vervaat (1983), Lemma 1.1 or Jurek (2007) (in the proof of Theorem 1) or Jurek (2012). Moreover, $(\widehat{I^{h,r}_{(a,b]}(\nu)})$ denotes here the characteristic function of the probability measure $I^{h,r}_{(a,b]}(\nu)$. In (5), we may write $\phi_\nu(-w) = \phi_{\nu^-}(w), w \in \mathbb{R}$, where ν^- is the reflected measure, that is, $\nu^-(B) := \nu(-B)$ for all Borel sets B.

For the purposes below we consider the following specific random integral mapping:

$$(ID, *) \ni \mu \to \mathcal{K}(\mu) \equiv I^{s, 1-e^{-s}}_{(0,\infty)}(\mu) = \mathcal{L}\big(\int_0^\infty s\, dY_\mu(1-e^{-s})\big) \in \mathcal{E}, \qquad (6)$$

from Jurek (2007), formula (17); ($\mathcal{E} := \mathcal{K}(ID)$). There, it was done in a generality of any real separable Hilbert space.

(In Barndorff-Nielsen and Thorbjornsen (2006), and in other works, the mapping (6) was denoted by the Greek letter Υ and (originally) was defined on the family of Lévy measures on a real line).

The mapping $\mathcal{K}$ is an isomorphism between convolution semigroup ID and $\mathcal{E}$ (range of the mapping $\mathcal{K}$). Moreover, if $\mu = [a, \sigma^2, M]$ then from (3) we get

$$\phi_{\mathcal{K}(\mu)}(t) = \exp\{ita - \sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} (\frac{1}{1-itx} - 1 - itx1_{\{|x|\leq 1\}}(x)) M(dx)\}; \quad (7)$$

cf. Jurek (2006), Corollary 5. For a general theory of the calculus on random integral mappings of the form (4) cf. Jurek (2012).

1.2. D. Voiculescu and others studying so called *free-probability* introduced new binary operations on probability measures and termed them accordingly *free-convolutions*; cf. Bercovici-Voiculescu (1993) and references therein. To recall the definition of additive $\boxplus$ convolution we need some auxiliary notions.

For a measure ν, its *Cauchy transform* is given as follows

$$G_\nu(z) := \int_{\mathbb{R}} \frac{1}{z-x} \nu(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (8)$$

Furthermore, having G_ν we define $F_\nu(z) := 1/G_\nu(z)$ and then *the Voiculescu transform* as

$$V_\nu(z) := F_\nu^{-1}(z) - z, \; z \in \Gamma_{\eta,M} := \{x + iy \in \mathbb{C}^+ : |x| < \eta y, y > M\}, \quad (9)$$

where a such region (called Stolz angle) exists and the inverse function is well defined on it; cf. Bercovici and Voiculescu (1993), Proposition 5.4 and Corollary 5.5.

The functional $\nu \to V_\nu(z)$ is an analogue of the classical Fourier transform $\nu \to \phi_\nu(t), t \in \mathbb{R}$. The fundamental fact is that, for two measures ν_1 and ν_2 one has

$$V_{\nu_1}(z) + V_{\nu_2}(z) = V_{\nu_1 \boxplus \nu_2}(z),$$

for a uniquely determined probability measure, denoted as $\nu_1 \boxplus \nu_2$. This property allowed to introduce the notion of $\boxplus$ free-infinite divisibility. For this new $\boxplus$ infinite divisibility we have the following analog of the Lévy-Khintchine formula (3):

Theorem 1. *A measure ν is $\boxplus$ infinitely divisible if and only if*

$$V_\nu(z) = b + \int_{\mathbb{R}} \frac{1+sz}{z-s} \rho(ds), \qquad z \in \mathbb{C} \setminus \mathbb{R}, \quad (10)$$

for some uniquely determined real constant b and a finite Borel measure ρ.

Cf. Bercovici and Voiculescu (1993). The integral formula (10), in complex analysis, is called *the Nevanlinna-Pick* formula.

Remark 1. Note that $V_\nu : \mathbb{C}^+ \to \mathbb{C}^-$ is an analytic function and for the mappings $T_c x := c\,x, x \in \mathbb{R}$, $(c > 0)$ and the image measure $T_c\nu$ we have $V_{T_c\nu}(z) = c\,V_\nu(c^{-1}z)$, for z in appropriate Stolz angle; cf. Bercovici-Voiculescu (1993) or Barndorff-Nielsen (2006), Lemma 4.20. This is in contrast to characteristic functions of measures where we have $\phi_{T_c\mu}(t) = \phi_\mu(ct)$, for all $t \in \mathbb{R}$.

1.3. For some analogies and comparison below, let us recall from Jurek (2006) that the restricted versions of G_ν and V_ν are just those functions considered only on the imaginary axis. Then we have that

$$\frac{1}{it}G_\nu(\frac{1}{it}) = \int_{\mathbb{R}} \frac{1}{1-itx}\,\nu(dx) = \phi_{e\cdot\eta}(t), \quad t \neq 0, \ \text{and} \ \lim_{t\to 0}\frac{1}{it}G_\nu(\frac{1}{it}) = 1 \quad (11)$$

where $e \cdot \eta$ means the probability distribution of a product of two stochastically independent rv's: the standard exponential e and the variable η with probability distribution ν.

The identity (11) means that we can retrieve a measure ν from the characteristic function $\phi_{e\cdot\eta}$; cf. Jurek (2006), the proof of Theorem 1, on p.189 and Examples on pp. 195-198. This is in sharp contrast with the classical Stieltjes inversion formula where one needs to know G_ν in strips of complex plane; see it, for instance, in Nica and Speicher (2006), p. 31.

In fact, we have even more straightforward relation. Namely,

$$\int_0^\infty \phi_\nu(s)e^{-ts}ds = \overline{iG_\nu(it)}, \quad \text{for} \quad t > 0, \qquad (12)$$

cf. Jankowski and Jurek (2012), Proposition 1. Thus, restricted Cauchy transforms are just Laplace transforms of characteristic functions.

In the spirit of (11) and (12), instead of (10), let us introduce *the restricted Voiculescu transform* as

$$V_\nu(it) \equiv k_{b,\rho}(it) := b + \int_{\mathbb{R}} \frac{1 + it\,s}{it - s}\,\rho(ds), \quad t \neq 0; \qquad (13)$$

(for a free infinitely divisible measure ν). From the inversion formula in Theorem 1 in Jankowski and Jurek (2012) and from (13), we have that

$$b = \Re k_{b,\rho}(i); \qquad \rho(\mathbb{R}) := -\Im k_{b,\rho}(i); \quad \text{and for the measure } \rho \text{ we have}$$
$$\int_0^\infty \phi_\rho(r)e^{-w\,r}dr = \frac{i\,k_{b,\rho}(-iw) - i\,\Re k_{b,\rho}(i) - w\Im k_{b,\rho}(i)}{w^2 - 1}, \quad w > 0. \quad (14)$$

(See there also the comment (a paradigm) in the first paragraph in the introduction on p. 298.)

In order to have (13) in a form more explicitly related to (7), let us define the new triple: shift a, the variance σ^2 and the Lévy spectral measure M, as follows:

$$\sigma^2 := \rho(\{0\}); \qquad \rho(dx) := \frac{x^2}{1+x^2} M(dx), \quad \text{on } \mathbb{R}\setminus\{0\};$$
$$a := b + \int_{\mathbb{R}} s\ \big(1_{|x|\leq 1}(s) - \frac{1}{1+s^2}\big)\ \rho(ds); \tag{15}$$

Then from (13), with some calculations, we get

$$it\, k_{b,\nu}(\frac{1}{it}) = ia\,t - \sigma^2 t^2 + \int_{\mathbb{R}} \big(\frac{1}{1-sit} - 1 - its\, 1_{|s|\leq 1}\big)\, M(ds), \text{ for } t<0; \tag{16}$$

For computational details cf. Barndorff-Nielsen and Thorbjornsen (2006), Proposition 4.16, p. 105

1.4. The functions $G_\nu(z)$ and $V_\nu(z)$ are analytic in some complex domains and thus are uniquely determined by their values on imaginary axis (more generally, on subsets with limiting points in their domains).
[Note that for $p(z) := i\Im z$ and $q(z) := z$ we have that $p(it) = q(it)$ although they are different. Of course, p is not an analytic function!]

1.5. Because of (6), (7) and (16), here is the explicit relation (an isomorphism) between the free- $\boxplus$ and the classical - $*$ infinite divisibility:

Theorem 2. *A probability measure ν is $\boxplus$-infinitely divisible if and only if there exist a unique $*$-infinitely divisible probability measure μ such that*

$$(it)\, V_\nu((it)^{-1}) = \log\big(\widehat{I^{s,\,1-e^{-s}}_{(0,\infty)}(\mu)}\big)(t) =$$
$$\log\Big(\mathcal{L}\big(\int_0^\infty s\, dY_\mu(1-e^{-s})\big)\Big)^{\widehat{}}(t) = \int_0^\infty \log\phi_\mu(ts)e^{-s}ds, \quad \textit{for} \ \ t<0, \tag{17}$$

where $(Y_\mu(u), u\geq 0)$ is a cadlag Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu$.

Equivalently, we have that for $\boxplus$-infinitely divisible ν its Voiculescu transform V_ν is of the form

$$V_\nu(it) = it\int_0^\infty \log\phi_\mu(-t^{-1}s)e^{-s}ds = it^2\int_0^\infty \log\phi_\mu(-v)e^{-tv}dv, \ t>0; \tag{18}$$

for a uniquely determined $*$- infinitely divisible measure μ.

This is a rephrased version of Corollary 6 in Jurek (2007). Also cf. O.E. Barndorff-Nielsen and S. Thorbjornsen (2004).

Statements (17) and (18) are equivalent as one can be deducted from the other. Moreover, they provide easy way of computing examples (classes) of free $\boxplus$-infinitely divisible measures from their counterparts in $(ID, *)$; cf. Propositions 1- 4 below.

Also note that in both cases (17) and (18) we have Laplace transform of functions $\log \phi_\mu(-t)$ of $*$-infinitely divisible measures μ.

Remark 2. From the first line in (17) we see that measures from $(ID, \boxplus)$ can be identified with measures from the semigroup $\mathcal{E} = I_{(0,\infty)}^{s,1-e^{-s}}(ID)$.

2. Examples of explicit relations between free $\boxplus$ - and classical $*$ - infinite divisible Voiculescu and Fourier transforms (probability measures).

In the following subsections probability measures are indentified by their corresponding Voiculescu V_ν and Fourier ϕ_μ transforms, respectively. In principle, inverting them one can get measures ν and μ, as is in Example 2 and Remark 6.

In the first three subsections, for a given class $\mathcal{C}$ classical $*$-infinitely divisible Fourier transforms (probability measures) we identify its counterpart $\mathfrak{C}$ of free $\boxplus$ - infinitely divisible Voiculescu transforms (probability measures).

2.1. For the free $\boxplus$ analog of s-selfdecomposable distributions we have

Proposition 1. *A probability distribution* $\mathfrak{u}$ *is free* $\boxplus$ s-selfdecomposable*, in symbols,* $\mathfrak{u} \in (\mathcal{U}, \boxplus)$*, if and only it there exist a unique* $\mu = [a, \sigma^2, M] \in (ID, *)$ *such that its Voilculescu transform has representation*

$$(a) \quad it\, V_{\mathfrak{u}}(\frac{1}{it}) = \log\big(\widehat{I_{(0,\infty)}^{v,r_u(v)}(\mu^-)}\big)(t), \quad \textit{with } r_u(v) := e^{-v} - v\Gamma(0,v); \qquad (19)$$

$$(b) V_{\mathfrak{u}}(it) = \frac{a}{2} + \frac{\sigma^2}{3}\frac{1}{it} + \int_{\mathbb{R}\setminus\{0\}} \Big(\frac{(it)^2\,[\log(it-x) - \log(it)]}{x} - it - \frac{1}{2}x 1_{(|x|\le 1)}(x)\Big) M(dx)$$

for $t > 0$.

(c) For $z \in \mathbb{C}^+$ *we have*

$$V_{\mathfrak{u}}(z) = \frac{a}{2} + \frac{\sigma^2}{3}\frac{1}{z} + \int_{\mathbb{R}\setminus\{0\}} \Big(z^2 \frac{[\log(z-x) - \log(z)]}{x} - z - \frac{1}{2}x 1_{(|x|\le 1)}(x)\Big) M(dx).$$

Above $\Gamma(0;x) := \int_x^\infty \frac{e^{-s}}{s} ds, x > 0$*, is the incomplete Euler gamma function.*

Proof. Recall that λ is classical $*$-s-selfdecomposable, i.e., $\lambda \in (\mathcal{U}, *)$ if and only if $\lambda = I^{s,s}_{(0,1]}(\mu)$ for some $\mu \in ID$; cf. Jurek (1984), Theorem 2.1; for different characterizations of this class cf. Jurek (1985). Thus for $w \neq 0$, using (3) and (5), we get

$$\log \phi_\lambda(w) = \log \big(\widehat{I^{s,s}_{(0,1]}(\mu)}\big)(w) = \int_0^1 \log \phi_\mu(w\,u)du$$
$$= i\frac{1}{2}aw - \frac{1}{6}\,\sigma^2 w^2 + \int_{\mathbb{R}\setminus\{0\}} [\frac{e^{iwu}-1}{iwu} - 1 - i\frac{1}{2}wu\,1_{|u|\leq 1}(u)]\,M(du). \quad (20)$$

[The formula (20) is as in Corollary 7.1 in Jurek (1984). However, it was obtained there by using Choquet's Theorem on extreme points in a subset of all Lévy spectral measures.]

From (17) and the first line in (20) we get

$$it\,V_{\mathfrak{u}}(\frac{1}{it}) = \int_0^\infty \log\phi_\lambda(t\,s)e^{-s}ds = \int_0^\infty\int_0^1 \log\phi_\mu(tus)e^{-s}duds, \quad (v := su)$$
$$= \int_0^\infty \int_0^s \log\phi_\mu(tv)\frac{e^{-s}}{s}dv\,ds = \int_0^\infty \log\phi_\mu(tv)\Gamma(0,v)dv. \quad (21)$$

Let us define the (decreasing) time change $r_u(v)$ for $v > 0$ as follows

$$r_u(v) := \int_v^\infty \Gamma(0,w)dw = \int_v^\infty \int_w^\infty \frac{e^{-s}}{s}ds\,dw$$
$$= \int_v^\infty \int_v^s \frac{e^{-s}}{s}dw\,ds = \int_v^\infty \frac{e^{-s}}{s}(s-v)ds = e^{-v} - v\Gamma(0,v).$$

Then taking into account (5) and putting r_u into (21) to get

$$it\,V_{\mathfrak{u}}(\frac{1}{it}) = \int_0^\infty \log\phi_{\mu^-}(-tv)(1)dr_u(v) = \log(\widehat{I^{v,\,r_u(v)}_{(0,\infty)}(\mu^-)})(t)$$

and this completes the proof of the part (a).

For part (b), using (17), (3) and the first line in (20), after interchanging the order of integration, we get for $t < 0$,

$$it\,V_{\mathfrak{u}}(\frac{1}{it}) = \int_0^\infty \log\phi_\lambda(t\,s)e^{-s}ds = \int_0^1\int_0^\infty \log\phi_\mu(tus)e^{-s}dsdu$$
$$= \frac{1}{2}iat - \frac{1}{3}\sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}}\int_0^1[\int_0^\infty \big(e^{itusx} - 1 - itusx1_{(|x|\leq 1)}(x)\big)e^{-s}ds]duM(dx)$$
$$= \frac{1}{2}iat - \frac{1}{3}\sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}}[\int_0^1 \frac{itux}{1-itux}du - \frac{1}{2}itx1_{(|x|\leq 1)}(x)]M(dx)$$
$$= \frac{1}{2}iat - \frac{1}{3}\sigma^2t^2 + \int_{\mathbb{R}\setminus\{0\}}[-\frac{\log(1-itx)}{itx} - 1 - \frac{1}{2}itx1_{(|x|\leq 1)}(x)]M(dx). \quad (22)$$

Substituting $-1/t$ for t in (22) we arrive at

$$V_{\mathfrak{u}}(it)=\frac{a}{2}+\frac{\sigma^2}{3}\frac{1}{it}+\int_{\mathbb{R}\setminus\{0\}}\Big((it)^2\frac{[\log(it-x)-\log(it)]}{x}-it-\frac{1}{2}x1_{(|x|\le 1)}(x)\Big)M(dx),$$

which gives (b). Part (c) is an analytic extension of (b) and this completes a proof of Proposition 1.

[Equality (b) can be also obtained by putting second line from (20) into (18).]

Remark 3. (i) There is the connection between the class $(\mathcal{U},\boxplus)$ and the class $I_{(0,\infty)}^{v,r_u(v)}(ID)$ via (a) in Proposition 1.

(ii) $I_{(0,\infty)}^{t,1-e^{-t}}\circ I_{(0,1]}^{s,s}=I_{(0,\infty)}^{v,r_u(v)}$ with $r_u(v):=e^{-v}-v\Gamma(0,v)$;

The above equality (ii), in terms of the random integrals (parameters r,h and $(a,b]$ and using (4) and (5)), can be written as follows. If $h_1\otimes h_2$ denotes the tensor product of functions and $\rho_1\times\rho_2$ denotes the product of measures then we have

Corollary 1. *For $h_1(t):=t\,1_{(0,1)}(t),\rho_1(dx):=1_{(0,1)}(x)dx$ and $h_2(s):=s\,1_{(0,\infty)}(s)$, and $\rho_2(dy):=e^{-y}dy$ we have that*

$$(h_1\otimes h_2)(\rho_1\times\rho_2)(dw)=1_{(0,\infty)}(w)\Gamma(0,w)dw$$

This is in fact the calculation performed in (21) starting with the first double integral; also see Jurek (2012), Theorem 2.

2.2. The analogue of free $\boxplus$ - selfdecomposable probability measures was introduced in O.E. Barndorff-Nielsen and S. Thorbjornsen (2002) via their decomposability property. Below, as in the previous subsection, free $\boxplus$ - selfdecomposable probability measures are identified via their Voiculescu transforms V_ν.

Proposition 2. *A probability distribution $\mathfrak{s}$ is* $\boxplus$-selfdecomposable, *in symbols, $\mathfrak{s}\in(L,\boxplus)$, if and only if there exist a unique $\mu=[a,\sigma^2,M]\in(ID_{\log},*)$ such that*

$$(a)\ itV_{\mathfrak{s}}(\frac{1}{it})=\log\big(\widehat{I_{(0,\infty)}^{w,\Gamma(0,w)}(\mu^-)}\big)(t),\ \textit{for } t<0;\quad \Gamma(0,w):=\int_w^\infty\frac{e^{-s}}{s}\,ds \quad (23)$$

Equivalently,

$$(b)\ \ V_{\mathfrak{s}}(it)=a+\frac{\sigma^2}{2}\frac{1}{it}+\int_{\mathbb{R}\setminus\{0\}}[it\ln\frac{it}{it-x}-x1_{\{|x|\le1\}}]M(dx),\ \ \textit{for }\ t>0. \quad (24)$$

That is, for $z\in\mathbb{C}^+$,

$$(c)\quad V_{\mathfrak{s}}(z)=a+\frac{\sigma^2}{2}\frac{1}{z}+\int_{\mathbb{R}\setminus\{0\}}[z\ln\frac{z}{z-x}-x1_{\{|x|\le1\}}]M(dx).$$

Proof. Recall that $\rho \in (L,*)$, in other words, ρ is $*$- selfdecomposable if and only if $\rho = I^{e^{-s},s}_{(0,\infty)}(\mu)$ for some $\mu \in ID_{\log}$; cf. Jurek-Vervaat (1983) or Jurek-Mason (1993), Chapter 3 (with operator $Q = I$). Hence

$$\log \phi_\rho(w) = \log \big(\widehat{I^{e^{-s},\,s}_{(0,\infty)}(\mu)}\big)(w) = \int_0^\infty \log \phi_\mu(we^{-u})du$$
$$= iaw - \frac{1}{4}\,\sigma^2 w^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{(0,1)} \frac{e^{iwrx}-1}{r}\,dr - iwx\,1_{|x|\le 1}(x)\Big)\,M(dx). \quad (25)$$

[The characterization in (25) of selfdecomposable distributions was first obtained by K. Urbanik (by the method of extreme points) and then by Jurek and Vervaat (1983), formula (4.5) on p. 255.]

From (17) and the first line in (25), for $t < 0$, we have

$$itV_{\mathfrak{s}}(\frac{1}{it}) = \int_0^\infty \log\phi_\rho(ts)e^{-s}ds = \int_0^\infty \int_0^\infty \log\phi_\mu(tse^{-u})e^{-s}duds \;(w := se^{-u})$$
$$\int_0^\infty [\int_0^s \log\phi(tw)\frac{1}{w}dw]e^{-s}ds = \int_0^\infty \log\phi(tw)\frac{1}{w}[\int_w^\infty e^{-s}ds]dw$$
$$= \int_0^\infty \log\phi_\mu(tw)\frac{e^{-w}}{w}dw = \int_0^\infty \log\phi_{\mu^-}(-tw)(-1)(\Gamma(0,w))'dw$$
$$= \log\big(\widehat{I^{w,\;\Gamma(0,w)}_{(0,\infty)}(\mu^-)}\big)(t), \quad (26)$$

(see the equality (5)), which gives (a).

Or equivalently, using the second line in (25) and (17) we get

$$itV_{\mathfrak{s}}(\frac{1}{it}) = \int_0^\infty \Big[iats - \frac{1}{4}\,\sigma^2(ts)^2$$
$$+ \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{(0,1)} \frac{e^{itsrx}-1}{r}\,dr - itsx\,1_{|x|\le 1}(x)\Big)\,M(dx)\Big]e^{-s}ds$$
$$= iat - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{(0,1)} (\frac{1}{1-itxr} - 1)\,\frac{1}{r}\,dr - itx\,1_{|x|\le 1}(x)\Big)\,M(dx)$$
$$= iat - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{(0,1)} \frac{itx}{1-itxr}dr - \;itx\,1_{|x|\le 1}(x)\Big)M(dx)$$
$$= iat - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(-\ln(1-itx) - \;itx\,1_{|x|\le 1}(x)\Big)M(dx).$$

Now substituting for $s := -1/t > 0$ w get

$$\frac{1}{is}V_{\mathfrak{s}}(is) = \frac{-ia}{s} - \frac{\sigma^2}{2s^2} + \int_{\mathbb{R}\setminus\{0\}} \Big(-\ln(1+ix/s) + \;ix/s\,1_{|x|\le 1}(x)\Big)M(dx),$$

and hence the equality (b).

Part (c) is an analytic continuation of (b) and this concludes a proof of Proposition 2.

Similarly, as in the analysis before Corollary 1, the calculations in (26), in terms of tensor product (and the parameters of random integrals), can be written as follows.

Corollary 2. *For* $h_1(t) := t,\ \rho_1(dt) := e^{-t},\ 0 < t < \infty$ *and* $h_2(s) := e^{-s};\ \rho_2(ds) := ds, 0 < s < \infty$ *then*

$$(h_1 \otimes h_2)(\rho_1 \times \rho_2)(du) = \frac{e^{-u}}{u}du$$

This follows from (26) starting from the double integral and the calculations that follows. See also Jurek (2012), Theorem 2.

Remark 4. (a) From (26) we get $I^{t,1-e^{-t}}_{(0,\infty)} \circ I^{e^{-t},t}_{(0\infty)} = I^{w,\ \Gamma(w)}_{(0,\infty)}$, on $ID_{\log}$. This is Thorin class $\mathcal{T}$; cf. Jurek (2012), Remark 5(d). Thus $\mathcal{T}$ can be identified with the class of free-selfdecomposable measures. (Compare Remarks 2 and 3(i).)

(b) For *the standard exponential measure* $\mathbf{e}$ and *standard Poisson measure* $e(\delta_1)$ we have

$$I^{w,\ \Gamma(w)}_{(0,\infty)}(e(\delta_1)) = \mathbf{e}. \tag{27}$$

To see (b) note that

$$\log \phi_{\mathbf{e}}(v) = \int_0^\infty (e^{ivx} - 1)\,\frac{e^{-x}}{x}dx, \quad \text{for } v \in \mathbb{R},$$

therefore for $e(\delta_1)$ we conclude

$$\log \big(\widehat{I^{w,\ \Gamma(w)}_{(0,\infty)}(e(\delta_1))}\big)(t) \\ = \int_0^\infty \log \phi_{e(\delta_1)}(t\,w)\frac{e^{-w}}{w}dw = \int_0^\infty (e^{itw} - 1)\,\frac{e^{-w}}{w}dw = \log \phi_{\mathbf{e}}(t),$$

which proves that (unexpected ?) relation (27) between Poisson (discrete) and exponential (continuous) distributions.

2.3. For free $\boxplus$-stable distributions we have:

Proposition 3. *A measure* ν *is non-Gaussian free-*$\boxplus$ *stable if and only if for* $t > 0$ *its Voiculescu transform* V_ν *should be such that it is EITHER*

$$V_\nu(it) = a - C\frac{\Gamma(2-p)}{1-p}\Gamma(p+1)\cos\frac{\pi p}{2}\ \big[i^{p-1}(it)^{1-p}(i - \beta\tan(\frac{\pi p}{2})\big], \tag{28}$$

where $a \in \mathbb{R}, C > 0,\ 0 < p < 1$ *or* $1 < p < 2$ *and* $|\beta| \leq 1$,

OR $p = 1$ *and*

$$V_\nu(it) = a - C\beta(1-\gamma) + \frac{C}{2}\big[2\beta \log(it) - i\pi(1+\beta)\big] \tag{29}$$

where $1 - \gamma = \int_0^\infty w \log w \, e^{-w} dw$ *(Euler constant* $\gamma \sim 0.577$*).*

Proof. For classical $*$-stable measures from Meerscheart and Scheffler (2001), Theorem 7.3.5, p. 265 we have that

μ *is non-Gaussian* $*-$*stable if and only if there exist*
$C > 0,\ a \in \mathbb{R},\ 0 < p < 1,\ 1 < p < 2,\ -1 \leq \beta \leq 1$ *such that for each* $t \in \mathbb{R}$

$$\log \phi_\mu(t) = ita - C\,\frac{\Gamma(2-p)}{1-p}\,\cos(\frac{\pi p}{2})\,|t|^p\Big(1 - i\beta\, sign(t)\tan(\frac{\pi p}{2})\Big); \tag{30}$$

and for $p = 1$ we have

$$\log \phi_\mu(t) = ita - C\,\frac{\pi}{2}\,|t|\Big(1 + i\,\beta\frac{2}{\pi} sign(t)\log|t|\Big),\ t \in \mathbb{R}, \tag{31}$$

where $\beta := 2\theta - 1$ is the skewness parameter; $0 \leq \theta \leq 1$ is the probability of the positive tail of Lévy measure M of μ, that is, for $r > 0$, we have $M(x > r) = \theta C r^{-p}$. And $1 - \theta$ is the probability of the negative tail of M.

In order to get (28) one needs insert (30) into first equality in (18) and perform some easy calculations. Similarly, putting (31) into (18) and using the identity $\log i = i\pi/2$ one gets equality (29), which completes a proof of Proposition 3.

Remark 5. (i) In some papers and books often there is a small but essential error. Namely, in (30), there is β instead of $(-\beta)$; cf. P. Hall (1981).
(ii) Note that the expressions in square brackets in (28) and (29) are identical, up to the sign, with those in Proposition 5.12 in Bercovici-Pata (1999). Also compare Biane's formulas for free-stable distributions in the Appendix there.

2.4. For a finite Borel measure m, let $e(m) := e^{-m(\mathbb{R})} \sum_{k=0}^{\infty} \frac{m^{*k}}{k!}$ denotes the $*$-compound Poisson probability measures.

Proposition 4. *A probability measure* ν *is* $\boxplus$*-compound Poison probability measure if and only if*

$$V_\nu(it) = it \int_{\mathbb{R}} \frac{x}{it - x} m(dx),\ \textit{for}\ t > 0, \tag{32}$$

for some finite Borel measure m on the real line. Moreover, ν is free-infinitely divisible if and only if

$$V_\nu(z) = b + c^2\,\frac{1}{z} + \int_{\mathbb{R}\setminus\{0\}} \frac{1}{z-x} m(dx) + z\int_{\mathbb{R}} \frac{x}{z-x} m(dx), \quad z \in \mathbb{C}\setminus\{0\} \tag{33}$$

for some $b, c \in \mathbb{R}$ and finite Borel measure m on $\mathbb{R}$.

Proof. Since $\log\phi_{e(m)}(t) = \int_{\mathbb{R}}(e^{itx}-1)m(dx),\ t\in\mathbb{R}$, therefore by (17)

$$\begin{aligned} V_\nu(it) &= it\int_0^\infty \log\phi_{e(m)}(-t^{-1}s)e^{-s}ds \\ &= it\int_{\mathbb{R}}\int_0^\infty (e^{-it^{-1}sx}-1)e^{-s}ds\, m(dx) = it\int_{\mathbb{R}} \frac{x}{it-x} m(dx), \text{ for } \quad t>0, \end{aligned}$$

which completes a proof of (32). For (33), first one takes a possible atom of m at zero (to get free Gaussian part of ν) and then splits the integrand in Theorem 1 to get free Poissonian part of ν (the second integral in (33)). [Also see pp. 203-206 in Nica-Speicher (2006) for the discussion of free compound Poisson distributions]

2.5. In this subsection, for a given three examples of $\boxplus$-infinitely divisible measures we identify their classical $*$- infinitely divisible companions. It is to illustrate the method presented in previous sections. Free analogous of those three distributions and their properties were known before.

Example 1. The probability measure $\mathbf{w}$ such that $V_{\mathbf{w}}(z) = \frac{1}{z}, z \neq 0$, is called *free-Gaussian measure*. Why such a term?

Note that from Theorem 2, we get $(it)V_{\mathbf{w}}(\frac{1}{it}) = -t^2$. On the other hand, taking standard normal distribution $N(0,1)$ for the measure μ we get

$$\int_0^\infty \log\phi_\mu(ts)e^{-s}ds = -t^2\int_0^\infty \frac{s^2}{2}e^{-s}ds = -t^2 = \frac{1}{it}V_{\mathbf{w}}(\frac{1}{it}).$$

So, it is right to call $\mathbf{w}$ an analogue of free Gaussian distribution.

More importantly, $\mathbf{w}$ is a weak limit of free-analog of CLT and $\mathbf{w}$ is *the standard Wigner's semicircle law* with the density $\frac{1}{2\pi}\sqrt{4-x^2}\,1_{[-2,2]}(x)$, mean value zero and variance 1. [Using the inversion formula in (14) for $V_{\mathbf{m}}$ we get $b=0$ and $\rho=\delta_0$ in (10).]

Example 2. The probability measure $\mathbf{c}$ with $V_{\mathbf{c}}(z) = -i$ is called *free-Cauchy distribution*. Why such a term?

From Theorem 2, $(it)V_{\mathbf{c}}(\frac{1}{it}) = t$. On the other hand, taking the standard Cauchy distribution (with the probability density $\frac{1}{\pi}\frac{1}{1+x^2}$) for the measure μ

we get

$$\int_0^\infty \log \phi_\mu(ts)e^{-s}ds = -|t| \int_0^\infty se^{-s}ds = t = (it)V_{\mathbf{c}}(\frac{1}{it}), \text{ for } t < 0,$$

so it justifies the term free - Cauchy measure. In fact, we have that

Remark 6. The measure $\mathbf{c}$ is the standard Cauchy distribution. To see that we use the inversion procedure from (14). Thus $b = 0, \mathbf{c}(\mathbb{R}) = 1$ and

$$\int_0^\infty \phi_{\mathbf{c}}(r)e^{-wr}dr = \frac{1}{w+1}; \text{ i. e., } \phi_{\mathbf{c}}(r) = e^{-r}, \text{ for } r > 0.$$

Consequently, $\phi_{\mathbf{c}}(r) = e^{-|r|}$, for $r \in \mathbb{R}$ and hence $\mathbf{c}$ is the standard Cauchy probability measure.

Example 3. The probability measure $\mathbf{m}$ such that $V_{\mathbf{m}}(z) = \frac{z}{z-1}$ is called *free-Poisson distribution.* Why?

From Theorem 2, $(it)V_{\mathbf{m}}(\frac{1}{it}) = \frac{it}{1-it} = \frac{1}{1-it} - 1$. On the other hand, if $\mu = e(\delta_1)$ is the standard Poisson distribution then $\log \phi_\mu(t) = e^{it} - 1$, and by Theorem 2,

$$\int_0^\infty \log \phi_\mu(ts)e^{-s}ds = \int_0^\infty (e^{its} - 1)e^{-s}ds = \frac{1}{1-it} - 1 = (it)V_{\mathbf{m}}(\frac{1}{it}).$$

In fact, $\mathbf{m}$ has the probability density $\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\, 1_{(0,4]}(x)$ (so called Marchenko-Pastur law); cf. Bozejko and Hasebe (2014).

Remark 7. Putting $m := \delta_1$ in Proposition 4 we retrieve the Example 3.

Acknowledgments. Author would like thank Professor Takahiro Hasebe (Japan) for his help with references to free probability theory and anonymous Referee whose comments have helped to clarify the presentation of this article.

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