Shot noise distributions and selfdecomposability

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Abstract

Stationary (limiting) distributions of shot noise processes, with exponential response functions, form a large subclass of positive selfdecomposable distributions that we illustrate by many examples. These shot noise distributions are described among selfdecomposable ones via the regular variation at zero of their distribution functions. However, slow variation at the origin of (an absolutely continuous) distribution function is incompatible with selfdecomposability and this is shown in three examples.

Key words: Shot noise processes · selfdecomposability · subordination · random integral representation · BDLP · regular variation ·

1 Introduction.

Let us recall that by *a general univariate shot noise process* we mean a stochastic process of the form

$$X(t) = \sum_{\tau_i \le t} R_i(t - \tau_i), \ t \in R, \text{ or } t \ge 0, \tag{1}$$

where τ_i's are the points of a renewal process (or of a point process) and $\{R_i(t), t \ge 0\}, i = 1, 2, ...$ is a sequence of independent identically distributed (in short: i.i.d.) measurable stochastic processes that are also independent of the renewal (point) process τ_i's. The stochastic processes $\{R_i(t), t \ge 0\}$, $i = 1, 2, ...$ are called the *response processes*.

Shot noise processes have been used to model a lot of diverse phenomena. Papers by Hsing & Teugels (1989) and Vervaat (1979) provide a variety of

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applications. For more recent results see, Doney & O'Brien (1991), Bondesson (1992) Samorodnitsky (1998), Klüppelberg et al. (2001).

Typically one discusses particular cases of the response processes. A very common class is the following one :

$$R_i(t) = \xi_i h(t), \ t \geq 0,$$

where $\{\xi_i\}$ is a sequence of independent and identically distributed random variables and $h(t), t \geq 0$, is a non-random measurable function. This allows to separate a response amplitude ξ from the dynamics of a response given by a function h. Thus we have

$$X(t) = \sum_{\tau_i \leq t} \xi_i h(t - \tau_i), \ t \in R \ \text{ or } t \geq 0. \tag{2}$$

Furthermore we will assume that in the process (2), $\{\tau_i, i \geq 1\}$ is a Poisson flow, i.e., arrival times in a Poisson process.

Our main aim here is to study the stationary distributions of some shot noise processes. In Section 2, we restrict our attention to distributions of stationary versions of (2) with the exponential response function $h(u) = \exp(-\omega u)$, where $\omega > 0$ is a fixed parameter. These still constitute a quite large subclass of the class of all positive selfdecomposable distributions, what is illustrated by many examples. Shot noise distributions, among the selfdecomposable ones are described by regular variation at zero of their distribution functions (Theorem 1). In Section 3, we show that slowly varying at zero (and absolutely continuous) distribution functions are not selfdecomposable (Theorem 2). This is illustrated by some examples. In Section 4, we present a general result on existence of the shot-noise transform , although their fixed points will be studied in a separate paper. Finally we would like to stress that our primary interest is in probabilistic (distributional) theoretical questions rather than in modelling a realistic phenomena.

To simplify some terminology and notations, we will use of the following abbreviations:

"d.f."- distribution function, "p.d.f."- probability density function, "LT"- Laplace transform, "i.i.d."- independent identically distributed, "r.v."- random variable, "ID"- infinitely divisible, "SD"- selfdecomposable or selfdecomposability, "$\stackrel{d}{=}$*"- equality in distribution, "SN"- shot noise.*

2 Exponential response and selfdecomposability.

The non-stationary SN process $X(t)$ with exponential impulse response function is given by the equality

$$X(t) := X(0)e^{-\omega t} + \int_0^t e^{-\omega(t-u)} dA_\xi(u) \stackrel{d}{=} X(0)e^{-\omega t} + \int_0^t e^{-\omega u} dA_\xi(u), \tag{3}$$

(the equality in distribution holds for each fixed t) where the Lévy process $A_\xi(t) = \sum_{k=1}^{N_\lambda(t)} \xi_k$ is the compound Poisson process and $N_\lambda(t)$ is a Poisson process with arrival times $\{\tau_i\}, i \geq 1$ and intensity $0 < \lambda < \infty$. Furthermore $\xi,\ \xi_1, \xi_2, ...$ are *positive* i.i.d. r.v., independent also of $N_\lambda(t)$; $0 < \omega < \infty$ and $X(0)$ are given non-random constants. Note that in the sequel we may drop the index ξ (jump variable) from the compound Poisson process $A_\xi(t)$.

It is known that the limit in (3) exists, as $t \to \infty$, if and only if $\mathbb{E} \ln(1 + A_\xi(1)) < \infty$. And this is equivalent to $\mathbb{E} \ln(1 + \xi) < \infty$; cf. Jurek & Vervaat (1983) or Jurek & Mason (1993), p. 122 for Banach space case. Furthermore, the limit of (3), as $t \to \infty$, is the unique stationary distribution of the SN process; cf. Jurek & Mason (1993), Proposition 3.7.10, p. 161, where there is even more general case discussed. Also cf. Sato & Yamazato (1984).

Below $F(x)$ denotes the SN distribution of $X(\infty)$ in (3). As it was mentioned above F can be used to define a stationary process

$$Y(t) = Y(0)e^{-\omega t} + \sum_{i=1}^{N_\lambda(t)} \xi_i e^{-\omega(t-\tau_i)}, \tag{4}$$

where $P\{Y(0) \leq x\} = F(x)$. However, this fact will not be discussed or used in this paper.

Recall that a probability distribution of a random variable η is called *self-decomposable* or *Lévy class L distribution* if, for every $0 < c < 1$, there exists a rv η_c, independent of η, such that the equality

$$\eta \stackrel{d}{=} c\eta + \eta_c,$$

holds true. A useful characterization of class L (or SD) distributions, crucial for this paper, is the following *random integral representation*:

the distribution of r.v. η is SD iff there exists a unique, in distribution, Lévy process Z with $E \ln(1 + |Z(1)|) < \infty$ and

$$\eta \stackrel{d}{=} \int_0^\infty e^{-u} dZ(u). \tag{5}$$

To the process Z we refer to as the *background driving Lévy process*, (in short: BDLP) of η. Cf. Jurek & Vervaat (1983) or Jurek & Mason, Chapter 3, or Sato & Yamazato (1984) or Wolfe (1982). The terminology of background driving Lévy process was introduced in Jurek (1996).

A relation between positive SD r.v. η with LT Φ and its BDLP $Z(1)$ with LT Ψ, is the following one:

$$\log \Phi(s) = \int_0^s \log \Psi(r) \frac{dr}{r}, \quad \text{and} \quad \log \Psi(s) = \frac{s\Phi'(s)}{\Phi(s)}. \tag{6}$$

cf. Jurek & Mason (1993) or Proposition 3 in Jurek (2001). Further, let us note that stationary (limiting) distributions of SN processes are SD and their BDLP's are the compound Poisson processes. This is because (4) can be rewritten as (5), with $Z(t)$ being compound Poisson processes $A_\xi(t), t \geq 0$.

Before presenting next result let us recall that a function L is called *slowly varying at 0* (in Karamata sense), if for each $\lambda > 0$, $\lim_{s\to 0} \frac{L(\lambda s)}{L(s)} = 1$. In a case, the limit is of the form λ^ρ we say that L is *regularly varying with index* ρ. For the theory of such functions we refer to Bingham, Goldie and Teugels (1987); in short: BGT(1987).

Theorem 1 *If $F(x)$ is the limiting distribution function of the SN process (3) then it is regularly varying at 0 with index λ/ω, i.e.,*

$$F(x) \sim x^{\lambda/\omega} L(x), x \to 0,$$

with $L(x)$ being slowly varying at zero. Conversely, if a positive SD r.v. η has the distribution function F regularly varying at zero, with positive index, then it is SN distribution.

Proof. From the formula (3) we see that $X(\infty)$ is SD r.v. with the compound Poisson process $A_\xi(t), t \geq 0$, as its BDLP. Consequently, from Remark 3.6.9 (4) in Jurek & Mason (1993), p. 126, we get its LT

$$\Phi(s) = E\exp(-sX(\infty)) = \exp\{-(\lambda/\omega)\int_0^s (1 - E\exp(-u\xi)du/u\}. \quad (7)$$

Cf. also Jurek & Vervaat (1983) for Banach space valued SD r.v. or Jurek (1996),(1997),(2001). Now let us note that (7) can be rewritten as follows

$$\Phi(s) = s^{-\lambda/\omega}\exp[\lambda/\omega(\log s - \int_0^s (1 - E\exp(-u\xi))du/u)] =$$

$$s^{-\lambda/\omega}\Big(\exp[-\lambda/\omega\int_0^1 (1-E\exp(-u\xi))du/u]\Big)\exp[\lambda/\omega\int_1^s E(\exp(-u\xi))du/u].$$

By comparing the above with the Representation Theorem (Thm 1.3.1 in BGT (1987), p. 12) one observes that the LT $\Phi(s)$ of $X(\infty)$ is regularly varying at infinity with index $(-\lambda/\omega)$. Consequently, by Karamata's Tauberian Theorem (Thm 1.7.1, in BGT(1987) on p. 38), one gets that the corresponding d.f. F is regularly varying at zero with the index λ/ω, which proves the first part of the theorem.

Conversely, suppose that $\eta > 0$ is SD r.v., its LT $\Phi(s)$ is given by (6) and that the d.f. $P\{\eta \leq x\}$ is regularly varying at zero with index $\rho > 0$, while its BDLP Z(t) is not a compound Poisson process. Thus Z(1) has an infinite spectral measure N and

$$\lim_{s\to+\infty}(-\ln\Psi(s)) = \lim_{s\to+\infty}\int_0^\infty (1 - e^{-sx})N(dx) = \infty,$$

by the Lebesgue monotone convergence theorem . Note that the first equality is just the representation of the LT for positive ID distribution (the drift is zero because of regular variation assumption). On the other hand, from (6), the Tauberian and the Monotone Density Theorems (cf. BGT(1987) Thm 1.7.1 and 1.7.2) we conclude

$$\lim_{s\to+\infty} \ln\Psi(s) = \lim_{s\to+\infty} \frac{s\Phi'(s)}{\Phi(s)} = -\rho,$$

which is a contradiction and thus completes the proof.

[The above result, in Ukrainian, is given in Iksanov(2001) but without a detailed proof.]

Corollary 2.1 *A positive SD rv with support* $(0,\infty)$ *and the Lévy spectral measure* $M(dx) = k(x)/xdx$ *is SN distributions if and only if* $k(0+) < \infty$.

Proof. From Vervaat(1979) we know that $X(\infty)$ has Lévy spectral measure of the form

$$M(dx) = (\lambda/\omega)P\{\xi > x\}dx/x.$$

This and Theorem 1 gives the proof.

Remark. Also in Vervaat (1979) we have that

$$X(\infty) \stackrel{d}{=} R^{\omega/\lambda}(X(\infty) + \xi)$$

where rv R has the the uniform distribution on $[0,1]$. Consequently, Khintchine-Shepp criterion gives the unimodality of $X(\infty)$ with mode at zero, provided $\lambda/\omega \le 1$. However, as we now know from Yamazato's (1978) result, all SD on real line are unimodal.

Examples of SN distributions.

Before giving some explicit examples of positive SN distributions, note that formula (7) (or (6)) implies that

$$E[e^{-s\xi}] = 1 + (\omega/\lambda)s\{d\Big(\log E[e^{-sx}dF(x)]\Big)/ds\}, \tag{8}$$

which allows the identification of *the generating distribution of jumps* ξ_i, which appears in compound Poisson process (BDLP) $A_\xi(.)$.

In the examples below we put $\rho = \lambda/\omega$ as a parameter.

1) Gamma distributions or rv's denoted by $\gamma_{\rho,\beta}$, (where ρ and β are called the *shape* and β *scale parameters*, respectively;) are given via their p.d.f.

$$f_{\rho,\beta}(x) = \beta^\rho/\Gamma(\rho)x^{\rho-1}\exp(-\beta x), x > 0,$$

are SN distributions generated by the exponential distributions $P\{\xi \le x\} = 1 - \exp(-\beta x)$. Cf. Jurek (1997), p. 97, and Jurek (2000) for more details.

2) Distribution functions of the following form

$$F_{\rho,\beta}(x) = 1 - \sum_{k=0}^{\infty} (-\beta)^{-k} x^{\rho k} / \Gamma(1+\rho k), \quad x \geq 0, \ 0 < \rho \leq 1, \ \beta > 0, \tag{9}$$

are called *positive Linnik distributions.* Their LT are given by

$$\varphi_{\rho,\beta}(s) = 1/(1+\beta s^{\rho}). \tag{10}$$

Using (8) we see that the positive Linnik distribution is generated by itself. It is not hard to prove that these are the only SN distributions with such a property (Iksanov (2001)).

The distributions with LT (10) are relatively well-known because $\eta \stackrel{d}{=} Z(\varepsilon)$, where η is the positive Linnik distribution with LT (10) and $Z(t)$ is a positive strictly ρ-stable Lévy process independent of ε that is an exponentially distributed r.v. with the scale parameter $1/\beta$. (A *'stochastic'* way of proving SD of the above η is presented in Jurek (2001), p. 244 .) Also note that the explicit form (9) was given in Pillai (1990) and Jayakumar and Pillai (1996). But let us add the following

Corollary 2.2 *For positive Linnik distributions with* $0 < \rho < 1$ *we have that*

$$\lim_{x\to\infty} \frac{1 - F_{\rho,\beta}(x)}{F_{\rho,\beta}(1/x)} = \beta^2 \Gamma(1+\rho)/\Gamma(1-\rho).$$

Proof. Apply Corollary 8.1.7 from BGT(1987), p. 334, to the the upper tail of F, and the Karamata Tauberian Theorem 1.7.1 in BGT(1987), p. 37, for the lower tail to conclude the proof.

3) *A generalized positive Linnik distribution* is given by the following LT

$$\varphi_{\rho,\rho_1,\beta}(s) = 1/(1+\beta s^{\rho_1})^{\rho/\rho_1}, 0 < \rho_1 \leq 1, \ \rho, \beta > 0.$$

Using (8) one notes it is generated by a positive Linnik distribution with parameters ρ_1, β.

Also, as before, a generalized positive Linnik distributions (with varying parameter ρ) can be realized as a law of the subordinated Lévy process arising from a positive strictly stable process and gamma process. (Comp. Jurek (2001), p. 244).

4) When one randomizes the shape parameter in the gamma distribution $\gamma_{k+\rho+1,1}$, with k distributed as Poisson rv, one gets the so called *Bessel distribution*; cf. Feller (1966), p. 58. Explicitly, it is the distribution with p.d.f. given by

$$f_\rho(x) = \exp(-\rho - x)(x/\rho)^{(\rho-1)/2} I_{\rho-1}(2\sqrt{\rho x}), \ x > 0, \ \rho > 0,$$

where $I_{\rho-1}(x)$ is the modified Bessel function with index $\rho - 1$. Bessel distribution is SN distribution when the generating rv ξ has gamma distribution, i.e., with d.f. $P\{\xi \le x\} = 1 - \exp(-x)(x+1)$. It follows from the fact that the LT of Bessel distribution is of the form from the Remark below with n=2.

Remark. 1). For BDLP of the form $A_{\gamma_{n,1}}(t)$, in (3), one gets that

$$E\exp(-sX(\infty)) = (1+s)^{-\rho} \prod_{i=1}^{n-1} \exp(-(\rho/i)(1-(1+s)^{-i})),$$

i.,e., the law of $X(\infty)$ is a convolution of $\gamma_{\rho,1}$ with $n-1$ compound Poisson laws. The i-th of them has intensity ρ/i and gamma distribution (with shape parameter i) as the law of its jumps.

2) For $n = 2$ the above reveals that Bessel distribution is a convolution of SD distribution (i.e. gamma distribution) and its BDLP evaluated at one . Two other such cases (SD distribution convoluted with its BDLP gives SD distribution) are given in Jurek (2001), p. 248. Both are related to some integral functionals of a Brownian motion .

In all SN distributions **1)-4)** we know explicitly the generating (the jumps) distribution ξ, while for those **5)-7)** below we do not. For the examples **5)** and **6)**, jumps ξ have d.f. with completely monotone derivatives ; cf. Bondesson (1992), p. 68 and p. 59 property (iv) for the example **5)** and p. 60 for example **6)**.

5) *Burr distributions or a generalized beta distributions of second type have d.f.* are given by

$$F_{\rho,\beta_1,\beta_2}(x) = 1 - (\beta_1/(x^\rho + \beta_1))^{\beta_2},\ x \ge 0,\ \beta_1 > 0, \beta_2 > \rho > 0.$$

These include Pareto when $\rho = 1$ and F-distribution for properly specified values of the parameters. From Theorem 1 it follows immediately that those are indeed SN distributions.

6) *Weibull distribution* is defined by d.f.

$$F_{\rho,\beta}(x) = 1 - \exp(-\beta x^\rho),\ x \ge 0, 0 < \rho \le 1,\ \beta > 0;$$

Remark. Except for the example **4)**, all the above distributions have completely monotone p.d.f.'s, whenever $\rho \le 1$. Goldie (1967) and Steutel (1970) proved that all non-negative r.v.'s whose d.f.'s have completely monotone derivatives are ID (infinitely divisible). However, they are not necessarily SD even after replacing "derivative" by "p.d.f.". As a counterexample take a distribution with p.d.f.

$$g_{a,b}(x) = \exp(-ax)(1-\exp(-x))^{b-1}/B(a,b),$$

where $b \leq 1$ and $2a + b \geq 1$; c.f. Bondesson (1992), p. 143. Similar examples are discussed in the following section.

Remark. Note that if the distribution with completely monotone p.d.f. is SD then it can realized as for $X(\infty)$, with $\rho \leq 1$. On the other hand, even for $\rho \leq 1$ the limiting laws of (3) do not necessarily have completely monotone p.d.f.'s . These is so because these p.d.f.'s can have discontinuous derivatives, as is in case of degenerated at 1 jumps ξ_i .

7). Absolute value of Cauchy rv leads to *half-Cauchy distribution with p.d.f.*

$$f_1(x) = 2(\pi(1+x^2))^{-1},\ x > 0,$$

that by Theorem 1 gives SN distribution. Comp. Diédhiou(1998), where the selfdecomposability was originally proved via an analytic argument.

Finally let us remark that all positive selfdecomposable distributions are weak limits of sequences of laws of $X(\infty)$, i.e., shot noise distributions.

3 Positive non-selfdecomposable distributions.

In this section we provide some tools for checking the non-selfdecomposabilty of a positive and absolutely continuous rv. We begin with the following.

Theorem 2 *If a positive rv η has the distribution function $F(x)$ slowly varying at zero then $F(x)$ is not selfdecomposable.*

Proof. Recall that SD distributions have LT of the from $\exp \int_0^\infty (e^{-sx}-1)\frac{k(x)}{x}dx$, with function $k(x)$ being positive and non-increasing. From Theorem 1 and Corollary 2.1., $k(0+)$ is finite iff the BDLP is compound Poisson. On the other hand, $k(0+) = \infty$ implies that the corresponding d.f. F is rapidly varying at zero (by rapidly we mean: regularly varying with index ∞). This completes the proof.

Examples of non-selfdecomposable distributions.

A). For a fixed $\epsilon > 0$, define a measure

$$M_\epsilon(dx) = (x \log^{1+\epsilon}(x))^{-1} 1_{(x>e)}(x)dx + x^{-1}1_{(0,e]}(x)dx,$$

which is a Lévy spectral measure of SD d.f., say F_ϵ; (note that $xdM_\epsilon(x)/dx$ is non-increasing function). Since the upper tail $M_\epsilon([x,\infty))$ is a slowly varying , then so is $1 - F_\epsilon(x)$, by Thm 8.2.1 in BGT(1987). Now let define rv η_ϵ whose tail is equal to LT of SD rv with d.f. F_ϵ, i.e.,

$$P(\eta_\epsilon > x) = \int_{0+}^\infty \exp(-xy)dF_\epsilon(y).$$

Applying Corollary 8.1.7 in BGT(1987) we see it is also slowly varying function. So, in view of the above Theorem 2 we conclude that η_ϵ is not SD.

Remark. Let us recall here, that p.d.f.'s which are "proportional" to LT's of SD distributions are SD; cf. Bondesson (1992), p. 28.

B). Before stating our next result let us recall that strictly stable processes "stopped" at independent positive SD rv give again SD rv.(cf. Jurek (2001), Proposition 1, or Bondesson (1992).) In other words, Lévy processes on R subordinated to a strictly stable process by a SD non-decreasing Lévy process (subordinator) is SD. Sato (2001) proved analogous result for a Brownian motion with drift as the subordinand. However, this is not true for non-Gaussian stable processes with drift. From Ramachandran (1997), Theorem 2.1, p. 302, we conclude that the Lévy process $Y_1(Y_2(t))$ is *not* SD , where $Y_2(t)$ is the gamma process and $Y_1(t)$ is the stable with the characteristic function $E\exp(izY_1(t)) = \exp t(i\mu z - b\,|z|^\alpha)$, $1 < \alpha < 2$, μ, $b > 0$. Next proposition provides similar cases.

Proposition 3.1 *If $X_1(t)$ and $X_2(t)$ are two independent non-decreasing Lévy SD processes , whose BDLP's are compound Poisson processes, then the subordinated Lévy process (subordinator) $X_3(t) = X_1(X_2(t))$ is not SD.*

Proof. It suffices to verify the case of $t = 1$. Put $\Phi_i(s) = E\exp(-sX_i(1))$, $i = 1, 2, 3$. These LT's are related as follows: $\Phi_3(s) = \Phi_2(\ln \Phi_1(s))$. From Theorem 1 we infer that there exist α_1,$\alpha_2 > 0$ such that $\Phi_i(s) \sim s^{-\alpha_i} L_i(s)$, $s \to \infty$, where $L_i(s)$ are slowly varying at infinity, $i = 1, 2$. Formula (7) with $X_1(1)$ instead of $X(\infty)$ and α_1 instead of λ/μ implies $\lim_{s\to\infty} s\Phi_1'(s)/\Phi_1(s) = -\alpha_1$. Now we can use de L'Hôpital's rule to get

$$\lim_{s\to\infty} \frac{-\ln \Phi_1(s)}{\alpha_1 \ln s} = \lim_{s\to\infty} \frac{-s\Phi_1'(s)/\Phi_1(s)}{s(\alpha_1/s)} = 1.$$

Thus $\ln \Phi_1(s)$ is a slowly varying. Condition $\lim_{s\to\infty} \ln \Phi_1(s) = -\infty$, guarantees the slow variation of $\Phi_3(s)$ as the superposition of regularly and slowly varying functions; cf. BGT (1987) Prop.1.5.7 . This with Theorem 2 completes the proof. ■

C). Consider the log-Cauchy law with p.d.f. $f(x) = (\pi x(1+\ln^2 x))^{-1}$, $x > 0$. Diédhiou (1998) points out without proof that this law is not SD. But this immediately follows from our Theorem 2.

D). Here is an example of a phenomena that *"stopping" SD Lévy process at non-infinitely divisible random time may get SD distribution.* Let $S_{1/2}(t)$ be a strictly stable subordinator with the index of stability 1/2 and let $s_{1/2,1}$, $s_{1/2,2}$ be independent copies of $S_{1/2}(1)$. Recall that the 1/2 - stable d.f. coincides with d.f. of N^{-2}, where N is a normal rv. Then the r.v. $s_{1/2,1}/s_{1/2,2} \stackrel{d}{=} S_{1/2}(s_{1/2,2}^{-1/2})$ has p.d.f. $p(x) = (\pi\sqrt{x}(x+1))^{-1}$, $x > 0$. This is p.d.f. of the beta distribution of the second kind (ratio of two independent gamma rv) and, hence, it is SD; cf. Bondesson (1992), p. 59 . On the other hand, we will verify that the r.v. $s_{1/2,2}^{-1/2}$ has LT $T_{1/2}(\lambda) := \sum_{k=0}^{\infty}(-1)^k\lambda^k(\Gamma(1+\alpha k))^{-1}$ and hence is not ID because the tail of its distribution tends to zero more rapidly than it is possible for a non-normal i.d. distribution; see Theorem 8.1.12 of BGT (1987). To this end, let $\eta_{1/2}$ be a r.v. having positive Linnik distribution with parameters 1/2, 1 and let ε be an independent exponentially distributed r.v. with scale 1. Then

$$\eta_{1/2} \stackrel{d}{=} S_{1/2}(\varepsilon) \stackrel{d}{=} s_{1/2,2}\varepsilon^2 \quad \text{or} \quad \eta_{1/2}^{1/2} \stackrel{d}{=} \varepsilon/s_{1/2,2}^{-1/2}.$$

This is equivalent to the following

$$T_{1/2}(x) = P\{\eta_{1/2}^{1/2} > x\} = \int_0^{\infty} \exp(-xy)dP\{s_{1/2,2}^{-1/2} \le x\}.$$

For other examples of similar kind cf. Kelker (1971) and Sato (2001).

4 A remark on a general shot noise transform.

To study fixed points of shot noise transforms one needs to find the corresponding domains. We will study the problem of fixed point in a separate paper, while in this section we focus on existence of some random integrals or infinite series of independent rv.

Fix probability space (Ω, $\mathcal{F}$, $\mathbb{P}$) and let $\mathcal{P}(\mathbb{R}^d) = \mathcal{P}$ denotes the set of all Borel probability measures on Euclidean space $\mathbb{R}^d$, let $A_\xi(t)$ denotes the compound Poisson process, with the jump rv ξ,and $0 < \lambda < \infty$ being the intensity parameter. Finally, let $h : (0, \infty) \to \mathbb{R}$ be a fixed measurable function and $t > 0$ fixed parameter . Now let us define *random integral mappings*

$$\mathcal{I}_h^{(t)} : \mathcal{P} \to ID \;\; \text{and} \;\; \mathcal{I}_h^{(t)}(\mu_\xi) := \mathcal{L}(\int_0^t h(s)dA_\xi(s)) = \mathcal{L}(\sum_{\tau_i \le t} h(\tau_i)\xi_i) = \nu_t \in ID,$$

where $\mu_\xi = \mathcal{L}(\xi)$. As in the beginning of Section 2, we refer to Jurek & Vervaat (1983), Lemma 1.1 or Jurek & Mason (1993), Chapter 3 , to conclude that ν_t is ID and that its *standard* Lévy-Khintchine representation has the following "triplet" : a shift $a_h^{(t)}$, a zero Gaussian part and the Lévy spectral measure $M_h^{(t)}$, and

$$a_h^{(t)} = \int_0^t \int_{\mathbb{R}^d} \frac{h(s)x}{1+h^2(x)||x||^2}\mu_\xi(dx)ds = \int_0^t \mathbb{E}\Big[\frac{h(s)\xi}{1+h^2(x)\xi^2}\Big]ds,$$
$$M_h^{(t)}(B) := \lambda\int_0^t T_{h(s)}\mu_\xi(B)ds = \lambda\int_0^t \mu_\xi(\frac{1}{h(s)}B)ds = \lambda\int_0^t \mathcal{L}(h(s)\xi)(B)ds,$$

where B is any Borel subset of $\mathbb{R}^d - \{0\}$ and t is fixed.

Note, however, because of the compound Poisson input $A_\xi(t)$, the stochastic process

$$t \to \int_0^t h(s)dA_\xi(s), \quad t \geq 0,$$

is with independent increments and has paths with bounded variation. Consequently, by Gihman and Skorohod (1975), Chapter IV, Theorem 8, p. 279, ν_t has "finite variation" Lévy-Khintchine representation $(0, 0, M_h^{(t)})$, with the integrability condition $\int_{\mathbb{R}^d}(\|x\| \wedge 1)M_h^{(t)}(dx) < \infty$, for each $t > 0$.

Proposition 4.1 *In order that* $X(\infty) := \lim_{t\to\infty} \int_0^t h(s)dA_\xi(s) = \sum_{k=1}^\infty h(\tau_k)\xi_k$ *exists in distribution (or in probability or a.s.) it is necessary and sufficient that*

$$\begin{aligned}\int_{\mathbb{R}^d}(1 \wedge ||y||)M_h^{(\infty)}(dy) &= \int_0^\infty \int_{\mathbb{R}^d}(1 \wedge h(s)||y||)\mu_\xi(dy)ds = \\ &= \int_0^\infty \mathbb{E}(1 \wedge h(s)||\xi||)ds < \infty. \qquad (11)\end{aligned}$$

Proof. Denoting by $< .,. >$ a scalar product in $\mathbb{R}^d$, we note that

$$\mathbb{E}\Big[\exp i < z, \int_0^t h(s)dA_\xi(s) > \Big] = \exp \int_0^t \log \phi_{A_\xi(1)}(h(s)z)ds =$$
$$= \exp \int_0^t \lambda \int_{\mathbb{R}^d}(e^{i<z,h(s)x>} - 1)\mu_\xi(dx)ds = \exp \int_{\mathbb{R}^d}(e^{i<z,x>} - 1)M_h^{(t)}(dx).$$

is a ch.f. of an ID distribution, without Gaussian part and a shift, and whose Lévy spectral measure $M_h^{(t)}$ satisfies the condition

$$\int_{\mathbb{R}^d}(\|x\| \wedge 1)M_h^{(t)}(dx) < \infty, \qquad (12)$$

for each $t > 0$. Since, $M_h^{(t)} = (\mathbb{P} \times Leb) \circ H^{-1}$, where $H : \Omega \times (0, \infty) \to \mathbb{R}^d$ is given by $H(w, s) := \xi(w)h(s)$ therefore the above is equivalent to

$$\int_0^t \mathbb{E}(1 \wedge h(s)||\xi||)ds < \infty, \text{ for each } t > 0.$$

Lévy spectral measures $M_h^{(t)}(.)$ are increasing in t. The limit, as $t \to \infty$, exists iff (12) is satisfied for $t = \infty$. Moreover,we infer that the limit is also without Gaussian part; cf. Araujo-Gine (1980), Corollary 5.3 in Chapter 2 or Theorem 4.7 in Chapter 3. Finally to see that all the three modes of convergence are equivalent use either Jurek-Vervaat (1983) or a theorem on convergence of infinite series of independent summands. This completes the proof.

Corollary 4.1 *An improper random integral* $\int_0^\infty h(s)dA_\xi(ds)$ *or an infinite series* $\sum_{k=1}^\infty h(\tau_k)\xi_k$ *exists a.s. (or in distribution or in probability) if and only if* $\int_0^\infty \mathbb{E}(1 \wedge h(s)||\xi||)ds < \infty$. *Furthermore, its characteristic function is of the form* $\exp \int_0^\infty \lambda \int_{\mathbb{R}^d}(e^{i<z,h(s)x>} - 1)\mu_\xi(dx)ds$.

Remark Our proof above is valid also for Hilbert and Banach space valued r.v.'s.

Examples. Below we use the condition from Corollary 4.1 but the explicit calculations are not given.

1) For $h(s) = e^{-s}$ one has that $\sum_{k=1}^{\infty} e^{-\tau_k}\xi_k$ exists iff $\mathbb{E}[log^+||\xi||] < \infty$ iff $\mathbb{E}[log(1 + ||\xi||)] < \infty$.

2) For $h(s) = s^{-\alpha}, \alpha > 1$, we $\sum_{k=1}^{\infty} \tau_k^{-\alpha}\xi_k$ exists iff $\mathbb{E}[||\xi||^{1/\alpha}] < \infty$.

3) For $h(s) = 1_{(0<s<1)}(s)$, the infinite series $\sum_{k=1}^{\infty} 1_{(0<s<1)}(\tau_k)\xi_k$ converges always.

The univariate general shot noise transform $\mathbb{T}_h$ is defined as follows

$$\mathbb{T}_h(\mathcal{L}(\xi)) = \mathcal{L}(\sum_{i=1}^{\infty} \xi_i h(\tau_i)) = \mathcal{L}(\int_0^{\infty} h(x) dA_\xi(dx)), \tag{13}$$

for ξ such that series or integral converge. We say that ξ is *a fixed point* of the shot noise transform $\mathbb{T}_h$ whenever $\mathcal{L}(\xi) = \mathbb{T}_h(\mathcal{L}(\xi))$. Those fixed points are, of course ID, and their existence and properties are studied in Iksanov and Jurek (2002).

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