On fixed points of Poisson shot noise transforms

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Abstract

Distributional fixed points of a Poisson shot noise transform (for non-negative, non-increasing and bounded by 1, response functions) are characterized. The tail behavior of fixed points is described. Typically they have either exponential moments or their tails are proportional to a power function, with exponent greater than minus one. The uniqueness of fixed points is also discussed. Finally it is proved that in most cases fixed points are absolutely continuous, apart from the possible atom at zero.

Key words: Shot noise transform · fixed points · regular variation · renewal theorem · absolute continuity· infinite divisibility · Banach Contraction Principle.

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1 Introduction and main results.

Throughout the paper $\mathcal{P}^+$ denotes the set of all Borel probability measures on non-negative half-line $\mathbb{R}^+ = [0, \infty)$ and $\delta_x \in \mathcal{P}^+$ denotes the measure concentrated at $x \geq 0$. Further, $A_{\xi,\lambda}(t) := \sum_{j=1}^{N_\lambda(t)} \xi_j, t \geq 0$, denotes a compound Poisson process. Here ξ_j's are independent and identically distributed copies of an $\mathbb{R}^+$-valued random variables ξ (random jumps) which are also independent of the standard Poisson process $N_\lambda(t), t \geq 0$, with the intensity parameter $0 < \lambda < \infty$. Finally, $\tau_i's$, $i = 1, 2, ...,$ denote *the arrival times* (the Poisson flow) in process $N_\lambda(t)$. We assume that all random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For a Borel measurable *response function* $h : (0, \infty) \to [0, \infty)$ it is easy to show that random series

$$\sum_{i=1}^{\infty} \xi_i h(\tau_i) < \infty \ \text{ almost surely iff } \int_0^{\infty} \mathbb{E}[1 \wedge h(s)\xi]ds < \infty. \tag{1}$$

This follows from the Lévy-Khintchine formula for non-negative infinitely divisible distributions. Furthermore, in (1) we also have convergence in distribution and in probability; cf. e.g. Iksanov and Jurek (2002) for even more general situation. On the domain $\mathcal{P}_h^+$ of convergence of (1), that is

$$\mathcal{P}_h^+ := \{\mu \in \mathcal{P}^+ : \int_0^\infty \int_0^\infty [1 \wedge h(s)y] ds \mu(dy) < \infty\},$$

we define *the Poisson shot noise transform* (in short: SNT) $\mathbb{T}_{h,\lambda} : \mathcal{P}_h^+ \to \mathcal{P}^+$ as follows

$$\mathbb{T}_{h,\lambda}(\mathcal{L}(\xi)) := \mathcal{L}\left(\sum_{i=1}^\infty \xi_i h(\tau_i)\right) = \mathcal{L}\left(\int_0^\infty h(s) dA_{\xi,\lambda}(s)\right), \tag{2}$$

where a parameter $\lambda > 0$ is fixed and $\mathcal{L}(\cdot)$ denotes the probability distribution of a random variable in question.
In other words, (2) is a limiting distribution of *a shot noise process* given by

$$X(t) = \sum_{\tau_i \le t} \xi_i h(t - \tau_i) = \int_0^t h(t-s) dA_{\xi,\lambda}(s) \stackrel{d}{=} \int_0^t h(s) dA_{\xi,\lambda}(s),$$

where "$\stackrel{d}{=}$" means the equality in distribution.

There are a lot of papers dealing with both theoretical and application features of the shot noise processes. Here we refer only to Vervaat (1979) and Bondesson (1992, Section 10) as they are the most appropriate references for our needs; cf. also Iksanov and Jurek (2002).

The purpose of this paper is to study *non-zero distributional fixed points* of the SNT $\mathbb{T}_{h,\lambda}$, i.e., distributions of ξ's that are invariant under the $\mathbb{T}_{h,\lambda}$. Explicitly, these are solutions to the following measure equation

$$\mu = \mathbb{T}_{h,\lambda}(\mu), \tag{3}$$

where $\mu = \mathcal{L}(\xi)$. For a given response function h, we will investigate the existence and the uniqueness of non-zero solutions to (3). Furthermore, we will show that in most cases those solutions are absolutely continuous with an exception of an atom at the origin.

In terms of the Laplace-Stieltjes transforms $\varphi_{L,h}$ of ξ (or the characteristic functions $\varphi_{C,h}$ or the moment generating functions $\varphi_{G,h}$(when exist)) we have that $\mu = \mathcal{L}(\xi)$ is a fixed point of the SNT $\mathbb{T}_{h,\lambda}$ if and only if

$$\varphi_{k,h}(s) = \exp\left(\lambda \int_0^\infty (\varphi_{k,h}(sh(u)) - 1) du\right), k = L, C, G. \tag{4}$$

In the sequel, we omit the index h and simply write $\varphi_k(s)$, $k = L, C, G$.

In what follows we will assume that the response function h is right-continuous and non-increasing. As it was shown by Vervaat (1979, p.768) (cf. also Bondesson (1992, p.155)) this assumption does not restrict the generality. Also we exclude response functions h of the form $h(u) = 1_{[0,a)}(u)$, for some $a > 0$, where $1_A(u)$ is the indicator function of set A, as they do not admit non-zero fixed points; see Lemma 3.1 below. All response functions h will be chosen from the following set

$$\begin{aligned}\mathcal{H} &= \{h : h \text{ is non-increasing, right continuous, } \ h(x) \leq 1, \\ \text{and } h &\neq 1_{[0,a)}, \text{ for some } \ a > 0\}.\end{aligned}$$

For our purposes below we will use a decomposition $\mathcal{H} = \mathcal{H}_{bs} \cup \mathcal{H}_{ubs}$, where $\mathcal{H}_{bs}$, $\mathcal{H}_{ubs}$ consist of response functions of bounded and unbounded support, respectively. Furthermore, in order to study the uniqueness of fixed points of the SNT we need to restrict the class $\mathcal{H}$ as follows

$$\begin{aligned}\mathcal{H}_{uniq} := \mathcal{H}_{bs} \cup \{h \in \mathcal{H}_{ubs} : \\ \text{if } \int_0^\infty h^\alpha(u)du &= \lambda^{-1} \text{ for some } \alpha \in (0,1] \text{ then} \\ \int_0^\infty h^{\alpha-\Delta}(u)du &< \infty \text{ for some } \Delta \in (0,\alpha)\}. \end{aligned} \tag{5}$$

Note that for $h \in \mathcal{H}_{bs}$ the condition (5) is automatically satisfied.

In order to consider fixed points with some moment conditions we introduce the following three subclasses of measures. Namely, for $1 < \delta < 2$, $m > 0$, $0 < \alpha < 1$, $\overline{s} = \overline{s}(\delta, m) > 0$ (an exact description of $\overline{s}$ is in Section 3) and for Euler Γ function, we define

$$\mathcal{P}_h^+(\delta, m) := \{\mu \in \mathcal{P}_h^+ : \int_0^\infty x\mu(dx) = m, \quad \int_0^\infty x^\delta \mu(dx) < \infty\},$$

$$\mathcal{P}_h^+(\delta, m, \overline{s}) := \{\mu \in \mathcal{P}_h^+ : \int_0^\infty x\mu(dx) = m, \int_0^\infty \exp(sx)\mu(dx) < \infty \text{ for } \ 0 < s \leq \overline{s}\},$$

$$\mathcal{U}^+(\alpha{,}m) := \{\mu \in \mathcal{P}_h^+ : \lim_{x\to\infty} x^\alpha \mu((x,\infty)) = \frac{m}{\Gamma(1-\alpha)}\}.$$

Here are the main results about 1) the existence and uniqueness and 2) absolute continuity of non-zero fixed points of the SNT.

Theorem 1.1 *Let $h \in \mathcal{H}$. Then*
a) the SNT $\mathbb{T}_{h,\lambda}$ has non-zero fixed point if and only if

$$\lambda \int_0^\infty h^\alpha(u)du = 1 \quad \textit{for some} \quad \alpha \in (0,1]. \tag{6}$$

b) If (6) holds for $\alpha = 1$ then, for each m, the SNT $\mathbb{T}_{h,\lambda}$ restricted to $\mathcal{P}_h^+(\delta, m)$ has a unique fixed point μ_*. In fact, $\mu_* \in \mathcal{P}_h^+(\delta, m, \overline{s})$ and is the weak limit of iterations $(\mathbb{T}_{h,\lambda}^n \rho)_{n\in\mathbb{N}}$, independent of the choice of $\rho \in \mathcal{P}_h^+(\delta, m)$.

c) If (6) holds for $0 < \alpha < 1$ then, for each m, the SNT $\mathbb{T}_{h,\lambda}$ restricted to $\mathcal{U}^+(\alpha, m)$ has a unique fixed point given by the equality

$$\mu_{*,\alpha}(x,\infty) = \int_0^\infty s_\alpha(xt^{-1/\alpha},\infty)\mu_*(dx), \quad x > 0, \tag{7}$$

where in (7), s_α is the strictly stable positive distribution with the index of stability α and μ_* is the fixed point for $\mathbb{T}_{h^\alpha,\lambda}$, with mean m, given in part b). Moreover, $\mu_{*,\alpha}$ is the weak limit of the iterations $(\mathbb{T}^n_{h,\lambda}\rho)_{n\in\mathbb{N}}$, independent of the choice of $\rho \in \mathcal{U}^+(\alpha,m)$, where $m = \int_0^\infty x\mu_*(dx) < \infty$.
d) If, additionally, $h \in \mathcal{H}_{uniq}$ then the SNT $\mathbb{T}_{h,\lambda}$, on $\mathcal{P}^+_h$, has no other fixed points than those described in parts b) and c).

As far as the absolute continuity of fixed points is concerned we have the following result.

Theorem 1.2 *i) The fixed points of the SNT $\mathbb{T}_{h,\lambda}$: μ_*, for $h \in \mathcal{H}_{uniq}$, and $\mu_{*,\alpha}$, given by (7), for $h \in \mathcal{H}$, are absolutely continuous except a possible atom at zero.*
ii) A fixed point has an atom at zero iff $h \in \mathcal{H}_{bs}$. Furthermore, the mass $q \in (0,1)$ of the atom is the unique solution to the equation $\exp(-\lambda a(1-x)) = x$, where $a_h \equiv a = \sup\{u > 0 : h(u) > 0\}$.
iii) Probability densities of $\mu_{*,\alpha}$'s are of the class C^∞. If $h \in \mathcal{H}_{bs}$ then the probability densities of μ_*'s, on $(0,\infty)$, are continuous and moreover continuously differentiable up to the order $[d]$, where $[d]$ denotes the integer part of d and $d \geq 1$ is such that $\lambda q \int_0^a h^{-d}(u)du < 1$.

Here are some additional comments regarding the fixed points of the SNT.

Remark 1.1 *The characteristic functions of fixed points $\mu_* \in \mathcal{P}^+_h(\delta, m, \overline{s})$ are analytic functions. Therefore, the fixed points are completely determined by their moments.*

Remark 1.2 *Let r.v.'s ξ^*, ξ^*_α and strictly α-stable process $S_\alpha(t)$ have the following distributions: $\mu_* = \mathcal{L}(\xi^*)$, $\mu_{*,\alpha} = \mathcal{L}(\xi^*_\alpha)$, $s_\alpha = \mathcal{L}(S_\alpha(1))$, respectively. Then the equality (7) can be expressed as follows : $\xi^*_\alpha \stackrel{d}{=} S_\alpha(1)\xi^{*1/\alpha} \stackrel{d}{=} S_\alpha(\xi^*)$.*

Remark 1.3 *For $h \in \mathcal{H}\backslash\mathcal{H}_{uniq}$ satisfying (6) may exist fixed points μ_α, $\alpha \in (0,1]$ which are not covered by Theorem 1.1(b,c). Although we have some doubts, our approach has not allowed us to exclude that possibility. But if they do exist, then these fixed points still would have the tail behavior described in Lemma 3.3(b). Hence by Tauberian Theorem (cf. Bingham, Goldie, Teugels (1989), Corollary 8.1.7) we would have :*
for $\alpha = 1$, $\lim_{x\to\infty} \frac{\int_0^x \mu_1(y,\infty)dy}{L(x)} = 1$; and for $\alpha \in (0,1)$, $\lim_{x\to\infty} \frac{\mu_\alpha(x,\infty)}{x^{-\alpha}l(x)} = 1$, where $L(x)$, $l(x)$ are slowly varying at ∞ and do not tend to finite limits as $x \to \infty$.

Our research on a general class $\mathcal{H}$ of response functions was motivated by some specific examples. One of them is given below and for others see Remark 2.1.

Example 1.1 *Let* $h(u) := \exp(-u)$. *Then the fixed points of the SNT exist if and only if* $\lambda \le 1$. *These are positive Linnik distributions with tails*

$$\mu_{*,\lambda}(x,\infty) = \sum_{k=0}^{\infty} (-\beta)^{-k} x^{\lambda k}/\Gamma(1+\lambda k), \quad x \ge 0, \quad \beta > 0,$$

and the Laplace-Stieltjes transforms

$$\int_0^\infty \exp(-zx)\mu_{*,\lambda}(dx) = (1+\beta z^\lambda)^{-1};$$

cf. Iksanov(2001). (For partial results cf. Bondesson(1982), Jayakumar, Pillai (1996) and Lin(2001)). Furthermore, we have $\mu_{*,\lambda} = \mathcal{L}(S_\lambda(\xi_1))$, *where* $S_\lambda(t)$ *is a strictly stable subordinator with index* λ, *and* $\mathcal{L}(\xi_1) = \mu_{*,1}$. *In other words, positive Linnik distribution is the distribution of strictly stable subordinator evaluated at exponential random time. Such distributions are usually called* geometric strictly stable laws. *Comp. Remark 1.2 and Theorem 1.1(c).*

We complete this section with a list of notations and conventions used throughout the paper.

First, here is a list of abbreviations:
"LT"("LST")-Laplace (-Stieltjes) transform, "ch.f."-characteristic function,
"m.g.f."-moment generating function, "r.v."- random variable,
"ID"- infinitely divisible, "c.m."-completely monotone, "a.s."-almost sure(ly),
"SN"- shot-noise, "(l.)r.h.s"-(left-) right-hand side.

Secondly, with a pair (λ, h), where $h \in \mathcal{H}$ and $\lambda \int_0^\infty h(u)du = 1$, we will associate:
i) fixed points μ_* of the SNT $\mathbb{T}_{h,\lambda}$ that are described in Theorem 1.1 and
ii) a probability measure ρ_h defined by

$$\rho_h(dz) := -\lambda z h^{\leftarrow}(dz),$$

where $h^{\leftarrow}$ is the right-continuous and non-increasing *generalized inverse function* of h given as follows $h^{\leftarrow}(z) := \inf\{u : h(u) < z\}$ for $z < h(0^+)$ and 0, otherwise. The measure ρ_h is concentrated on $[h(a^-), h(0^+)] \subseteq [0,1]$, where $a_h := \sup\{u > 0 : h(u) > 0\}$, and ρ_h has no atom at zero. Cf. Remark 3.1. Note that for a non-negative Borel measurable function g we have

$$\int_{h(a_h^-)}^{h(0^+)} g(u)\rho_h(du) = \lambda \int_0^\infty g(h(s))h(s)ds.$$

Finally, ϑ will denote random variables such that $\mathcal{L}(\vartheta) = \rho_h$; φ_L^* , φ_C^* , φ_G^* be the LST, ch.f. and m.g.f. of μ_*, respectively.
Similarly, for each pair (λ, h), where $h \in \mathcal{H}$ and $\lambda \int_0^\infty h^\alpha(u)du = 1$ for some $\alpha \in (0,1)$, we will write $\mu_{*,\alpha}$ for fixed points of $\mathbb{T}_{h,\lambda}$ and $\varphi_{L,\alpha}^*$, $\varphi_{C,\alpha}^*$, $\varphi_{G,\alpha}^*$ for their LST, ch.f. and m.g.f., respectively.

2 Some auxiliary results.

Let us define shot noise transforms in a more general framework than is needed in what follows and derive some of their general properties.

Consider *a multidimensional Poisson shot noise transform* $\mathbb{T}_{h,\lambda} : \mathcal{P}_h^d \to \mathcal{P}^d$, defined for fixed λ in the same manner as in (2), that is

$$\mathbb{T}_{h,\lambda}(\mathcal{L}(\xi)) = \mathcal{L}\left(\sum_{i=1}^{\infty} \xi_i h(\tau_i)\right) = \mathcal{L}\left(\int_0^{\infty} h(s) dA_{\xi,\lambda}(s)\right), \tag{8}$$

with the only difference being that $\mathcal{P}^d$ is the set of all probability measures on the Borel subsets of $\mathbb{R}^d$, $h : (0,\infty) \to \mathbb{R}$, ξ_j's are copies of an $\mathbb{R}^d$-valued random vector ξ, and

$\mathcal{P}_h^d := \{\mu \in \mathcal{P}^d : \int_{\mathbb{R}^d} \int_0^{\infty} [1 \wedge |h(s)| ||y||] ds \mu(dy) < \infty\}$.

The definition of $\mathcal{P}_h^d$ is explained in Corollary 4.1 of Iksanov and Jurek (2002), where in essence it was proved that a pair (h, ξ) is in the domain of the SNT $\mathbb{T}_{h,\lambda}$, i.e., the series or integral in (8) converge a.s. (or in probability or in distribution) if and only if $\int_0^{\infty} \mathbb{E}(1 \wedge |h(s)| ||\xi||) ds < \infty$. This implies that convergence in (8) is independent of λ and more importantly

$$\text{if } \mathbb{T}_{h,\lambda}(\mathcal{L}(\xi)) \text{ exists then } \lim_{s\to\infty} h(s) = 0, \text{ provided } \xi \text{ is not 0 a.s.}$$

Distributional fixed points of the SNT $\mathbb{T}_{h,\lambda}$ are laws "solving" an equation

$$\mu = \mathbb{T}_{h,\lambda}(\mu),$$

where $\mu = \mathcal{L}(\xi)$. Of course, $\mathcal{L}(\xi)$ is a fixed point of the SNT whenever $\mathcal{L}(c\xi)$ is, where c is an arbitrary constant.

Furthermore, if $\phi_\eta(z), z \in \mathbb{R}^d$ denotes the ch.f.(Fourier transform) of a random vector η then we have

$$\begin{aligned} \phi_{\mathbb{T}_{h,\lambda}(\mathcal{L}(\xi))}(z) &= \exp\left(\int_0^{\infty} \log \phi_{A_{\xi,\lambda}(1)}(h(s)z) ds\right) = \\ \exp\left(\lambda \int_0^{\infty} [\phi_\xi(h(s)z) - 1] ds\right) &= \exp\left(\int_0^{\infty} [\phi_\xi(h(s\lambda^{-1})z) - 1] ds\right). \end{aligned} \tag{9}$$

Hence, without loss of a generality, we could assume $\lambda = 1$ or otherwise change the scale in the response function h. However, in some places of this paper the presence of λ in the model under consideration is essential. Taking this into account and in order to preserve a unified approach everywhere the above mentioned possibility will not be used here.

¿From (9), for any $c > 0$, we have

$$(\mathcal{L}(\sum_{i=1}^{\infty} \xi_i h(\tau_i)))^{\star c} = \mathcal{L}(\int_0^{\infty} h(s) dA_{\xi,\lambda}(cs)) = \mathcal{L}(\sum_{i=1}^{\infty} \xi_i h(\tau_i/c)),$$

which means among others that (multidimensional) Poisson shot noise distributions are ID.

Further in terms of ch.f.'s we have that $\mu = \mathcal{L}(\xi)$ is a fixed point of the SNT $\mathbb{T}_{h,\lambda}$ iff

$$\phi_\xi(z) = \exp\left(\lambda \int_0^\infty [\phi_\xi(h(s)z) - 1]ds\right),$$

for all $z \in \mathbb{R}^d$. From the above it follows that fixed points μ are ID without shifts and Gaussian parts, and thus are completely characterized by their Lévy spectral measures M. In particular, if $d = 1$ then

$$M(dx) = \lambda \int_0^\infty \mu(dx/h(u), \infty)du.$$

Above we use term "shift" in the context of representation of Lévy processes with bounded variation paths; cf. Gihman and Skorohod (1975), Chapter IV.1, Theorems 7 and 8. In the general Lévy-Khintchine representation shifts are of the form

$$\lambda \int_0^\infty \int_{\mathbb{R}^d} \frac{h(s)||x||}{1 + h^2(s)||x||^2} \mu_\xi(dx)ds < \infty.$$

Some application of our criterions are given below.

Remark 2.1 *a) If $h(s) := 1_{[0,a)}(s)$, for some $a > 0$, and μ is fixed point of $\mathbb{T}_{h,\lambda}$ then $\mu = \delta_0$ (delta measure at zero) a.s., for any $\lambda > 0$. There are no non-zero fixed points (see our Lemma 3.1 for one-dimensional case).*
b) If $h(s) := e^{-s}$ then fixed points μ, or more precisely ch.f.'s $\phi_\xi(.)$ of random vectors ξ with $\mathcal{L}(\xi) = \mu$ are characterized as the solutions to the Bernoulli differential equation $xdy/dx - \lambda y^2 + \lambda y = 0$ with initial condition $y(0) = 1$ (see Example 1.1 for more details on the line).
c) If $h(s) := s^{-\alpha}, \alpha > 1$ then changing variable in (9) leads to formula $\phi_\xi(z) = \exp[-c_{\alpha,\lambda}|z|^{1/\alpha}]$, where $c_{\alpha,\lambda}$ is a positive constant. So, it would require $\mathcal{L}(\xi)$ to be stable distribution with index $1/\alpha$ but it is not in the domain of the SNT for $h(s) := s^{-\alpha}$ because $\mathbb{E}||\xi||^{1/\alpha} = \infty$; cf. Iksanov and Jurek (2002, p.12). Thus there are no fixed points .

Let us recall here that the integrability of sub-multiplicative functions of an ID random vector is equivalent to the integrability of the function in question with respect to the corresponding Lévy spectral measure. Suppose $g : \mathbb{R}^d \to [0, \infty)$ is sub-multiplicative, i.e., for some constant $c > 0$ we have $g(x + y) \leq cg(x)g(y)$, for all $x, y \in \mathbb{R}^d$, and let η be an ID random vector with the Lévy spectral measure N. Then

$$\mathbb{E}[g(\eta)] < \infty \ \text{ iff} \int_{||x||>1} g(x)N(dx) < \infty.$$

Of course 1 can be replaced by any positive number. Examples of functions g include: $g_1(x) = \exp(a||x||^p), 0 < p \leq 1, a > 0$; $g_2(x) = 2^p(2 + ||x||^p)$, $p > 0$; $g_3(x) = 2 + \log(1 + ||x||)$; cf e.g. Jurek & Mason (1993), Chapter 1, p. 35-36.

Consequently, for any $p > 0$ and any fixed point $\mu = \mathcal{L}(\xi)$ of the SNT $\mathbb{T}_{h,\lambda}$ we have

$$\mathbb{E}[||\xi||^p] < \infty \ \text{ iff } \int_{\{(s,\omega):|h(s)||\xi(\omega)||>1\}} |h(s)|^p||\xi(\omega)||^p ds\mathbb{P}(d\omega) < \infty.$$

Since $1 \wedge |h(s)|^2||\xi(\omega||^2 \leq 1 \wedge |h(s)|||\xi(\omega)|| \leq |h(s)||\xi(\omega)|| \leq 1 \vee |h(s)|^p||\xi(\omega)||^p$, for $p \geq 1$, therefore from Iksanov and Jurek (2002) we may conclude that $h \in L_1(ds)$ and $\xi \in L_1(\Omega, \mathbb{P})$ imply that series and integral in (8) converge a.s. Similarly we have $h \in L_p(ds)$ and $\xi \in L_p(\Omega, \mathbb{P})$ imply that (8) is well-defined. Furthermore, the infinite series converges in L_p-norm. In particular for $p = 1$, we have

$$\mathbb{E}\Big[\sum_{i=1}^{\infty} \xi_i h(\tau_i)\Big] = \lambda \mathbb{E}[\xi] \int_0^{\infty} h(s)ds. \tag{10}$$

We now return to the basic assumptions for this paper as they were described in Section 1, and give two more auxiliary lemmas.

Lemma 2.1 *a) If $\mathbb{E}\xi_1 < \infty$ and $\int_0^\infty h(u)du < \infty$ then the series (1) converges a.s. Moreover, $\mathbb{E}\sum_{i=1}^\infty \xi_i h(\tau_i) = \lambda\mathbb{E}\xi_1 \int_0^\infty h(u)du$.*
b) Let h be a positive bounded function such that the series (1) converges. Then $\mathbb{E}\xi_1^p < \infty$, $p > 1$, implies $\mathbb{E}\left(\sum_{i=1}^\infty \xi_i h(\tau_i)\right)^p < \infty$.

Part a) follows from the discussion above formula (10). Part b) is a partial case of Samorodnitsky (1998, Section 2).

Let us recall that a positive and measurable function L, defined in some neighborhood of zero, is said to be *slowly (or regularly with index ρ) varying at zero* if for all $y > 0$,

$$\lim_{x\to+0} \frac{L(xy)}{L(x)} = 1 \ \ (\text{or } \ y^\rho \ \text{ for a fixed real } \ \rho).$$

Here are some of basic properties of those functions that are used later on .

Lemma 2.2 *a)If L is slowly varying at zero and $\delta > 0$ then (1) $\lim_{x\to+0} x^\delta L(x) = 0$; (2) $\lim_{x\to+0} \frac{L(xy)}{L(x)} = 1$ locally uniformly in y on $(0,\infty)$.*
b)(Monotone Density Theorem). If for some $c \in R$, $\rho \geq 0$ $F(x) = \int_0^x f(u)du \sim$ $\sim cx^\rho L(x)$, $x \to +0$ where f is monotone in some right neighborhood of zero, then $f(x) \sim c\rho x^{\rho-1}L(x)$, $x \to +0$.
c) If in (b) $c\rho > 0$ then the converse assertion holds true without assumption on monotonicity.

We refer to Bingham, Goldie, Teugels (1989), where are studied behaviour at infinity. However, note that if $l(x)$ is slowly varying at infinity then $l(1/x)$ is slowly varying at zero.

3 Tail behavior of fixed points and the proof of Theorem 1.1.

It is always assumed tacitly that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is large enough to accommodate independent copies of r.v.'s and to guarantee existence of iterates of the SNT $\mathbb{T}_{h,\lambda}$.
We begin with some lemmas that explain our choice of the space $\mathcal{H}$ of response functions.

Lemma 3.1 *For $h(u) = 1_{[0,a)}(u)$, with fixed $a > 0$, and for any $\lambda > 0$, the SNT $\mathbb{T}_{h,\lambda}$ has no non-zero fixed points.*

Proof. ¿From (4), if φ_L is the LST of a fixed point then $\varphi_L(z) = \exp(-\lambda a(1 - \varphi_L(z)))$. Hence differentiation gives $[\varphi_L(z)]' =$ $= \lambda a \varphi_L(z)[\varphi_L(z)]'$ and this implies that the fixed point is δ_0. ■

Lemma 3.2 *If $h \in \mathcal{H}$ vanishes on half-line $[c, \infty)$, $c > 0$, and is positive, otherwise, and the SNT $\mathbb{T}_{h,\lambda}$ has non-zero fixed point then $\lambda c > 1$.*

Proof. Let $\mu \neq \delta_0$ with the LST φ_L be a fixed point. Then

$$\varphi_L(s) = \exp\left(-\lambda \int_0^c (1 - \varphi_L(sh(u)))du\right),$$

and from this we infer that μ is a compound Poisson distribution. Consequently, $\mu\{0\} = \gamma \in (0,1)$ and equivalently $\lim_{s \to +\infty} \varphi_L(s) = \gamma$. Taking the limits above, as $s \to \infty$, gives $\gamma = \exp(-\lambda c(1 - \gamma))$ and hence $\lambda c > 1$. ■

The following two Lemmas 3.3 and 3.4 are very crucial steps in our paper. In both of their proofs we will only consider the case of $h \in \mathcal{H}_{ubs}$, as the other one $h \in \mathcal{H}_{bs}$ can be treated similarly, and much simpler.
In Lemmas 3.3 and 3.4, $\varphi^*_{L,1}(s) := \varphi^*_L(s)$ denotes the LST of $\mu_{*,1} := \mu_*$.

Lemma 3.3 *Let $h \in \mathcal{H}$ and the SNT $\mathbb{T}_{h,\lambda}$ has a non-zero fixed point. Then*
a) $\lambda \int_0^\infty h^\alpha(u)du = 1$, for some $\alpha \in (0,1]$.
b) If $\varphi^*_{L,\alpha}(s)$ denotes the LST of a fixed point, then the function $s^{-\alpha}(1-\varphi^*_{L,\alpha}(s))$ is a slowly varying at zero.

Proof. *Part a).* Fix a response function $h \in \mathcal{H}_{ubs}$. Let μ be a non-zero fixed point of $\mathbb{T}_{h,\lambda}$ with the LST $\varphi(s)$. Put $b := h(+0)$, and let us define the function $\psi(s) := \dfrac{1 - \varphi(s)}{s}$ and a positive Borel measure θ_h on $[0,b]$ by formula $\theta_h(dz) = -\lambda z h^{\leftarrow}(dz)$; note that we cannot write μ_* or $\mu_{*,\alpha}$ and ρ_h, since we still do not know if $\lambda \int_0^\infty h^\alpha(u)du = 1$ for some $\alpha \in (0,1]$ (cf. our convention at the end of Section 1).

Since $\int_0^b \theta_h(dz) = \lambda \int_0^\infty h(u)du$, and we do not assume that h is integrable at the vicinity of infinity, we conclude that the measure θ_h is σ-finite.

¿From the equation (4), we get

$$\lim_{s\to+0}\frac{-\ln\varphi(s)}{1-\varphi(s)}=1=\lim_{s\to+0}\lambda\int_0^\infty\frac{1-\varphi(sh(z))}{1-\varphi(s)}dz=\lim_{s\to+0}\int_0^b\frac{\psi(sz)}{\psi(s)}\theta_h(dz).$$

Since

$$0<\frac{\psi(sz)}{\psi(s)}\le 1,\quad\text{for all}\quad z\ge 1,\tag{11}$$

then by the selection principle for any sequence $0<s_n\to 0$, as $n\to\infty$, there exists a subsequence s_{m_n} such that for $t_n:=s_{m_n}\to 0$, $n\to\infty$, and $z>1$, $\frac{\psi(t_nz)}{\psi(t_n)}$ converges to some finite limit $\Lambda(z)$, as $n\to\infty$. On the other hand, since each of $\psi(t_nz)$, for $n=1,2,...$, is c.m. function in $z\in(0,\infty)$, and this property is preserved under the limits, therefore $\Lambda(z)$ is c.m., and thus, in particular, it is continuous on $(0,\infty)$. Furthermore,

$$\lim_{n\to\infty}\frac{\psi(t_nz)}{\psi(t_n)}=\Lambda(z)\text{ everywhere on }(0,\infty).\tag{12}$$

Now by an extension of Dini's theorem (cf. Bingham, Goldie, Teugels (1989), p.55) the convergence in (12) is locally uniform on $(0,\infty)$. Also for fixed $v>0$, the convergence in $\lim_{n\to\infty}\frac{\psi(t_nvz)}{\psi(t_nv)}=\frac{\Lambda(vz)}{\Lambda(v)}$ is locally uniform in $z\in(0,\infty)$ as well. In Remark 3.1 below, we prove that $\theta_h\{0\}=0$. Thus by the local uniform convergence get

$$\lim_{n\to\infty}\int_{0+}^b\frac{\psi(t_nvz)}{\psi(t_nv)}\theta_h(dz)=\int_0^b\frac{\Lambda(vz)}{\Lambda(v)}\theta_h(dz)=1.$$

In fact, the last equality means that

$$\int_0^b\Lambda(vz)\theta_h(dz)=\Lambda(v),\quad\text{for}\quad v>0.$$

The above integral equation can be written in additive form by the change of variable. Namely, putting $\Psi(z):=\Lambda(e^{-z})$ and $\pi_h(dz):=-\theta_h(de^{-z})$ we get

$$\int_{-\ln b}^\infty\Psi(v+z)\pi_h(dz)=\Psi(v)\quad v\in\mathbb{R}.$$

¿From Lau and Rao (1982) we know that solutions to such an equation are of the form

$$\Psi(v)=p(v)e^{(1-\alpha)v},\quad\text{a.e. on }\mathbb{R};\quad p(v)=p(v+w)>0\quad\text{for all}\quad w\in supp(\pi_h),$$

where α is determined by the equation

$$1=\int_{-\ln b}^\infty e^{(1-\alpha)z}\pi_h(dz)\quad(=\int_0^b z^{\alpha-1}\theta_h(dz)=\lambda\int_0^\infty h^\alpha(u)du).\tag{13}$$

Our next aim is to show that $0 < \alpha \leq 1$. To this end let us introduce $k(v) := p(-\ln v)$. Then $\Lambda(v) = k(v)v^{\alpha-1}$ holds everywhere on $\mathbb{R}$, as Λ is a continuous function. Furthermore, since it is also non-increasing and k is periodic we have $\alpha \leq 1$.

Since $k(v) = v^{1-\alpha}\Lambda(v)$, thus it is differentiable and
$k^{'}(v) = v^{1-\alpha}((1-\alpha)v^{-1}\Lambda(v) + \Lambda^{'}(v))$.

Because of the differentiability and periodicity of $k(v)$ there exists $v_0 > 0$ such that $k^{'}(v_0) = 0$. In fact, $k^{'}(u^n v_0) = 0$, for $u \in supp(\theta_h)$ and $n = 1, 2, ...$

On the other hand, both functions $v^{-1}\Lambda(v)$ and $(-\Lambda^{'}(v))$ are positive, non-increasing and convex. Consequently, for $0 \leq \alpha \leq 1$, the equation
$(1-\alpha)v^{-1}\Lambda(v) = -\Lambda^{'}(v)$ either holds identically or has at most two solutions (graphs of the l.h.s. and the r.h.s. may either coincide or intersect at most at two points). However, the latter means that $k^{'}(v) = 0$ at most at two points, which contradicts $k^{'}(u^n v_0) = 0$ for $n = 1, 2, ...$ Thus $(1-\alpha)v^{-1}\Lambda(v) \equiv -\Lambda^{'}(v)$ which implies $k(v) = const$. If $\alpha < 0$ then k is non-increasing. Since it is also periodic thus $k = const$. Note however that α cannot be negative, as $v\Lambda(v) = kv^{\alpha}$ would be the limit of *non-decreasing* functions $\dfrac{1-\varphi(t_n v)}{1-\varphi(t_n)}$. Finally case $\alpha = 0$ is excluded by condition (13) if $h \in \mathcal{H}_{ubs}$. Similarly for h with support $[0, c)$, α can not be zero as well by Lemma 3.2. The later requires $\lambda c > 1$, but (13) gives $\lambda c = 1$. All in all we have $0 < \alpha \leq 1$, which proves the part a) of the Lemma.

Part b). Since, by (12), $\Lambda(1) = 1$ we conclude $k(v) = 1$, for $v \geq 0$, or equivalently $\Lambda(v) = v^{\alpha-1}$. Furthermore, appealing to (12) again we get

$$\lim_{n\to\infty} \frac{1-\varphi(t_n v)}{1-\varphi(t_n)} = v\Lambda(v) = v^{\alpha}, \quad \text{for all} \quad v \geq 0.$$

However, as previously below (11), the same argument can be repeated for any subsequence, therefore we conclude

$$\lim_{s\to+0} \frac{1-\varphi(sz)}{1-\varphi(s)} = z^{\alpha}, \quad \text{for all} \quad z \geq 0. \tag{14}$$

Because of our convention and already established equality (13), with $0 < \alpha \leq 1$, we may replace the LST φ by $\varphi^{*}_{L,\lambda}$ in (14). Finally, since (14) can be rewritten as

$$s^{-\alpha}(1-\varphi^{*}_{L,\alpha}(s)) \sim L(s), \quad s \to +0, \text{ where } L(s) \text{ is slowly varying at zero,}$$

we get the part b) of the Lemma.
Since very analogous reasoning can be used for $h \in \mathcal{H}_{bs}$, the proof of Lemma 3.3 is completed. ■

Remark 3.1 *To have a complete proof of Lemma 3.3 we still need to show that the measure θ_h, which appeared in the proof above, has no atom at the origin.*

Let the fixed point μ has the Lévy spectral measure M. Then M is of the form as it is above Remark 2.1. So, if we introduce an associated measure

$\overline{M}(dx) := xM(dx)$ *then*

$$\overline{M}(dx) = \int_0^b \overline{\mu}(dx/z)\theta_h(dz), \tag{15}$$

where $\overline{\mu}(dx) := x\mu(dx)$ *and* $b := h(0+)$*. Since* $\overline{M}(dx)$ *is finite in neighbourhood of* $0+$ *then* $\overline{M}\{0\} = \theta_h\{0\}\overline{\mu}[0,\infty] \in [0,\infty]$*. By Sato (1999), Theorem 51.1,* $\overline{M}\{0\}$ *is the shift in ID* μ*. However, any shot noise distribution has zero shift (cf. Section 2), therefore* $\theta_h\{0\} = 0$*.*

Here is our second key lemma for the paper. It strengthens part b) of the previous lemma under additional restriction on the response function.

Lemma 3.4 *Let* $h \in \mathcal{H}_{uniq}$ *and let assume that there exists a fixed point, with the LST* $\varphi^*_{L,\alpha}$*, for the SNT* $\mathbb{T}_{h,\lambda}$*. Then there is a constant* $0 < C = C(\alpha) < \infty$ *such that*

$$\lim_{s\to+0} s^{-\alpha}(1 - \varphi^*_{L,\alpha}(s)) = C.$$

Proof. From Lemma 3.3(a) we conclude that there exists $\alpha \in (0,1]$ such that $\lambda\int_0^\infty h^\alpha(u)du = 1$ and there exists a fixed point (probability measure) $\mu_{*,\alpha}$ of the SNT $\mathbb{T}_{h,\lambda}$ such that

$$\varphi^*_{L,\alpha}(s) = \exp\left(-\lambda\int_0^\infty (1 - \varphi^*_{L,\alpha}(sh_1^{1/\alpha}(u)))du\right), \tag{16}$$

where $h_1(u) := h^\alpha(u) \in \mathcal{H}$ and $\varphi^*_{L,\alpha}$ is the LST of $\mu_{*,\alpha}$. (Here we don't need to have the additional information that $h \in \mathcal{H}_{uniq}$). Furthermore, without the loss of generality we will assume $h_1(0+) = 1$ and to simplify notation we will write $\varphi_\alpha(s) \equiv \varphi^*_{L,\alpha}(s)$. Note that Lemma 3.3(b) gives that

$$s^{-\alpha}(1 - \varphi_\alpha(s)) \sim L(s), \text{ as } s \to +0, \tag{17}$$

where $L(s)$ is a slowly varying at 0.

In (16) we may change the order of integration with differentiation and obtain that

$$\begin{aligned} -\varphi_\alpha'(s) &= \lambda\varphi_\alpha(s)\int_0^\infty (-\varphi_\alpha'(sh_1^{1/\alpha}(u)))h_1^{1/\alpha}(u)du \\ &= \lambda\varphi_\alpha(s)\int_0^1 (-\varphi_\alpha'(sz^{1/\alpha}))z^{1/\alpha-1}\rho_{h_1}(dz) \end{aligned} \tag{18}$$

To see this, recall that the associated measure $\overline{M}_{*,\alpha}$ with (an ID distribution) $\mu_{*,\alpha}$, defined in Remark 3.1, is equal to

$$\overline{M}_{*,\alpha}(dx) = \lambda\int_0^\infty \overline{\mu}_{*,\alpha}(dx/h_1^{1/\alpha}(u))h_1^{1/\alpha}(u)du = \int_0^1 \overline{\mu}_{*,\alpha}(dx/z^{1/\alpha})z^{1/\alpha-1}\rho_{h_1}(dz),$$

where, as before, $\overline{\mu}_{*,\alpha}(dx) := x\mu_{*,\alpha}(dx)$. In terms of LST's this can be expressed as follows

$$\int_0^\infty e^{-sx}\overline{M}_{*,\alpha}(dx) = \lambda \int_0^\infty (-\varphi_\alpha^{'}(sh_1^{1/\alpha}(u)))h_1^{1/\alpha}(u)du.$$

On the other hand,

$$\int_0^\infty e^{-sx}\overline{M}_{*,\alpha}(dx) = -{\varphi_\alpha}^{'}(s)/\varphi_\alpha(s),$$

is true for any non-negative ID distribution; cf. for example, Sato(1999), p.385. Thus (18) is justified.

Let us introduce new function $K_\alpha(s) := -\alpha^{-1}s^{1-\alpha}\varphi_\alpha^{'}(s)$, for $s > 0$. Using (18) we get the functional equation

$$K_\alpha(s) = \varphi_\alpha(s)\int_0^1 K_\alpha(sz)\rho_{h_1,\alpha}(dz), \quad \text{where} \quad \rho_{h_1,\alpha}(dz) := \rho_{h_1}(dz^\alpha). \tag{19}$$

Note that if $\rho_{h_1} = \mathcal{L}(\vartheta)$ then $\rho_{h_1,\alpha} = \mathcal{L}(\vartheta^{1/\alpha})$. From (17) and Lemma 2.2(b) we get that

$$K_\alpha(s) \sim L(s), \text{ as } s \to +0 \tag{20}$$

Obviously (19) can be rewritten as

$$K_\alpha(s) = \int_0^1 K_\alpha(sz)\rho_{h_1,\alpha}(dz) - (1-\varphi_\alpha(s))\int_0^1 K_\alpha(sz)\rho_{h_1,\alpha}(dz),\ s > 0, \tag{21}$$

and then transformed into the renewal equation

$$K_\alpha(e^{-s}) = \int_{-\infty}^0 K_\alpha(e^{-s+t})\widehat{\rho}_{h_1,\alpha}(dt) - (1-\varphi_\alpha(e^{-s}))\int_{-\infty}^0 K_\alpha(e^{-s+t})\widehat{\rho}_{h_1,\alpha}(dt), \tag{22}$$

where $\widehat{\rho}_{h_1,\alpha} := \mathcal{L}(\alpha^{-1}\ln\vartheta)$. For notational simplicity we will write $\vartheta_\alpha := \vartheta^{1/\alpha}$.

Let us denote by $R_\alpha(e^{-s})$ the second summand in (22). We will show that $K_\alpha(s)$ has a finite limit at zero by considering two cases : $\widehat{\rho}_{h_1,\alpha}$ is non-arithmetic or arithmetic distribution. Furthermore from now on we assume that $h \in \mathcal{H}_{uniq}$.
CASE 1. Assume $\widehat{\rho}_{h_1,\alpha}$ is a non-arithmetic distribution.

If the function $R_\alpha(e^{-s})$ is *directly Riemann-integrable* (in short*: dRi;* cf. Feller (1966), p.348-349, for more details) then the key renewal theorem for the whole line implies that

$$\lim_{s\to\infty} K_\alpha(e^{-s}) = \frac{\int_{-\infty}^{+\infty} R_\alpha(e^{-s})ds}{\int_{-\infty}^0 t\widehat{\rho}_{h_1,\alpha}(dt)} = C_1, \tag{23}$$

for some $C_1 \geq 0$; cf. Feller(1966), Chapter XI, Theorem 1, p. 368.
Since $h \in \mathcal{H}_{uniq}$ we have additionally that

$$\int_0^\infty h^{\alpha-\Delta}(u)du < \infty \text{ for some } \Delta \in (0,\alpha). \tag{24}$$

We claim that if (24) holds then the denominator in (23) is a negative finite number, which implies that $C_1 \in (0, +\infty)$. Indeed, since $\mathcal{L}(\vartheta_\alpha) = \rho_{h_1,\alpha}$ is concentrated on $[0,1]$ and $\rho_{h_1,\alpha} \neq \delta_1$, which follows from Lemma 3.1, then the r.v. ϑ_α has finite moments of all orders, and moreover,

$$\mathbb{E}(\vartheta_\alpha^\gamma) < 1, \text{ for any } \gamma > 0. \tag{25}$$

On the other hand, for some $\Delta \in (0, \alpha)$ satisfying (24),

$$B_\Delta := \mathbb{E}\vartheta_\alpha^{-\Delta} = \int_0^1 z^{-\Delta} \rho_{h_1,\alpha}(dz) < \infty \tag{26}$$

which follows from equalities

$$\mathbb{E}(\vartheta_\alpha^{-\Delta}) = \int_0^1 z^{-\Delta/\alpha} \rho_{h_1}(dz) = -\lambda \int_0^1 z^{1-\Delta/\alpha} h_1^{\leftarrow}(dz) = \lambda \int_0^\infty h^{\alpha-\Delta}(u)du < \infty.$$

Thus, with the same choice of γ and Δ as above, by Jensen's inequality we obtain

$$-\Delta \log \mathbb{E}(\vartheta_\alpha^{-\Delta}) \leq \mathbb{E} \log \vartheta_\alpha \leq \gamma \log \mathbb{E}(\vartheta_\alpha^\gamma),$$

where the r.h.s. is negative in view of (25), and the l.h.s. is finite (negative) by (26). Hence, $\int_{-\infty}^0 t\widehat{\rho}_{h_1,\alpha}(dt) = \mathbb{E} \log \vartheta_\alpha \in (-\infty, 0)$. Thus $0 < C_1 < \infty$.

To complete Case 1, we still have to show that $R_\alpha(e^{-s})$ is *dRi*. We will do this in two steps.

Step 1. If for some $\delta_1 \in (0,1)$ and $\delta_2 > 0$,

$$-R_\alpha(s) = O(s^{1-\delta_1}),\ s \to +0\ ;\ -R_\alpha(s) = o(s^{-\delta_2}),\ s \to +\infty, \tag{27}$$

then $R_\alpha(e^{-s})$ is *dRi*.

To see this, let us note that, if the conditions (27) are fulfilled then
$-R_\alpha(e^{-s}) = O(e^{-(1-\delta_1)s})$, $s \to +\infty$ and $-R_\alpha(e^{-s}) = o(e^{\delta_2 s})$, $s \to -\infty$.
Thus, for large $|s|$, the function $-R_\alpha(e^{-s})$ is bounded by the *dRi* functions. This together with the local Riemann integrability of $R_\alpha(e^{-s})$, (which is so as it is continuous and bounded on every finite interval), implies that $-R_\alpha(e^{-s})$ is indeed *dRi* function.

Step 2. Conditions (27) holds true.
Since, by (17), $s^{-\alpha}(1-\varphi_\alpha(s))$ is slowly varying at zero, then by Lemma 2.2(a(1)) one can choose $\varepsilon_1 \in (0, \alpha)$ such that $\lim_{s \to +0} s^{-\alpha+\varepsilon_1}(1 - \varphi_\alpha(s)) = 0$. Consequently, there exist $A_1 > 0$ and $s_1 = s_1(A_1) > 0$ such that $s^{-\alpha+\varepsilon_1}(1 - \varphi_\alpha(s)) \leq A_1$, for all $s \in (0, s_1)$.

Using (26), one can choose $\varepsilon_2 > 0$ small enough that $0 < \varepsilon_1 + \varepsilon_2 < \alpha$ and $\mathbb{E}\vartheta_\alpha^{-\varepsilon_2} < \infty$. The slow variation of $K_\alpha(s)$ and Lemma 2.2(a(1)) imply that there exist $A_2 > 0$ and $s_2 = s_2(A_2) > 0$ such that $s^{\varepsilon_2} K_\alpha(s) \leq A_2$, for all $s \in (0, s_2)$. Put $s_0 := \min(s_1, s_2)$. Then

$$s^{-\alpha+\varepsilon_1}(1 - \varphi_\alpha(s)) \leq A_1, \quad s^{\varepsilon_2} K_\alpha(s) \leq A_2, \quad \text{for all} \quad s \in (0, s_0). \tag{28}$$

Now (21) and (28) imply that for all $s \in (0, s_0)$,
$-s^{-1}R_\alpha(s) = \dfrac{1}{s^{1-\alpha+\varepsilon_1+\varepsilon_2}} \dfrac{1-\varphi_\alpha(s)}{s^{\alpha-\varepsilon_1}} \int_0^1 s^{\varepsilon_2} z^{\varepsilon_2} K_\alpha(sz) z^{-\varepsilon_2} \rho_{h_1,\alpha}(dz) \le k s^{-\delta_1}$,
where $k := A_1 A_2 E \vartheta_\alpha^{-\varepsilon_2} < \infty$, $0 < \delta_1 = 1 - \alpha + \varepsilon_1 + \varepsilon_2 < 1$.

Thus we have proved the first part of (27).

As for the second part of (27) we proceed as follows. Since $\varphi_\alpha(s)$ is c.m., as the LST of a probability measure, therefore $-\varphi_\alpha^{'}(s)$ is non-increasing. Hence $s/2(-\varphi_\alpha^{'}(s/2)) \le \int_{s/2}^{s}(-\varphi_\alpha^{'}(u))du$ implies that $\lim\limits_{s\to+\infty}(-s\varphi_\alpha^{'}(s)) = 0$.

¿From this and for Δ chosen in (24) we have $\lim\limits_{s\to+\infty} s^\Delta K_\alpha(s) = 0$, which implies that there exist $A > 0$ and $s_3 = s_3(A) > 0$ such that

$$s^\Delta K_\alpha(s) < A, \quad \text{for all} \quad s > s_3. \tag{29}$$

Because of (26), we may and do introduce yet another probability measure $\rho_{h_1,\alpha,\Delta}(dz) := B_\Delta^{-1} z^{-\Delta} \rho_{h_1,\alpha}(dz)$. Then for a fixed $0 < s_4 < s_3$ we may write

$$-s^{-1}R_\alpha(s) \le B_\Delta s^{-1-\Delta} \int_0^1 s^\Delta z^\Delta K_\alpha(sz) \rho_{h_1,\alpha,\Delta}(dz) =$$

$$= \; B_\Delta s^{-1-\Delta}\left(\int_0^{s_4/s} + \int_{s_4/s}^{s_3/s} + \int_{s_3/s}^{1}\right) = B_\Delta s^{-1-\Delta}\left(I_1 + I_2 + I_3\right),$$

where each of integrals $I_k \to 0$ as $s \to +\infty$, for $k = 1, 2, 3$. This is so because :

$$I_1 = s_4^\Delta K_\alpha(s_4) \int_0^{s_4/s} \rho_{h_1,\alpha,\Delta}(dz) = o(1), \quad s \to +\infty;$$

$$I_2 = \int_{s_4/s}^{s_3/s} s^{1-\alpha+\Delta} z^{1-\alpha+\Delta} (-\varphi_\alpha^{'}(sz)) \rho_{h_1,\alpha,\Delta}(dz) \le$$

$$\le s_3^{1-\alpha+\Delta}(-\varphi_\alpha^{'}(s_4)) \int_{s_4/s}^{s_3/s} \rho_{h_1,\alpha,\Delta}(dz) = o(1), \quad s \to +\infty.$$

For I_3 we have
$$I_3 = o(1), \quad s \to +\infty$$
by the dominated convergence theorem and (29). Finally choosing $\delta_2 := \Delta$ proves the second part of (27).

All in all we have proved (23), i.e., $\lim_{s\to+0} K_\alpha(s) = C_1 \in (0, \infty)$ in Case 1.

CASE 2. Let $\widehat{\rho}_{h_1,\alpha} = \mathcal{L}(\alpha^{-1}\ln\vartheta)$ be an arithmetic distribution with the span λ. Then $\lim\limits_{s\to+0} K_\alpha(s) = C_2$, for some $C_2 \in (0, \infty)$.

Since we already have proved the function $R_\alpha(e^{-s})$, $s \in \mathbb{R}$ is dRi, then by the renewal theorem for the whole line

$$\lim_{n\to\infty} K_\alpha(e^{-h-\lambda n}) = \frac{-\lambda\sum_{k=-\infty}^{\infty} R_\alpha(e^{-h-\lambda k})}{\mathbb{E}[\alpha^{-1}\ln\vartheta]} = C_2(h) \ge 0, \tag{30}$$

for any real h; cf. Feller (1966), Chapter XI, Theorem 1, p.368.
Furthermore, the denominator is finite negative number; cf. below (24). Now since $-R_\alpha(e^{-\lambda k}) \geq 0$, and there exist $k \in Z$ such that $-R_\alpha(e^{-\lambda k}) > 0$ thus $C_2(h) \in (0, \infty)$, for any real h. In particular, $C_2 := C_2(0) \in (0, \infty)$.
By (20) and (30), we have

$$1 = \lim_{n\to\infty} \frac{K_\alpha(e^{-h-\lambda n})}{K_\alpha(e^{-\lambda n})} = \frac{C_2(h)}{C_2(0)},$$

for any $h \in \mathbb{R}$, and consequently

$$\lim_{n\to\infty} K_\alpha(e^{-h-\lambda n}) = C_2. \tag{31}$$

The convergence is locally uniform in e^{-h} on $(0, \infty)$ by Lemma 2.2(a)(part 2). To extend that convergence to any $s \to +0$, i.e., to get

$$\lim_{s\to+0} K_\alpha(s) = C_2. \tag{32}$$

we use a standard approximation. Namely, for $u < 1$ and for $s \leq e^{-\lambda}$, define $n(s) := [-\lambda^{-1} \log s]$, where $[\cdot]$ stands for the integer part. Then $e^{-\lambda(n(s)+1)} < s \leq e^{-\lambda n(s)}$ and consequently, $\dfrac{us}{e^{-\lambda n(s)}} = e^{-\lambda}\dfrac{us}{e^{-\lambda(n(s)+1)}} \in [ue^{-\lambda}, 1]$. Finally the local uniform convergence in (31) implies $\lim_{s\to+0} K_\alpha(us) = \lim_{s\to+0} K_\alpha(\dfrac{us}{e^{-\lambda n(s)}} e^{-\lambda n(s)}) = C_2$, which proves (32). This with (20) and Lemma 2.2(c) completes the proof of Lemma 3.4 in Case 2. ■

Lemma 3.4 can be expressed in terms of the tails or moments of fixed points as follows.

Corollary 3.1 *Let $h \in \mathcal{H}_{uniq}$ and $\mu_{*,\alpha}$ and μ_* are the fixed points of the SNT $\mathbb{T}_{h,\lambda}$.*

a) If $\alpha = 1$ then fixed points μ_* have finite mean.

b) If $0 < \alpha < 1$ then the fixed points $\mu_{*,\alpha}$ have tails

$$\lim_{x\to\infty} x^\alpha \mu_{*,\alpha}(x, \infty) = \frac{C}{\Gamma(1-\alpha)}, \tag{33}$$

where constant C is defined in Lemma 3.4 and Γ stands for the Euler gamma function.

Proof. For part a) recall that $\int_0^\infty x\mu_*(dx) = \lim_{s\to+0} s^{-1}(1 - \varphi_L^*(s))$, and the r.h.s. is finite by Lemma 3.4.

Similarly, Lemma 3.4 for $\alpha \in (0, 1)$ and the Tauberian Theorem (cf. Bingham, Goldie, Teugels (1989) Corollary 8.1.7), gives part b). ■

FINALLY WE ARE READY TO BEGIN THE PROOF OF Theorem 1.1.
Proof of the part a) in Theorem 1.1. The necessity part, i.e., condition (6), is proved in Lemma 3.3(a). The sufficiency of (6) follows from parts b) and c)

bellow. In both cases we provide a construction or a procedure leading to fixed points of the SNT $\mathbb{T}_{h,\lambda}$. Of course, in proofs of b) and c) we use only the necessity part of a).

Proof of the part b) in Theorem 1.1. Before starting the proof we need to introduce new spaces of probability measures. Since we are going to use the Banach Contraction Principle–a fundamental tool in proving existence of fixed points–we need to have complete metric spaces and contracting transforms on those spaces, cf. Rudin (1966), Theorem 9.23. Below are two lemmas providing those properties.

1) For fixed $1 < \delta < 2$, and $m > 0$, let us consider the set $\mathcal{P}_h^+(\delta, m)$ of probability measures defined as follows

$$\mathcal{P}_h^+(\delta, m) := \{\mu \in \mathcal{P}_h^+ : \int_0^\infty x\mu(dx) = m, \int_0^\infty x^\delta \mu(dx) < \infty\}.$$

Note that the SNT

$$\mathbb{T}_{h,\lambda} \quad : \mathcal{P}_h^+(\delta, m) \to \mathcal{P}_h^+(\delta, m),$$

(i.e., it is mapping into $\mathcal{P}_h^+(\delta, m)$) provided $h \in \mathcal{H}$ is such that $\lambda \int_0^\infty h(u)du = 1$. This is indeed true because

$$\int_0^\infty x(\mathbb{T}_{h,\lambda}\mu)(dx) = \int_0^\infty x\mu(dx) = m, \quad \text{by Lemma 2.1(a)}$$

and the part b) of the same lemma guarantees the existence of the other moments. For $\mu_1, \mu_2 \in \mathcal{P}_h^+(\delta, m)$, let us define function

$$r_\delta(\mu_1, \mu_2) := \int_0^\infty s^{-\delta-1} \left| \int_0^\infty \exp(isx)\mu_1(dx) - \int_0^\infty \exp(isx)\mu_2(dx) \right| ds.$$

¿From Lemma 3.1 in Baringhaus, Grübel (1997) we know that r_δ is a metric on $\mathcal{P}_h^+(\delta, m)$ and that $(\mathcal{P}_h^+(\delta, m), r_\delta)$ is a complete metric space.

Lemma 3.5. Let $h \in \mathcal{H}$ and $\lambda \int_0^\infty h(u)du = 1$. Then the SNT $\mathbb{T}_{h,\lambda}$, on $(P_h^+(\delta, m), r_\delta)$, is a strictly contractive mapping.

Proof. For ν_1, $\nu_2 \in \mathcal{P}_h^+(\delta, m)$ with ch.f.'s $\psi_{C,1}(s)$, $\psi_{C,2}(s)$, respectively, let us denote by $\varphi_{C,i}(s)$ the ch.f. of $\mathbb{T}_{h,\lambda}\nu_i$, $i = 1, 2$. We will show that

$$r_\delta(\mathbb{T}_{h,\lambda}\nu_1, \mathbb{T}_{h,\lambda}\nu_2) \le \left(\lambda \int_0^\infty h^\delta(u)du\right) r_\delta(\nu_1, \nu_2)$$

or equivalently that

$$\int_0^\infty s^{-\delta-1} |\varphi_{C,1}(s) - \varphi_{C,2}(s)| \, ds \le$$

$$\le \left(\lambda \int_0^\infty h^\delta(u)du\right) \int_0^\infty s^{-\delta-1} |\psi_{C,1}(s) - \psi_{C,2}(s)| \, ds.$$

Notice that for any complex, non-zero z_1, z_2

$$|z_1 - z_2| \leq |\ln z_1 - \ln z_2| \max(|z_1|, |z_2|).$$

Then for $z_1 := \varphi_{C,1}(s)$ and $z_2 := \varphi_{C,2}(s)$ and using (4) we obtain

$$\begin{aligned} |\varphi_{C,1}(s) - \varphi_{C,2}(s)| &\leq |\ln \varphi_{C,1}(s) - \ln \varphi_{C,2}(s)| \leq \\ &\leq \lambda \int_0^\infty |\psi_{C,1}(sh(u)) - \psi_{C,2}(sh(u))| \, du. \end{aligned}$$

Hence,

$$\int_0^\infty s^{-\delta-1} |\varphi_{C,1}(s) - \varphi_{C,2}(s)| \, ds$$

$$\leq \lambda \int_0^\infty \int_0^\infty s^{-\delta-1} |\psi_{C,1}(sh(u)) - \psi_{C,2}(sh(u))| \, ds \, du =$$

$$\left(\lambda \int_0^\infty h^\delta(u)du\right) \int_0^\infty z^{-\delta-1} |\psi_{C,1}(z) - \psi_{C,2}(z)| \, dz.$$

Finally note that the contracting constant is $d := \lambda \int_0^\infty h^\delta(u)du < 1$ (because $\delta > 1, h(u) \leq 1$ and $h(u_0) < 1$ for some $u_0 > 0$) and this completes the proof of the Lemma.
2) Here is the second fact related to the contraction transform.
Lemma 3.6. Assume that $h \in \mathcal{H}$ and $\lambda \int_0^\infty h(s)ds = 1$. Then for each $\mu_0 \in \mathcal{P}_h^+(\delta, m)$, the sequences of iterations $\mathbb{T}_{h,\lambda}^n(\mu_0)$, as $n \to \infty$, converge weakly to the same limit μ_*. It is the only fixed point of the SNT restricted to $\mathcal{P}_h^+(\delta, m)$. Furthermore, those fixed points have some exponential moments finite, i.e., they are in set $\mathcal{P}_h^+(\delta, m, \overline{s})$; cf. Section 1.

Proof. The first part is just restatement of the Banach Contraction Principle. We only need to show the existence of exponential moments.
For $\beta_\delta := \int_0^\infty h^\delta(u)du / \int_0^\infty h(u)du < 1$, let us define the function $f_\delta(s) := \exp\{ms + \beta_\delta s^\delta\}$. Of course, $f_\delta(s) = 1 + ms + \beta_\delta s^\delta + o(s^\delta)$, as $s \to +0$, therefore there exists $s_0 \in (0,1)$ such that for $s \in (0, s_0]$ $f_\delta(s) \leq 1 + ms + \gamma_\delta s^\delta$, whenever $\gamma_\delta \geq \beta_\delta$. Choose

$$\overline{s} := \overline{s}(\delta, m) := \sup\{s_0 \in (0,1) : f_\delta(s) \leq 1 + ms + s^\delta \quad \text{for} \quad 0 < s \leq s_0\}.$$

Hence,

$$f_\delta(s) \leq 1 + ms + s^\delta, \text{ for all } s \in (0, \overline{s}].$$

For any $\mu_0 \in \mathcal{P}_h^+(\delta, m)$ and its m.g.f. $\psi_G(s)$ we have that
$\psi_G(s) := \int_0^\infty \exp(sx)\mu_0(dx) \leq 1 + ms + s^\delta$, for $s \in (0, \overline{s}]$.

¿From (4), for $s \in (0, \overline{s}]$ we get

$$\varphi_G(s) = \int_0^\infty \exp(sx)(\mathbb{T}\mu_0)(dx) = \exp\left(\lambda \int_0^\infty (\psi_G(sh(u)) - 1)du\right) \leq$$

$$\leq \exp\left(\lambda ms \int_0^\infty h(u)du + \lambda s^\delta \int_0^\infty h^\delta(u)du\right) = f_\delta(s) \leq 1 + ms + s^\delta.$$

In the same way, as for μ_0, for all $s \in (0, \overline{s}]$,

$$\int_0^\infty \exp(sx)(\mathbb{T}^n \mu_0)(dx) \le 1 + ms + s^\delta, \; n = 1, 2, ... \tag{34}$$

¿From above and the Markov inequality

$$(\mathbb{T}^n \mu_0)(x, \infty) \le (1 + m\overline{s} + \overline{s}^\delta) \exp(-\overline{s}x),$$

for all $x > 0$ and $n = 1, 2, ...$

Since $\overline{s}$ does not depend on n, taking the limit $n \to \infty$ gives

$$\mu_*(x, \infty) \le (1 + m\overline{s} + \overline{s}^\delta) \exp(-\overline{s}x),$$

for all $x > 0$. This means that $\mu_* \in \mathcal{P}_h^+(\delta, m, \overline{s})$; cf. Section 1. This completes the proof of the Lemma. ■

Obviously the previous lemma is in fact the part b) of Theorem 1.1.

Proof of the part c) of Theorem 1.1. For $h_1(s) := h^\alpha(s)$ and $0 < \alpha < 1$, we have $\lambda \int_0^\infty h_1(s)ds = 1$ and, in view of the part b), there exists a fixed point μ_* of the SNT $\mathbb{T}_{h_1,\lambda}$ with given mean value m.

Claim 1. If r.v. ξ_* is non-negative with the distribution μ_* and is stochastically independent of an α-stable r.v. S_α then $\overline{\mu}_\alpha := \mathcal{L}(\xi_* \cdot S_\alpha)$ is a fixed point of the SNT $\mathbb{T}_{h_1,\lambda}$.

Proof of Claim 1. Let $\varphi_L^*(s)$, $s \ge 0$, denotes the LST of the fixed point μ_*. Thus from (4) we have equation

$$\varphi_L^*(s) = \exp\left(-\lambda \int_0^\infty (1 - \varphi_L^*(sh_1(u)))du\right) \text{ for all } \; s \ge 0. \tag{35}$$

Note that the LST $\overline{\varphi}_{L,\alpha}$ of $\overline{\mu}_\alpha$ is of the form

$$\overline{\varphi}_{L,\alpha}(s) := \mathbb{E}[\exp(-s(\xi_*)^{1/\alpha} S_\alpha)] = \mathbb{E}[e^{-s^\alpha \xi}] = \varphi_L^*(s^\alpha).$$

Substituting s^α for s in (35), one gets equality

$$\overline{\varphi}_{L,\alpha}(s) = \exp\left(-\lambda \int_0^\infty (1 - \overline{\varphi}_{L,\alpha}(sh(u)))du\right), \tag{36}$$

which, by (4), means that $\overline{\mu}_\alpha$ is a fixed point of the SNT $\mathbb{T}_{h,\lambda}$. This completes the proof of Claim 1.

For $0 < m < \infty$ let us define a set

$$\mathcal{U}^+(\alpha, m) = \{\mu \in \mathcal{P}_h^+ : \lim_{x\to\infty} x^\alpha \mu(x, \infty) = \frac{m}{\Gamma(1-\alpha)}\}$$

and a set of the corresponding LST's

$$\mathcal{V}(\alpha, m) = \{\varphi : \varphi(s) = \int_0^\infty e^{-sx}\mu(dx) : \lim_{s\to+0} s^{-\alpha}(1 - \varphi(s)) = m\}.$$

Claim 2. If μ_* is the fixed point of $\mathbb{T}_{h^\alpha,\lambda}$ with mean value m then $\overline{\mu}_\alpha$, constructed above, is the unique fixed point of the SNT $\mathbb{T}_{h,\lambda}$ when restricted to $\mathcal{U}^+(\alpha,m)$.
Proof of Claim 2. Suppose that there exists another fixed point $\widetilde{\mu}_\alpha \in \mathcal{U}^+(\alpha,m)$, with LST $\widetilde{\varphi}_\alpha \in \mathcal{V}(\alpha,m)$. Let us introduce $M_\alpha(s) := \dfrac{\left|\widetilde{\varphi}_\alpha(s) - \overline{\varphi}_{L,\alpha}(s)\right|}{s^\alpha}$, for $s > 0$.
Since both LST's satisfy (35) then, using the inequality $|e^{-x} - e^{-y}| \leq |x - y|$, $x, y \in \mathbb{R}$, we arrive at

$$M_\alpha(s) \leq \int_0^1 M_\alpha(sz)\rho_{h^\alpha,\alpha}(dz) \leq ... \leq \mathbb{E}M_\alpha(s\vartheta_{1,\alpha}...\vartheta_{n,\alpha}), \tag{37}$$

where $\rho_{h^\alpha,\alpha}$ is a probability measure defined as $\rho_{h^\alpha,\alpha}(dz) := \rho_{h^\alpha}(dz^\alpha)$, and $\vartheta_{1,\alpha}, \vartheta_{2,\alpha}, ...$ are independent copies of r.v. ϑ_α with $\mathcal{L}(\vartheta_\alpha) = \rho_{h^\alpha,\alpha}$. Further we proceed as in Athreya (1969), Theorem 1. Since
$M_\alpha(s) \leq |m - s^{-\alpha}(\widetilde{\varphi}_\alpha(s) - 1)| + \left|s^{-\alpha}(1 - \overline{\varphi}_{L,\alpha}(s)) - m\right|$, for $s > 0$,
and both $\widetilde{\varphi}_\alpha$ and $\overline{\varphi}_{L,\alpha}$ are in $\mathcal{V}(\alpha,m)$ we obtain that $\lim_{s\to+0} M_\alpha(s) = 0$.
Since $\mathbb{E}\log\vartheta_\alpha < 0$ we have that $\vartheta_{1,\alpha}\vartheta_{2,\alpha}...\vartheta_{n,\alpha} \overset{a.s.}{\to} 0$, as $n \to \infty$, by the strong law of large numbers. This and bounded convergence in (37) imply that $M_\alpha(s) = 0$ for $s > 0$. Consequently, $\widetilde{\varphi}_\alpha(s) \equiv \overline{\varphi}_{L,\alpha}(s)$ and the uniqueness, on $\mathcal{U}^+(\alpha,m)$, of the fixed point is proved.

Claim 3. The fixed point $\overline{\mu}_\alpha \in \mathcal{U}^+(\alpha,m)$ is a limit of the iterations $\mathbb{T}^n_{h,\lambda}(\mu^1_\alpha)$,as $n \to \infty$, for any $\mu^1_\alpha \in \mathcal{U}^+(\alpha,m)$.
Proof of Claim 3. For fixed $\mu^1_\alpha \in \mathcal{U}^+(\alpha,m)$ and its LST $\varphi_{1,\alpha} \in \mathcal{V}(\alpha,m)$, let us define $\varphi_{n+1,\alpha}$ as the LST of $\mathbb{T}^n_{h,\lambda}$, for $n = 1, 2, ...$ Thus we have relations

$$\varphi_{n+1,\alpha}(s) = \exp\left(-\lambda\int_0^\infty (1 - \varphi_{n,\alpha}(sh(u)))du\right), n = 1, 2, ...$$

For $K_{n,\alpha}(s) := -\alpha^{-1}s^{1-\alpha}\varphi'_{n,\alpha}(s)$, we have

$$K_{n+1,\alpha}(s) = \varphi_{n+1,\alpha}(s)\int_0^1 K_{n,\alpha}(sz)\rho_{h^\alpha,\alpha}(dz),\ s > 0.$$

Hence the dominated convergence theorem implies that $\varphi_{n,\alpha} \in \mathcal{V}(\alpha,m)$, for each $n = 1, 2, ...$ Our aim is to show $\lim_{n\to\infty}\varphi_{n,\alpha}(s) = \overline{\varphi}_{L,\alpha}(s)$, for $s \geq 0$. Or equivalently, for each $\varepsilon > 0$, there exists a positive integer N such that

$$\left|\varphi_{n,\alpha}(s) - \overline{\varphi}_{L,\alpha}(s)\right| < \varepsilon, \text{ for all } n \geq N,\ \ s \geq 0. \tag{38}$$

Let us put $M_{n,\alpha}(s) := s^{-\alpha}|\varphi_{n,\alpha}(s) - \overline{\varphi}_{L,\alpha}(s)|$, for $s > 0$ and $n = 1, 2, ...$ For these functions we have inequalities

$$M_{n+1,\alpha}(s) \leq \int_0^1 M_{n,\alpha}(sz)\rho_{h^\alpha,\alpha}(dz) \leq ... \leq \mathbb{E}M_{1,\alpha}(s\vartheta_{1,\alpha}...\vartheta_{n,\alpha}),\ n = 1, 2, ... \tag{39}$$

which can be proved in the same manner as the inequality (37).
Since for any $s > 0$, $M_{1,\alpha}(s) \le |m - s^{-\alpha}(\varphi_{1,\alpha}(s) - 1)| + |s^{-\alpha}(1 - \overline{\varphi}_{L,\alpha}(s)) - m|$, and both $\varphi_{1,\alpha}$ and $\overline{\varphi}_{L,\alpha}$ are in $\mathcal{V}(\alpha,m)$, we conclude $\lim_{s\to+0} M_{1,\alpha}(s) = 0$. For $\varepsilon_1 > 0$ let us choose $s_0 > 0$ such that $M_{1,\alpha}(s) < \varepsilon_1$, whenever $0 < s \le s_0$. Furthermore, $\overline{\vartheta}_{n,\alpha} := \vartheta_{1,\alpha}...\vartheta_{n,\alpha} \to 0$ a.s., as $n \to \infty$, by the strong law of large numbers. Hence, for arbitrary $\varepsilon_2 > 0$ one can choose a positive integer N such that for all $n > N$ and fixed $s > 0$, $\mathbb{P}\{s\overline{\vartheta}_{n,\alpha} > s_0\} < \varepsilon_2$. Therefore (39) can be rewritten as follows

$$M_{n+1,\alpha}(s) \le \mathbb{E}M_{1,\alpha}(s\overline{\vartheta}_{n,\alpha}) =$$
$$= \mathbb{E}[1_{\{s\overline{\vartheta}_{n,\alpha}\le s_0\}} M_{1,\alpha}(s\overline{\vartheta}_{n,\alpha})] + \mathbb{E}[1_{\{s\overline{\vartheta}_{n,\alpha}> s_0\}} M_{1,\alpha}(s\overline{\vartheta}_{n,\alpha})] <$$
$$< \varepsilon_1 + \varepsilon_2 \max_{s\ge s_0} M_{1,\alpha}(s) =: \varepsilon,$$

which proves (38) holds for $s > 0$. This completes the proof of Claim 3.
Thus the proof of part c) of Theorem 1.1 is completed as well.

Proof of the part d) of Theorem 1.1. Let $h \in \mathcal{H}_{uniq}$ and (6), in Theorem 1.1(a), holds for $\alpha = 1$. Then by Corollary 3.1(a) we know that all fixed points have finite first moment and hence are in $\mathcal{P}_h^+$ by Lemma 2.1(a). But all of such fixed points on whole $\mathcal{P}_h^+$ are of the form as described in Theorem 1.1(b).
In case $0 < \alpha < 1$ and $h \in \mathcal{H}_{uniq}$, Corollary 3.1(b) says that all fixed points have tails proportional to a power function and, moreover, are in $\mathcal{P}_h$, which easily follows from (36) and dominated convergence theorem. But those fixed points are of the form given in Theorem 1.1(c). Thus this completes the proof of part d).

4 Absolute continuity and the proof of Theorem 1.2

Let us begin with an example of the shot noise distribution $\rho := \mathbb{T}_{h,\lambda}(\mu)$ (not a fixed point) that is continuous singular.
For the response function $h(u) := 2^{-n}$, for $u \in [n, n+1)$, $n = 0, 1, ...$, and for the random jumps variable ξ with $\mathcal{L}(\xi_1) := \delta_1$, we obtain

$$\phi_\rho(z) = \exp[\lambda \sum_{n=0}^{\infty} (\exp(i2^{-n}z) - 1)],$$

which is continuous and singular, for a Poisson flow with any intensity λ; cf. Example 4.3 in Watanabe (2000). In fact, he proved that ρ^{*s} are singular continuous for all $s > 0$. Note that function h integrates to 2.

First, let us discuss the possibility of atoms at zero of the SNT's fixed points. As we know this is the case when $a := \sup\{u > 0 : h(u) > 0\} < \infty$ because the fixed points are compound Poisson distribution; cf. Lemma 3.2. Recall that atoms at zero have mass $0 < q < 1$, that is a solution to the equation $exp(-\lambda a(1 - x)) = x$, provided $\lambda a > 1$.

Secondly, we present the known fact that will be useful for investigating the absolute continuity of fixed points of the form $\mu_{*,\alpha}$.

Lemma 4.1 *Suppose that r.v.'s η_1 and η_2 are independent and such that η_1 is a non-negative r.v. with a probability distribution μ_1 and a probability density function $m_1(\cdot)$ and that η_2 is an r.v. concentrated on $[0,c]$, $0 < c < \infty$, with a probability distribution μ_2. Then $\mu := \mathcal{L}(\eta_1\eta_2)$ is of the form $\mu = p\delta_0 + m(x)1_{(0,\infty)}$, where $p := \mu_1\{0\} + \mu_2\{0\} - \mu_1\{0\}\mu_2\{0\} \in (0,1)$ and $m(x) := \int_{0+}^{c} y^{-1}m_1(xy^{-1})\mu_2(dy)$ is the density of μ on $(0,\infty)$.*
Furthermore, if $m_1^{(n)}$ (n-th derivative) is bounded and continuous then $m^{(n)}$ exists and is continuous.

Proof. Independence and conditioning on η_2 allows to write $\mu(0,x] := \mathbb{P}(0 < \eta_1\eta_2 \le x) = \mathbb{E}_{\eta_2}(\mu_1(0,x/\eta_2]) = \int_{(0,c]} \mu_1(0,x/y]\mu_2(dy)$.
Hence the probability density $m(x)$ of μ on positive half-line is equal $m(x) = \int_{(0,c]} m_1(x/y)y^{-1}\mu_2(dy)$. As for the atom at zero note that $p := \mathbb{P}(\eta_1 = 0 \ \text{or} \ \eta_2 = 0)$ ■

For the SNT's fixed points of the form μ_* we will consider two cases: 1) $h \in \mathcal{H}_{uniq}\backslash\mathcal{H}_{bs}$ and 2) $h \in \mathcal{H}_{bs}$. Although such a choice is not necessary (as both cases can be treated in a quite similar way) we decided to do so, because the second case can be derived from the result by Athreya (1969) (cf. Example 5.1 for more details).

Let $h \in \mathcal{H}_{uniq}\backslash\mathcal{H}_{bs}$, that is $h \in \mathcal{H}_{ubs}$ and $\int_0^\infty h^{1-\Delta}(u)du < \infty$, for some $\Delta \in (0,1)$. Then fixed points μ_* have finite mean by Theorem 1.1(b). Hence their Lévy spectral measures integrate function $g(x) := x$; cf. the paragraph below Remark 2.1.

As before, let $\varphi_C^*(z)$ be the ch.f. of μ_*. Our proof of Theorem 1.2 is essentially based on the following three lemmas.

Lemma 4.2 *For fixed points μ_* their ch.f.'s have property: $\limsup_{|z|\to\infty} |\varphi_C^*(z)| < 1$.*

Proof. First note that, if μ_* were a lattice distribution (i.e., a measure concentrated on points kd, $k = 0,1,...,$ where $d > 0$ is fixed span) then it must be a compound Poisson, as it is non-negative ID distribution. Consequently, its Lévy spectral measure M_* would be finite. However, using formula before Remark 2.1 and Fatou's Lemma we get

$$\liminf_{x\to 0} M_*(x,\infty) \ge \lambda \int_0^\infty \liminf_{x\to 0} \mu_*(x/h(u),\infty)du = \infty,$$

which contradicts that M_* is a finite measure.
Thus for μ_*, as non-lattice distribution, we have

$$|\varphi_C^*(z)| < 1, \quad \text{for all } z \neq 0, \tag{40}$$

because of general properties of ch.f.'s of non-lattice distributions; see e.g. Lukacs (1970), pp.19-20.

¿From the equation (4), for distributional fixed points, we get

$$|\varphi_C^*(z)| = \exp\left(-\lambda \int_0^\infty (1 - Re\ \varphi_C^*(zh(u)))du\right), \tag{41}$$

where $Re\ u$ means the real part of u.
¿From now on we follow Athreya's (1969, Lemma 6) way of reasoning. Suppose, in contrary, that

$$\limsup_{|z|\to\infty} |\varphi_C^*(z)| = 1. \tag{42}$$

For any $z_0 > 0$, (40) implies that $|\varphi_C^*(z_0)| < 1$. From (42) one has that for a fixed $\varepsilon \in (0, 1 - |\varphi_C^*(z_0)|)$ there exist $z_1 < z_0 < z_2$ such that

$$0 < Re\varphi_C^*(z) \leq |\varphi_C^*(z)| < 1 - \varepsilon, \text{ for all } \ z \in (z_1, z_2), \tag{43}$$

and $|\varphi_C^*(z_1)| = |\varphi_C^*(z_2)| = 1 - \varepsilon$. From (41) one has

$$-\ln(1-\varepsilon) = -\ln|\varphi_C^*(z_0)| = \lambda \int_0^\infty (1 - Re\varphi_C^*(z_0 h(u)))du =$$

$$= -\lambda \int_0^{h(+0)} (1 - Re\varphi_C^*(z_0 u)))h^{\leftarrow}(du) \geq$$

$$\geq -\lambda \int_{z_1/z_0}^{h(+0)} (1 - Re\varphi_C^*(z_0 u)))h^{\leftarrow}(du) > \ \lambda\varepsilon h^{\leftarrow}(z_1/z_0)$$

Letting $\varepsilon \to 0$ implies $z_1 \to 0$ and leads to a contradiction $1 > \infty$. This completes the proof of the Lemma. ■

Remark 4.1. At the beginning of the previous proof we have shown that μ_* is non-lattice. A stronger version of this result follows from Theorem 3.2 of Vervaat (1979). Namely, since both μ_* and $\overline{\mu}_*(dx) = x\mu_*(dx)$ are non-degenerate then $\overline{\mu}_*$ is either absolutely continuous or continuous singular. Clearly, so is μ_*. However we do not use this observation.

Before next lemmas let us recall the following sufficient condition for absolute continuity. For an arbitrary ID probability measure μ on $\mathbb{R}^d$ with Lévy spectral measure M we know that the absolute continuity of μ follows from the absolute continuity of $(\widetilde{M})^{*k}(\cdot)$, for some k, where $\widetilde{M}(dx) := x^2/(1+x^2)M(dx)$; cf. Sato (1999), Theorem 27.7 or Jurek and Mason (1993), Theorem 3.8.1, p.163. However, for our purposes on positive half-line, the above reduces to

Lemma 4.3. If ID ν has no atom at zero, and its Lévy spectral measure σ is such that $\overline{\sigma}^{*k}$ is absolutely continuous for some positive integer k than ν is absolutely continuous.

Recall that $\overline{\sigma}(dx) = x\sigma(dx)$. This is applicable criterium because of the following.

Lemma 4.4. For a fixed point μ_* with the Lévy spectral measure M_* there exists $n \in \mathbb{N}$ such that $\overline{M}_*^{*n}$ is absolutely continuous with bounded continuous density vanishing at ∞.

Proof. The ch.f.'s of the fixed points μ_* are analytic, cf. Remark 1.1, therefore are differentiable infinitely many times. To establish the Lemma we will prove that some their nth derivatives are integrable. From the equality (4) we infer that

$$(\varphi_C^*)'(z) = \varphi_C^*(z) \int_0^{h(+0)} (\varphi_C^*)'(zu)\rho_h(du). \tag{44}$$

(the interchange of integration and differentiation is justified in the same way as we did in establishing (18)). In what follows, without loss of generality, we will assume that $\int_0^\infty x\mu_*(dx) = 1$. Therefore $\overline{M}_*$ is a probability measure with the ch.f. $f_C^*(z) = -i\int_0^{h(+0)}(\varphi_C^*)'(zu)\rho_h(du)$. Hence, by recursive nature of equality (44), one gets

$$f_C^*(z) = \int_0^{h(+0)} \varphi_C^*(zu) f_C^*(zu)\rho_h(du). \tag{45}$$

In view of Lemma 4.2 and (40) there exists an $l \in (0,1)$ such that

$$|\varphi_C^*(z)| < l, \tag{46}$$

whenever $|z| \geq 1$. By assumptions of Theorem 1.2 we may choose $\tau \in (0,\Delta)$ so small to guarantee $lm_\tau < 1$, where

$$m_\tau := \int_0^{h(+0)} u^{-\tau}\rho_h(du) = \lambda \int_0^\infty h^{1-\tau}(u)du < \infty.$$

It is also important to notice that Markov inequality implies

$$\rho_h[0,1/z) \leq m_\tau z^{-\tau}. \tag{47}$$

We claim that the following inequality holds true:

$$|f_C^*(z)| \leq l \int_0^{h(+0)} |f_C^*(zu)|\,\rho_h(du) + m_\tau\,|z|^{-\tau}, \tag{48}$$

for $z \neq 0$. To this end notice that by (45)-(47) we get

$$|f_C^*(z)| \leq \int_0^{h(+0)} |\varphi_C^*(zu)|\,|f_C^*(zu)|\,\rho_h(du) = \int_{1/|z|}^{h(+0)} + \int_0^{1/|z|} \leq$$

$$l \int_0^{h(+0)} |f_C^*(zu)|\,\rho_h(du) + \rho_h[0,1/|z|) \leq l \int_0^{h(+0)} |f_C^*(zu)|\,\rho_h(du) + m_\tau\,|z|^{-\tau}.$$

Iterating (48) $n-1$ times one gets

$$|f_C^*(z)| \leq \varepsilon^n \mathbb{E}\,|f_C^*(z\vartheta_1...\vartheta_n)| + m_\tau \sum_{i=0}^{n-1}(lm_\tau)^i\,|z|^{-\tau},$$

and hence, by letting $n \to \infty$, we get the required inequality

$$|f_C^*(z)| \leq \frac{m_\tau}{1-lm_\tau}\,|z|^{-\tau};$$

(as before, for first term one uses the strong law of large numbers to obtain $\vartheta_1...\vartheta_n \stackrel{a.s.}{\to} 0$ and bounded convergence). Now choosing $n \in \mathbb{N}$ such that $n\tau > 1$ we conclude $|(f_C^*(z))^n| = |f_C^*(z)|^n = O(|z|^{-n\tau})$. This and classical argument from the Fourier transform theory, cf. Sato (1999) Proposition 2.5(xii), completes the proof of Lemma 4.4.

With all of those preparations we are now ready for the proof of our second theorem.

Proof of Theorem 1.2 i) and iii) Since the fixed points $\mu_{*,\alpha}$ are probability distributions of a product of two independent r.v.'s (cf. Remark 1.2), where one of them is the stable distribution with exponent α, we get our statements from Lemma 4.1. Since a positive stable distribution has a density (let us denote it by g_α), which is also infinitely differentiable with $g_\alpha^{(n)}$ being continuous and bounded for each $n \in \mathbb{N}$, then similar holds for $\mu_{*,\alpha}$ by Lemma 4.1.

Now let us consider fixed points μ_*. Suppose $h \in \mathcal{H}_{bs}$ or equivalently $a < \infty$. Furthermore, without loss of generality, one may assume that $a = 1$, because pairs $(\lambda, h(u))$ and $(\lambda t, h(tu))$, $t > 0$ generate the same fixed points. Thus $\lambda > 1$, by Lemma 3.2. Let q be the unique solution, in the interval $(0, 1)$, to the equation $\exp(-\lambda(1 - x)) = x$.

Now we proceed as in Athreya (1969). See Example 5.1 for the description and the notations we use below.

R.v. X has an atom of mass q at zero and is absolutely continuous elsewhere with continuous density $u(x)$. One checks that $\mathcal{L}(X) = M_*$ and therefore $\overline{M}_*$ is absolutely continuous everywhere with continuous (uniformly) density $xu(x)$. The standard representation of non-negative ID distributions (Steutel (1970), Theorem 4.2) can be rewritten in our circumstances as follows $\overline{\mu}_* = \mu_* * \overline{M}_*$. This formula reveals that: firstly, $\overline{\mu}_*$ is absolutely continuous, and secondly that its density w is continuous. Consequently, μ_* is absolutely continuous, except a possible atom at zero and its density $x^{-1}w(x)$ is continuous on $x > 0$. For the smoothness property one argues in the same way as in the proof of Theorem 1.1(ii) of Liu (1999). This completes the proof in case $h \in \mathcal{H}_{bs}$.

If $h \in \mathcal{H}_{uniq} \backslash \mathcal{H}_{bs}$ then the proof of absolute continuity of μ_* is merely a combination of Lemmas 4.2-4.4.

For the *proof of Theorem 1.2 ii)* see the second paragraph of this Section.

5 Relations to other results and notions.

Besides the theoretical interest in the notion of fixed points of the SNT, we would like to point out that it is also related to many other concepts studied previously. Here are only two examples. Note that the first one was useful in proving our Theorem 1.2 and it is given here in a slightly more extended form.

Example 5.1 *(Bellman-Harris process) Let $X(t)$ denotes an age-dependent branching process (number of particles alive at time $t > 0$), where each particle, at the end of its life, produces N new offsprings, with N being a Poisson r.v. with mean $1 < \lambda < \infty$ (supercritical case). Furthermore we assume that all*

particles have (random) life-length κ.
For a unique α *such that* $\mathbb{E}\exp(-\alpha\kappa) = \lambda^{-1}$ *let us define an r.v.* κ_α *such that* $\kappa_\alpha \stackrel{d}{=} e^{-\alpha\kappa}$. *It is known that there exist an r.v.* X, *with mean value one, such that a.s.* $X(t)/\mathbb{E}X(t) \to X$, *as* $t \to \infty$, *and whose LT* $\theta(s) := \mathbb{E}\exp(-sX)$ *satisfies the equation*

$$\theta(s) = \int_0^\infty \exp(-\lambda(1-\theta(s\exp(-\alpha x))))dP\{\kappa \le x\},$$

or, equivalently,

$$\theta(s) = \int_0^1 \exp(-\lambda(1-\theta(sz)))dP\{\kappa_\alpha \le x\}. \qquad (49)$$

For a full account of this result see Athreya (1969).
Let $X_1, X_2, ...$ *and* $\kappa_{\alpha,1}, \kappa_{\alpha,2}, ...$ *be independent copies of* X *and* κ_α, *respectively, and let* $t_1, t_2, ...$ *be an infinite sample from the uniform distribution on* $[0,1]$. *Also we assume that r.v.'s* X_i, $\kappa_{\alpha,i}$, t_i, $i \ge 1$, N, *are independent. Then the equality (49) can be rewritten in terms of r.v.'s as follows:*

$$X \stackrel{d}{=} Y\kappa_\alpha, \qquad (50)$$

where $Y \stackrel{d}{=} \sum_{i=1}^N X_i$ *and* $\mathbb{E}Y = \lambda$. *Furthermore assuming that* $P\{\kappa_\alpha = 0\} = 0$, *and defining* $h(u) := \inf\{w\colon P\{\kappa_\alpha \le w\} \ge 1-u\}$ *(i.e., h is the generalized inverse of* $P\{\kappa_\alpha \le u\ \}$ *evaluated at* $1-u$*), one observes that the fixed points of the SNT and those whose LT's are given by (49) are related by (50). Indeed, for* h *chosen above we have* $\kappa_{\alpha,i} \stackrel{d}{=} h(t_i)$ *(note that* h *excludes the possibility of an atom at zero for* κ_α*), and, hence,* $Y \stackrel{d}{=} \sum_{i=1}^N X_i \stackrel{d}{=} \sum_{i=1}^N Y_i\kappa_{\alpha,i} \stackrel{d}{=} \sum_{i=1}^N Y_i h(t_i) \stackrel{d}{=} \sum_{i=1}^N Y_i h(\tau_i)$. *The latter equality follows from the well-known fact that, conditionally on* $N = n$, *the vector* $(\tau_1, ..., \tau_n)$ *has the same distribution as the order statistics from a sample of size* n *drawn from the uniform distribution on* $[0,1]$.

The second example deals with so called *perpetuity laws* that have origin in the insurance mathematics. However, there is also a wide range of non-insurance applications and many of them are given in Vervaat (1979) and Embrechts, Goldie (1994). Relations of perpetuities to selfdecomposable laws were investigated in Jurek (1999). A comparatively new field of applications of perpetuity laws is in an analysis of probability algorithms; cf. Rösler, Rüschendorf (2001).

Example 5.2 *(Perpetuity) Let us consider the distributional equality*

$$\eta \stackrel{d}{=} A\eta + B, \qquad (51)$$

where random vector (A, B) *is independent of r.v.* η *(laws of* η *are called perpetuities). In other words, a distribution of* η *is a fixed point of* a random affine

mapping (transform).
Perpetuities arise in two distinct ways from our model.
1. Set $A = h(\tau_1)$, $B = \sum_{i=2}^{\infty} \xi_i h(\tau_i)$, *then the fixed point equation* $\mathbb{T}_{h,\lambda}\mu = \mu$ *expressed in terms of r.v.'s can be rewritten as (51) with* η *replaced by* ξ_1 *and* A, B *being dependent.*
2. Let $h \in \mathcal{H}$ *and* $\lambda \int_0^\infty h(u)du = 1$. *Then by Proposition 1.1(b) there exists a fixed point* μ_* *of finite mean* m, *say. Let* η *be a r.v. with the probability distribution* $\overline{\mu}_*(dx) := m^{-1}x\mu_*(dx)$ *and let* A *be a r.v. with the probability distribution* $\rho_h(dx) := -\lambda x h^{\leftarrow}(dx)$ *(where as before* $h^{\leftarrow}$ *denotes the generalized inverse function of* h*), and finally let* $B \overset{d}{:=} \xi_1$. *So defined r.v.'s satisfy the distributional equation (51). In fact, this is rewritten representation of non-negative ID distributions due to Steutel (1970, Theorem 4.2.4).*

The fixed points of other transforms have received much of attention as well. Typically one discusses so called *smoothing transforms.* These are solutions of the equation

$$\zeta \overset{d}{=} \sum_{n=1}^{M} v_i \zeta_i, \tag{52}$$

where M is finite and perhaps random. The basic methods for finite non-random M were developed by Durrett and Liggett (1983). Liu (1998) extended their results for finite and random M; cf. also Baringhaus and Grübel (1997) and Biggins and Kyprianou (2001). Some authors considered partial cases of the smoothing transforms in the context of the sorting algorithm *Quicksort*; cf. Rösler, Rüschendorf (2001) and also the series of paper by Fill and Janson available at http://www.math.uu.se/ ˜svante. Our aim here was the case $v_i = h(\tau_i)$ and $M = \infty$ (if $h \in \mathcal{H}_{ubs}$) that, in the context of the SNT, appears to be more involved and interesting to work with. However, as a by-product, the general presentation covers also the case of Poisson r.v. M (if $h \in \mathcal{H}_{bs}$), independent of both v_i'e and ζ_i's.

Now we would like to indicate some specific connections of our paper to the ones of the others.

1) If $h \in \mathcal{H}_{ubs}$ then the Remark to Theorem 4 in Rösler (1992) implies that the SNT has a unique fixed point on the space of non-negative probability distributions with fixed mean and finite second moment. However, it is not clearly if there are no fixed points on the complement of that space. Cf. also Lyons (1997) for a result related to our Theorem 1.1(b). The fixed points $\mu_{*,\alpha}$ given by (7) are closely related to the notion of "canonical fixed point" introduced for the smoothing transforms; cf. Guivarc'h (1990) and Liu (1998). We are not aware of any papers dealing with solutions to (52) with $M = \infty$ and $\mathbb{E}\zeta = \infty$.

2) When $h \in \mathcal{H}_{bs}$ then there is a close relation between our fixed points and those appearing in the context of the Bellman-Harris processes; cf. Example 5.1. We have used that observation in proving the corresponding part of Theorem 1.2. In fact, for response functions with compact support, Theorem 1.2 can be partially deduced from the powerful Theorem 0 of Athreya (1969). It seems

that part of our Theorem 1.1 can be indirectly (via Example 5.1) obtained from Liu (1998). However, this possibility is not investigated here. After the first version of our paper was completed, we have learned about results in Liu (1999, 2001). Note that if $a := sup\{u > 0 : h(u) > 0\} < \infty$ and $\lambda \int_0^a h(u)du = 1$ then some parts of our Theorem 1.1(a,b) can be derived (via Example 5.1) from Proposition 2.2(ii) of Liu (1999). The analyticity of a ch.f., stated in our Remark 1.1, can be obtained by means of Theorem 1.6 of the paper in question. Despite of this our approaches are quite different. On the other hand, the proof of our Lemma 4.3 and the part of Liu's (1999) proof of his Theorem 4.1 are rather similar. We think that this is so, because both are inspired by the proof of Lemma 9 in Athreya (1969). We are using Theorem 1.1(ii) from Liu (1999) or Theorem 2.2 from Liu (2001) to strengthen the statement of our Theorem 1.2. The remaining part of Theorem 1.2 seems to be completely new.

The key steps for the present paper are Lemmas 3.3 and 3.4. Note that the way of establishing Lemma 3.3 is based on the result from Lau-Rao (1982). On the other hand, the proof of Lemma 3.4 is based on a renewal argument that was used in similar situations before. This idea goes back to Grincevičius (1975) for real r.v.'s and to Kesten (1973) for random vectors. Note however, that the well-known Theorem 2 of Grincevičius (1975) is not applicable here.

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