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The random integral representation hypothesis revisited : new classes of s-selfdecomposable laws.

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ABSTRACT. For $0 < \alpha \leq \infty$, new subclasses $\mathcal{U}^{<\alpha>}$ of the class $\mathcal{U}$, of s-selfdecomposable probability measures, are studied. They are described by random integrals, by their characteristic functions and their Lévy spectral measures. Also their relations with the classical Lévy class L of selfdecomposable distributions are investigated.

Key words and phrases: s-selfdecomposable distributions; the class $\mathcal{U}$ background driving Lévy process; class L; Lévy spectral measure; Lévy exponent; random integrals.

Limit distribution theory belongs to the core of probability and mathematical statistics. Often limit laws are described by *analytical tools* such as Fourier or Laplace transforms, but a more stochastic approach (e.g., like stochastic integration, stopping times, random functionals etc.), seems more natural for probability questions. Some illustrations of this paradigm are given in the last paragraph of this note. In a similar spirit, in Jurek (1985) on page 607 (and later repeated in Jurek (1988) on page 474), the following hypothesis was formulated:

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Each class of limit distributions, derived from sequences of independent random variables, is the image of some subset of ID (the infinitely divisible probability measures) *by some mapping defined as a random integral.*

Random integral representations, when they can be established, would provide descriptions of limiting laws via *stochastic methods*, i.e., as the probability distributions of the random integrals of form

$$\int_{(a,b]} h(t)\, dY(r(t)) = \int_{(r(a),r(b)]} h(r^{-1}(s))\, dY(s), \tag{1}$$

where h and r are deterministic functions, $h : (a,b] \to \mathbb{R}$, $r : (a,b] \to (0,\infty)$ and $Y(s), 0 \leq s < \infty$, is a stochastic process with independent and stationary increments and cadlag (right continuous with left hand limits) paths; in short, we refer to Y as *a Lévy process*. In this note we provide new examples of classes of limit distributions for which the above hypothesis holds true. The main results here are Propositions 3, 4 and 5, and Corollaries 5, 6 and 7.

1. Introduction and notation.

Let E denotes a real separable Banach space, E' its conjugate space, $< \cdot, \cdot >$ the usual pairing between E and E', and $||.||$ the norm on E. The σ-field of all Borel subsets of E is denoted by $\mathcal{B}$, while $\mathcal{B}_0$ denotes Borel subsets of $E \setminus \{0\}$. By $\mathcal{P}(E)$ we denote the (topological) semigroup of all Borel probability measures on E, with convolution "$*$" and the weak topology, in which convergence is denoted by "$\Rightarrow$". Similarly, by $ID(E)$ we denote the topological convolution semigroup of all *infinitely divisible* probability measures, i.e.,

$$\mu \in ID(E) \;\text{ iff }\; \forall(\text{natural } \;k \geq 2)\;\; \exists(\mu_k \in \mathcal{P}(E)) \;\; \mu = \mu_k^{*k}.$$

Recall also here that $ID(E)$ is a closed topological subsemigroup of $\mathcal{P}(E)$. Finally on a Banach space E we define the transforms T_r, for $r > 0$, as follows: $T_r x := rx,\ x \in E$, and define $\mathcal{L}(\xi)$ as the probability distribution of an E-valued random variable ξ.

A probability measure $\mu \in \mathcal{P}(E)$ is said to be *s-selfdecomposable* on E, and we will write $\mu \in \mathcal{U}(E)$, if there exists a sequence $\rho_n \in ID(E)$ such that

$$\nu_n := T_{\frac{1}{n}}(\rho_1 * \rho_2 * ... * \rho_n)^{*1/n} \Rightarrow \mu, \quad \text{as} \;\; n \to \infty. \tag{2}$$

Since we begin with infinitely divisible measures ρ_n we do not include the shifts δ_{x_n} in (1), and do not assume that the triangle system $\{T_{\frac{1}{n}}\, \rho_j^{*1/n} : 1 \leq$

$j \leq n; n \geq 1\}$ is uniformly infinitesimal, as is usually done in the general limiting distribution theory.
Also let us note that our definition (2) is, in fact, the result of Theorem 2.5 in Jurek (1985). There s-selfdecomposability was defined in many different but equivalent forms. Finally, s-selfdecomposable distributions appeared in the context of an approximation of processes by their discretization; cf. Jacod, Jakubowski and Mémin (2001).

Originally the s-selfdecomposable distributions were introduced as limit distributions for sums of shrunken random variables in Jurek (1981). The *'s'* stands here for *shrinking operation* defined as follows:

$$U_r(x) := max(||x|| - r, 0)\, x/||x||, \quad \text{for} \quad r > 0 \quad \text{and} \quad x \in E \setminus \{0\}.$$

Also see the announcement in Jurek (1977). On the real line similar distributions, but not related to s-operation, were studied in O'Connor (1979).

In the present paper we will repeat the scheme (2) successively and will assume that ρ_k are chosen from a previously obtained class of limit laws. Such an approach, for another scheme of limiting procedure was introduced by K. Urbanik (1973) and then continued by K. Sato, A. Kumar and B. M. Schreiber, N. Thu, with the most general setting, up to now, described in Jurek (1983), where there is also a list of related references.

For easy reference we collect below some of the known characterizations of the class $\mathcal{U}(E)$ of s-selfdecomposable probability measures and indicate only the main steps in the corresponding proofs.

PROPOSITION 1. *The following statements are equivalent:*

(i) $\mu \in \mathcal{U}(E)$.

(ii) $\forall (0 < c < 1)\ \exists (\mu_c \in ID(E))\ \mu = T_c\mu^{*c} * \mu_c$.

(iii) *there exists a unique Lévy process Y such that* $\mu = \mathcal{L}(\int_{(0,1)} t\, dY(t))$.

Sketch of proofs. Characterizations (i) and (ii) are equivalent by Theorem 2.5 and Corollary 2.3 in Jurek (1985). Equivalence of (ii) and (iii) follows from Theorem 1.1 and Theorem 1.2(a) in Jurek (1988), where one needs to take the constat $\beta = 1$ and the linear operator $Q = I$.

For our purposes we define *random integrals* by the formal formula of integration by parts:

$$\int_{(a,b]} h(t)\, dY(r(t)) := h(b)Y(r(b)) - h(a)Y(r(a)) - \int_{(a,b]} Y(r(t))\, dh(t),$$

where the later integral is defined as a limit of the appropriate Rieman-Stieltjes partial sums. This "limited" approach to integration is sufficient for our purposes; cf. Jurek and Vervaat (1983) or Jurek and Mason (1993), Section 3.6. On the other hand, since Lévy processes are semi-martingales, the integrals (1) or the above, can be defined as the stochastic integrals as well.

COROLLARY 1. *The class $\mathcal{U}$ of s-selfdecomposable probability measures is closed topological convolution subsemigroup of ID. Moreover, it also is closed under the convolution powers (i.e, for $t > 0$ and μ we have that $\mu \in \mathcal{U}$ if and only if $\mu^{*t} \in \mathcal{U}$) and the dilations T_d, for $d \in \mathbb{R}$ (i.e., $\mu \in \mathcal{U}$ if and only if $T_d\mu \in \mathcal{U}$).*

Proof. Both algebraic properties follow from (ii) in Proposition 1 and the following identities $(T_d(\nu * \rho))^{*t} = T_d\nu^{*t} * T_d\rho^{*t}$, for $t > 0$, $d \in \mathbb{R}$, and $\nu, \rho \in ID$. To show that $\mathcal{U}$ is closed in weak convergence topology we use again the factorization (ii) together with Theorem 1.7.1 in Jurek an Mason (1993) or cf. Chapter 2 in Parthasarathy (1967).

In view of the property (iii), in Proposition 1, we define the following integral mapping

$$\mathcal{J} : ID(E) \to \mathcal{U}(E) \text{ given by } \mathcal{J}(\rho) := \mathcal{L}(\int_{(0,1)} s\, dY_\rho(s)), \qquad (3)$$

where $Y_\rho(\cdot)$ is a Lévy process (i.e., a process with independent and stationary increments, starting from zero and with cadlag paths) such that $\mathcal{L}(Y_\rho(1)) = \rho$. We refer to $Y(\cdot)$ as the **b**ackground **d**riving **L**évy **p**rocess (in short, the BDLP) for the s-selfdecomposable measure $\mathcal{J}(\rho)$.

REMARK 1. The random integral mapping $\mathcal{J}$ is an isomorphism between the closed topological semigroups $ID(E)$ and $\mathcal{U}(E)$; cf. Jurek (1985), Theorem 2.6.

Finally, let

$$\hat{\mu}(y) := \int_E e^{i<y,x>} \mu(dx), \quad y \in E',$$

be *the characteristic function* (the Fourier transform) of a measure μ. Then for random integrals (1) we infer that

$$\Big(\mathcal{L}(\int_{(a,b]} h(t) dY_\rho(r(t)))\Big)^{\widehat{}}(y) = \exp \int_{(a,b]} \log \hat{\rho}(h(t)y) dr(t), \qquad (4)$$

when h is a deterministic function, r is an increasing (or monotone) time change in $(0, \infty)$ and $Y_\rho(.)$ a Lévy process; cf. Lemma 1.2 in Jurek and Vervaat (1983) or Lemma 1.1 in Jurek (1985) or simply approximate the right-hand integral by Rieman-Stieltjes partial sums.

Our results are given in the generality of a Banach space E, however, below in many formulas we will skip the dependence on E.

2. m-times s-selfdecomposable probability measures.

Let us put $\mathcal{U}^{<1>} := \mathcal{U}(E)$ and for $m \geq 2$, let $\mathcal{U}^{<m>}$ denotes the class of limiting measures in (2), when $\rho_k \in \mathcal{U}^{<m-1>}$, for $k = 1, 2, ...$. As a convention we assume that $\mathcal{U}^{<0>} := ID$. Our first characterization is proved along the lines of the proofs of Theorem 1.1 and 1.2 in Jurek (1988), however one needs not to confuse the classes $\mathcal{U}_\beta$ introduced there, with those of $\mathcal{U}^{<m>}$ investigated here. Needed changes in arguments are explained as they are deemed.

PROPOSITION 2. *For $m = 1, 2, ...$, the following are equivalent descriptions of m-times s-selfdecomposable probability measures:*

(i) $\mu \in \mathcal{U}^{<m>}$.

(ii) $\forall (0 < c < 1)\ \exists (\mu_c \in \mathcal{U}^{<m-1>})\ \mu = T_c \mu^{*c} * \mu_c$.

(iii) *There exists a unique (in distribution) Lévy process Y_ρ such that*

$\mu = \mathcal{L}\Big(\int_{(0,1)} t dY_\rho(t) \Big)$, *where* $\mathcal{L}(Y_\rho(1)) = \rho \in \mathcal{U}^{<m-1>}$.

Moreover, in (ii) we have $\mu_c = \mathcal{L}\Big(\int_{[c,1)} t dY_\rho(t) \Big)$, *for* $0 < c < 1$.

Proof. For $m = 1$, the above is just the Proposition 1. Now suppose that the proposition is proved for m. If $\mu \in \mathcal{U}^{<m+1>}$ then, by the definition (formula (1)), $\rho_k \in \mathcal{U}^{<m>}$, for $k = 1, 2, ...$. For given $0 < c < 1$, let us choose natural numbers m_n such that $1 \leq m_n \leq n$ and $m_n/n \to c$, as $n \to \infty$. From (2) we have

$$\nu_n = T_{m_n/n} \nu_{m_n}^{*m_n/n} * T_{1/n}(\rho_{m_n+1} * ... * \rho_n)^{*1/n}. \quad (5)$$

By Theorems 1.2 and 2.1 in Parthasarathy (1967), the second convolution factor in (5) converges, say to μ_c, which must be in $\mathcal{U}^{<m>}$ by Corollary 1. Thus we get the factorization (ii) for $m + 1$, i.e., (i) implies (ii).

If (ii) holds we have a family $\mathcal{C} := \{\mu_c : 0 \leq c \leq 1\} \subset \mathcal{U}^{<m>}$, where $\mu_1 = \delta_0$ and $\mu_0 = \mu$, from which we construct sequence (ρ_k) as follows

$$\rho_1 := \mu \ \text{ and } \rho_k := T_k \mu^{*k}_{(k-1)/k} \ \text{ for } \ k \geq 2.$$

Using the factorization (ii) for $c = (k-1)/k$, then applying to both sides the dilation T_k and then raising to the (convolution) power k, gives the equality $T_k\mu^{*k} = T_{k-1}\mu^{*(k-1)} * \rho_k$, or in terms of Fourier transforms

$$\hat{\rho}_k(y) = [\hat{\mu}(ky)]^k / [\hat{\mu}((k-1)y)]^{k-1}, \ \text{ for } \ k \geq 2.$$

Hence

$$\rho_1 * \rho_2 * ... * \rho_n = T_n \mu^{*n} \ \text{ i.e., } \ \mu \in \mathcal{U}^{m+1},$$

which completes the proof that (ii) implies (i).

Since we have

$$\mu = \mathcal{L}(\int_{(0,1)} t dY_\rho(t)) = \mathcal{L}(\int_{(0,c)} t dY_\rho(t)) * \mathcal{L}(\int_{[c,1)} t dY_\rho(t))$$
$$= T_c(\mathcal{L}(\int_{(0,1)} t dY_\rho(t)))^{*c} * \mathcal{L}(\int_{[c,1)} t dY_\rho(t)),$$

we infer that (iii) implies (ii). To prove the converse that (ii) implies (iii) we proceed as in Jurek (1988), page 482 (formula (3.1)) till page 484, taking $\beta = 1$ and $Q = I$ (identity operator). Thus we construct process $Z(t)$ with independent increments and cadlag paths such that $\mathcal{L}(Z(t)) = \mu_{e^{-t}} \in \mathcal{U}^{<m>}$. Because of Corollary 1 we conclude that

$$\tilde{Y}(t) := \int_{(0,t]} s dZ(s), \ \text{ for } \ t \geq 0,$$

has increments with probability distributions in $\mathcal{U}^{<m>}$. All in all we have proved (iii).

COROLLARY 2. *(a) The classes $\mathcal{U}^{<m>}$, $m = 1, 2, ...,$ of the m-times s-selfdecomposable probability measures are closed convolution subsemigropus , closed under convolution powers and the dilations T_d.*

(b) $\mathcal{U}^{<m>} = \mathcal{J}(\mathcal{J}(...(\mathcal{J}(ID))))$, *(m-times composition),*

$$L_{m+1} \subset \mathcal{U}^{<m+1>} \subset \mathcal{U}^{<m>} \subset ID, \ \ \textit{for} \ \ m = 0, 1, 2, ... \ , \tag{6}$$

where L_k, $k = 1, 2, ...,$ are the convolution semigroups of k-times selfdecomposable probability distributions.

Proof. Part (a) follows from the characterization (ii) in Proposition 2. To prove that $\mathcal{U}^{<m>}$ are closed we use Theorem 1.7.1 in Jurek and Mason (1993) or cf. Chapter 2 in Parthasarathy (1967).

Part(b). Since $\mathcal{U} \subset ID$, therefore applying successively the random integral mapping $\mathcal{J}$ to both sides gives the inclusion $\mathcal{U}^{<m+1>} \subset \mathcal{U}^{<m>}$.
For the second inclusion $L_k \subset \mathcal{U}^{<k>}$, note that it is true for $k = 1$, cf. Corollary 4.1 in Jurek (1985). Assume it is true n, i.e., $L_n \subset \mathcal{U}^{<n>}$ and let $\mu \in L_{n+1} \subset L_n$. Then for any $0 < c < 1$ there exits $\nu_c \in L_n$ such that

$$\mu = T_c\mu * \nu_c = T_c\mu^{*c} * \rho_c \text{ with } \ \rho_c := T_c\mu^{*(1-c)} * \nu_c \in L_n \subset \mathcal{U}^{<n>},$$

because, by the induction assumption, ν_c and μ are in L_n. Consequently, by (ii), in Proposition 2, $\mu \in \mathcal{U}^{<n+1>}$ and this completes the proof.

Our next aim is to describe m-times s-selfdecomposability in terms parameters of infinitely divisible laws. Recall that each ID distribution μ is uniquely determined by a triple: a shift vector $a \in E$, a Gaussian covariance operator R, and a Lévy spectral measure M; we will write $\rho = [a, R, M]$. These are the parameters in the Lévy-Khintchine representation of the characteristic function $\hat{\mu}$, namely

$$\begin{aligned} &\rho \in ID \text{ iff } \hat{\rho}(y) = \exp(\Phi(y)), \quad \text{where} \\ &\qquad\qquad \Phi(y) := i < y, a > +1/2 < Ry, y > + \\ &\int_{E\setminus\{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_{||x||\leq 1}(x)]M(dx), \quad y \in E'; \quad (7) \end{aligned}$$

Φ is called the *Lévy exponent* of $\hat{\rho}$ (cf. Araujo and Giné (1980), Section 3.6). Furthermore, by the *Lévy spectral function* of ρ we mean the function

$$L_M(D, r) := -M(\{x \in E : \|x\| > r \text{ and } x\,||x||^{-1} \in D\}),$$

where D is a Borel subset of unit sphere $S := \{x : ||x|| = 1\}$ and $r > 0$. Note that L_M uniquely determines M.

Since the Lévy processes have infinitely divisible increments (from the class ID) and ID is a topologically closed convolution semigroup, and also closed under dilations T_a (a multiplication of random variable by a scalar a), therefore the random integrals $\int_{(a,b]} h(t)\, dY(r(t))$ have probability distributions in ID as well . If $[a_{h,r}, R_{h,r}, M_{h,r}]$ denotes the triple corresponding to the probability distribution of the integral in question, and $[a, R, M]$ denotes the one corresponding to the law of $Y(1)$ then (4) and (7) give the following equation:

$$R_{h,r} = \big(\int_{(a,b]} h^2(t)dr(t)\big)\, R, \tag{8}$$

$$M_{h,r}(A) = \int_{(a,b]} M((h(t))^{-1}A)dr(t) \quad \text{for} \quad A \in \mathcal{B}_0, \tag{9}$$

and finally for the shift vector we have

$$a_{h,r} = \big(\int_{(a,b]} h(t)dr(t)\big)\, a + \int_{E\setminus\{0\}} x \int_{(a,b]} h(t)[1_B(h(t)x) - 1_B(x)]dr(t)M(dx). \tag{10}$$

Specializing the above for the functions $h(t) = r(t) = t,\ 0 \le t \le 1$, and, for the simplicity of notations, putting $\rho = [a, R, M]$ and $[a', R', M'] = \mathcal{J}(\rho)$, we get from (8)-(10) the following relations

$$R' = 1/3R, \tag{11}$$

$$M'(A) = \int_{(0,1)} M(t^{-1}A)dt, \quad \text{for} \quad A \in \mathcal{B}_0, \tag{12}$$

$$a' = \frac{1}{2}a + \int_{(0,1)} t \int_{1<||x||\le t^{-1}} xM(dx)dt = \frac{1}{2}[a + \int_{\{||x||>1\}} x\,||x||^{-2}M(dx)]. \tag{13}$$

In order to get the second equality in (13) one needs to observe that $1_{\{t||x||\le 1\}}(x) = 1_{\{0<t\le ||x||^{-1}\}}(t)$ or to change the order of integration. Thus

$$\int_{(0,1)} t \int_{\{1<||x||\le t^{-1}\}} xM(dx)dt = \int_{\{||x||>1\}} \int_{(0,1)} tx1_{\{t||x||\le 1\}}(x)dtM(dx)$$
$$= \int_{\{||x||>1\}} x \int_0^{||x||^{-1}} tdtM(dx) = 1/2 \int_{\{||x||>1\}} x\,||x||^{-2}M(dx).$$

Now we may characterize the m-times s-selfdecomposable distributions in terms of the triples in their Lévy-Khintchine formula.

PROPOSITION 3. *For $m = 1, 2, ...,$ let $\rho = [a, R, M]$ and $[a^{<m>}, R^{<m>}, M^{<m>}] = \mathcal{J}^m(\rho)$ be m-times s-selfdecomposable probability measures. Then*

$$R^{<m>} = (1/3)^m R, \tag{14}$$

$$M^{<m>}(A) = ((m-1)!)^{-1} \int_{(0,1)} M(t^{-1}A)(-\ln t)^{m-1} dt, \quad for \quad A \in \mathcal{B}_0, \tag{15}$$

$$a^{<m>} = (1/2)^m [a + \int_{\{||x||>1\}} x\, ||x||^{-2} \sum_{j=0}^{m-1} \frac{(2\ln ||x||)^j}{j!} M(dx). \tag{16}$$

Proof. For $m = 1$ the above are just the formulae (11)-(13).
Assume that (14)-(16) holds form m. Since $[a^{<m+1>}, R^{<m+1>}, M^{<m+1>}] = \mathcal{J}([a^{<m>}, R^{<m>}, M^{<m>}])$, therefore, by (11)-(13), we get

$$R^{<m+1>} = (1/3) R^{<m>}, \tag{17}$$

$$M^{<m+1>}(A) = \int_{(0,1)} M^{<m>}(t^{-1}A) dt, \quad \text{for} \quad A \in \mathcal{B}_0, \tag{18}$$

$$a^{<m+1>} = \frac{1}{2} [a^{<m>} + \int_{\{||x||>1\}} x\, ||x||^{-2} M^{<m>}(dx)]. \tag{19}$$

Obviously, by the induction assumption, we infer that (14) holds for $m+1$. Similarly from (8) and (11), and from the change of the order of integration, we get

$$\begin{aligned} M^{<m+1>}(A) &= ((m-1)!)^{-1} \int_{(0,1)} \int_{(0,1)} M(t^{-1}s^{-1}A)(-\ln t)^{m-1} dt ds \\ &= ((m-1)!)^{-1} \int_{(0,1)} \int_{(0,t)} M(u^{-1}A) \frac{du}{t} (-\ln t)^{m-1} dt \\ &= ((m-1)!)^{-1} \int_{(0,1)} M(u^{-1}A) [\int_{(u,1)} \frac{1}{t} (-\ln t)^{m-1} dt] du \\ &= (m!)^{-1} \int_{(0,1)} M(u^{-1}A)(-\ln u)^m du, \end{aligned}$$

which proves (15).

In order to prove the formula for the shift, first note that by (11) and by change of order of integration, we have

$$
\begin{aligned}
w_m &:= \int_{\{||x||>1\}} x\,||x||^{-2} M^{<m>}(dx) \\
&= ((m-1)!)^{-1} \int_{(0,1)} \int_E 1_{\{\{||x||>1\}\}}(tz) z\,||z||^{-2}\, t^{-1}(-\ln t)^{m-1} dt\, M(x) \\
&= ((m-1)!)^{-1} \int_{\{||z||>1\}} z\,||z||^{-2} \big[\int_{||z||^{-1}}^{1} t^{-1}(-\ln t)^{m-1} dt \big] M(dz) \\
&= (m!)^{-1} \int_{\{||z||>1\}} z\,||z||^{-2} (\ln ||z||)^m M(dz),
\end{aligned}
$$

for $m = 1, 2, \ldots$. Note that for $m = 0$ the above formula gives the second summand in (13). In terms of w_m, (19) gives the recurrence relation

$$
a^{<m>} = 1/2(a^{<m-1>} + w_{m-1}), \quad \text{for} \quad m = 1, 2, \ldots,
$$

where $a^{<0>} := a$. Thus, if the formula for the shifts (16) holds for m, then the above gives that it also holds for $m+1$, which completes the proof the proposition.

Let us recall that the functions

$$
\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt, \quad x > 0, \; \alpha > 0, \tag{20}
$$

are called *the incomplete gamma functions*. Simple calculations shows that

$$
\Gamma(m, x) = (m-1)!\, e^{-x} \sum_{j=0}^{m-1} \frac{x^j}{j!}, \quad \text{for} \quad m = 1, 2, \ldots \; . \tag{21}
$$

Consequently, the formula (16) may be written as

$$
a^{<m>} = (1/2)^m \, [\, a + \frac{1}{\Gamma(m)} \int_{\{||x||>1\}} x\, \Gamma(m, 2 \ln ||x||)\, M(dx)\,]. \tag{22}
$$

Let us introduce *rescales of time* in the interval $(0, 1)$ as follows

$$
\tau_\alpha(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (-\ln u)^{\alpha-1} du, \quad 0 < t \le 1. \tag{23}
$$

Note that τ_α is the cumulative probability distribution function of the random variable $g_\alpha := e^{-G_\alpha}$, where G_α is the gamma random variable with the probability density $(\Gamma(\alpha))^{-1}\, x^{\alpha-1}e^{-x}$ for $x > 0$, and zero elsewhere. Hence

$$\tau_\alpha(t) = P\{g_\alpha \le t\}, \quad \text{and} \quad \mathbf{E}[g_\alpha^s] = (\Gamma(\alpha))^{-1} \int_0^1 t^s(-\ln t)^{\alpha-1}dt = (s+1)^{-\alpha};$$
$$\int_0^c t^s\, d\tau_\alpha(t) = (s+1)^{-\alpha}\, \frac{\Gamma(\alpha, -(s+1)\ln c)}{\Gamma(\alpha)}, \quad \text{for} \quad s > 0,\ 0 < c < 1, \quad (24)$$

and (15) can rewritten as

$$M^{<m>}(A) = \int_{(0,1)} M(t^{-1}A)d\tau_m(t) = \mathbf{E}[M(g_m^{-1}A)]. \qquad (25)$$

Now we can establish *the random integral representation* for the subclasses $\mathcal{U}^{<m>}$ of s-selfdecomposable probability measures.

PROPOSITION 4. *(a) The class $\mathcal{U}^{<m>}$ of m-times s-selfdecomposable probability measures coincides with the class of probability distributions of random integrals $\int_{(0,1)} tdY(\tau_m(t))$, where $Y(\cdot)$ is an arbitrary Lévy process.*

(b) The class of Fourier transforms of measures from $\mathcal{U}^{<m>}$ coincides with the class of functions $\exp \mathbf{E}[\Psi(g_m y)], y \in E'$, *where Ψ is an arbitrary Lévy exponent of an infinitely divisible probability measure and the random variable $g_m := \exp(-G_m)$, with G_m being the standard gamma random variable. In fact, the exponent Ψ is that of the random variable $Y(1)$ from (a).*

Proof. Let $[b_m, S_m, N_m]$ and $[a, R, M]$ are the triples describing the probability distribution of the integral $\int_{(0,1)} tdY(\tau_m(t))$ and $Y(1)$, respectively. Then from (8)-(10) and (24) we have

$$S_m = \big(((m-1)!)^{-1} \int_{(0,1)} t^2(-\ln t)^{m-1}dt\big)\, R = \frac{1}{3^m}\, R = R^{<m>},$$
$$N_m(A) = ((m-1)!)^{-1} \int_{(0,1)} M(t^{-1}A)(-\ln t)^{m-1}\, dt = M^{<m>}(A),$$
$$b_m = \big(((m-1)!)^{-1} \int_{(0,1)} t(-\ln t)^{m-1}dt\big) a$$
$$+ ((m-1)!)^{-1} \int_{(0,1)} t \int_{1<||x||\le t^{-1}} x\, M(dx)(-\ln t)^{m-1}\, dt$$

$$= 2^{-m}\, a + \int_{\{||x||>1\}} x\big[((m-1)!)^{-1}\int_{(0,||x||^{-1})} t(-\ln t)^{m-1}dt\big]\, M(dx)$$
$$= 2^{-m}a + 2^{-m}\int_{\{||x||>1\}} x\,[((m-1)!)^{-1}\,\Gamma(m, 2\ln||x||)\,]\, M(dx) = a^{<m>},$$

which completes the proof of the part (a).

For the part (b) we need to combine the formulae (4) (for $h(t) = t, r(t) = \tau_m(t)$) and (14)-(16), and use (24).

COROLLARY 3. *A function $\phi : E' \to \mathbb{C}$ is a Fourier transform of an m-times s-selfdecomposable probability measure if and only if there exist unique shift $a \in E$, a Gaussian covariance operator R and a Lévy spectral measure M such that*

$$\phi(y) = \exp\{i < y, a > +2^{-1} < Ry, y > +$$
$$\int_{E\setminus\{0\}} [\,\big(\widehat{\mathcal{L}(g_m)}\big)(< y, x >) - 1 - 2^{-m}i < y, x > 1_B(x)\,]M(dx)\},$$

where $g_m = e^{-G_m}$ and G_m is the gamma random variable.

Proof. Use Proposition 4 together with the formula (4). Note that there are not restrictions on a shift vector and a Gaussian covariance operator R. Finally, for m=1 this is Theorem 2.9 in Jurek (1985).

REMARK 2. Using the series representation of the exponential function and (24) we get

$$\widehat{\mathcal{L}(g_\alpha)}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!(n+1)^\alpha}, \quad t \in \mathbb{R}.$$

Since the previous characterization, of m-times s-selfdecomposability, has only a restriction on the Lévy spectral measure, therefore we have a characterization of $\mathcal{U}^{<m>}$ in terms of Lévy spectral functions.

COROLLARY 4. *An infinitely divisible $[a, R, M]$ probability measure is m-times s-selfdecomposable if and only if there exists a unique Lévy spectral measure G such that*

$$L_M(D, r) = ((m-1)!)^{-1} r \int_r^{\infty} (\ln w - \ln r)^{m-1} L_G(D, w)\frac{dw}{w^2},$$

for all sets D and all $r > 0$, or equivalently

$$L_M(D,r) = r\int_r^\infty x_{m-1}^{-1}\int_{x_{m-1}}^\infty \dots x_1^{-1}\int_{x_1}^\infty w^{-2}L_G(D,w)dw\,dx_1\dots dx_{m-1},$$

for all sets D and all $r > 0$.

Proof. In view of the Proposition 3 we have that $M = G^{<m>}$ for a unique Lévy spectral measure G, and (15) gives the first part of the corollary. Since the relation (18), in terms Lévy spectral functions, reads

$$L_{M^{<j>}}(D,r) = r\int_r^\infty L_{M^{<j-1>}}(D,x_{j-1})\frac{dx_{j-1}}{x_{j-1}^2}, \quad r>0, \text{ with } \; M^{<0>} := G,$$

for $j = 1, 2...$, therefore the inductive argument proves the second part of the corollary.

COROLLARY 5. *In order that a Lévy spectral measure G to be a Lévy spectral measure of an m-times s-selfdecomposable probability measure, it is necessary and sufficient that its Lévy spectral functions $r \to L_G(D,r)$ are m-times differentiable, except at countable many points r, and the function*

$$L(D,r) = (\mathcal{A}^m(L_G(D,\cdot)))(r) \quad \textit{is a Lévy spectral function.}$$

The operator $\mathcal{A}^m$ is the m-time composition of the linear differential operator $\mathcal{A}$, which is defined as follows

$$(\mathcal{A}(h))(x) := xh'(x) - h(x),$$

for once differentiable real-valued functions h defined on $(0,\infty)$.

Proof. If measures M' and M are related as in (12) then their corresponding spectral functions (tails) $L_{M'}$ and L_M satisfy equality

$$L_{M'}(D,r) = \int_{(0,1)} L_M(D,r/t)dt = r\int_r^\infty w^{-2}L_M(D,w)dw.$$

Hence $L_{M'}$ is at least once differentiable (except on a countable set) and

$$-L_M(D,r) = r\frac{d}{dr}L_{M'}(D,r) - L_{M'}(D,r) = (\mathcal{A}(L_{M'}(D,\cdot))(r). \tag{26}$$

Moreover, by Theorem 1.3 in Jurek (1985), we have the left-hand side is a Lévy spectral measure (on Banach space) if and only if so is a measure on the right-hand side. Because of the recurrence equation (18) we have proved the corollary.

REMARK 3. The random integral mapping $\mathcal{J}$ is defined on infinitely divisible measures $\rho = [a, R, M]$. If one assume that the formula (12) defines the mapping $\mathcal{J}$ on the measure M or its spectral function L_M, then $\mathcal{A}$ may be viewed as its inverse mapping.

Before the next characterization, of the class $\mathcal{U}^{<m>}$ distributions, let us recall that a Lévy exponent is just the logarithm of an infinitely divisible Fourier transform; cf. formula (7). Let us note that, if Ψ is the Lévy exponent of ρ and Φ is that of $\mathcal{J}(\rho)$, then (3) and (4) give the following

$$\Phi(ty) = \int_{(0,1)} \Psi(sty)ds = \frac{1}{t}\int_{(0,t)} \Psi(sy)ds,$$

and consequently

$$\Psi(y) = \Phi(y) + d(\Phi(ty))/dt|_{t=1}.$$

With these equalities and the recursive relation between classes $\mathcal{U}^{<m>}$ we have

COROLLARY 6. *A function $\Phi : E' \to \mathbb{C}$ is a Lévy exponent of an m-times s-selfdecomposable probability measure if and only if there exists a unique Lévy exponent Ψ such that the function*

$$E' \ni y \to \mathcal{D}^m(\Psi)(y) \quad \textit{is a Lévy exponent.}$$

The operator $\mathcal{D}^m$ is the m-time composition of the following linear differential operator

$$(\mathcal{D}g)(y) := g(y) + d(g(ty))/dt|_{t=1},$$

where $g : E' \to \mathbb{C}$ is once differentiable in each direction $y \in E'$ and $t \in \mathbb{R}$.

Note that in a particular case one has $(\mathcal{D}g)(y) := g(y) + y\,dg(y))/dy$, when $y \in E' = \mathbb{R}$ and it differs from $\mathcal{A}$ in Corollary 5 only by a sign.

REMARK 4. If ones defines $\mathcal{J}$ on Lévy exponents by (4) then the operator $\mathcal{D}$ can be viewed as its inverse, i.e., $\mathcal{D} = \mathcal{J}^{-1}$, on Lévy exponents on a Banach space.

PROPOSITION 5. *A probability measure $\mu = [a, R, M]$ is completely s-selfdecomposable, i.e., $\mu \in \mathcal{U}^{<\infty>} := \bigcap_{m=1}^{\infty} \mathcal{U}^{<m>}$ if and only if there exists a unique bi-measure $\sigma(\cdot, \cdot)$ on $S \times (0, 2)$ such that*

$$M(A \cdot D) = \int_{(0,2)} \int_D \int_A w^{-(z+1)}\, dw\, \sigma(du, dz) = \int_{(0,2)} \int_A w^{-(z+1)}\, dw\, \sigma(D, dz), \tag{27}$$

where $A \cdot D := \{x \in E : x/||x|| \in D, \quad ||x|| \in A\}$ *and for each Borel* $D \subset S$, $\sigma(D, \cdot)$ *is a finite Borel measure on the interval* $(0, 2)$ *and for each Borel subset* $A \subset (\epsilon, \infty)$ *for some* $\epsilon > 0$, $\sigma(\cdot, A)$ *is a finite Borel measure on the unit sphere* S. *Moreover, we have that*

$$\int_{(0,2)} \int_S | < y, u > |^2 \frac{1}{2 - z} \sigma(du, dz) < \infty,$$

for all $y \in E'$.

Proof. If $\mu = [a, R, M]$ is completely s-selfdecomposable then by Proposition 3 or Corollary 4, for each m there exists a unique Lévy measure G such that

$$M(A) = ((m-1)!)^{-1} \int_{(0,1)} G(t^{-1}A)(-\ln t)^{m-1} dt, \text{ for } \ A \in \mathcal{B}_0.$$

or for all D and $r > 0$

$$L_G(D, r) = r \int_r^\infty x_{m-1}^{-1} \int_{x_{m-1}}^\infty ... \, x_1^{-1} \int_{x_1}^\infty w^{-2} L_G(D, w) dw \, dx_1 \, ... \, dx_{m-1}.$$

Hence, for each set D, the functions $f(x) := -e^{-x} L_G(D, e^x), \quad x \in \mathbb{R}$, are m-times differentiable and

$$(-1)^m d^m f(x)/dx^m = -e^{-x} \, L_G(D, e^x) \geq 0.$$

In other words, f is completely monotone and by Bernstein's Theorem, there exists a unique finite Borel measure $\sigma^\sim(D, \cdot)$ on $(0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xz} \sigma^\sim(D, dz), \quad \text{i.e.,} \quad L_M(D, r) = -\int_0^\infty \frac{1}{r^{z-1}} \sigma^\sim(D, dz). \quad (28)$$

Since Lévy spectral functions vanish at ∞ and $\sigma^\sim(D, \cdot)$ are finite measures, therefore they must be concentrated on half-line $(1, \infty)$. Consequently, from (28) we get

$$M([r, s) \cdot D)) = L_M(D, r) - L_M(D, s) = \int_1^\infty (z - 1) \int_r^s \frac{1}{w^z} \, dw \, \sigma^\sim(D, dz),$$

for all $0 < r < s < \infty$ and all Borel sets $D \subset S$. Since for $y \in E'$, $(\pi_y M)(C) := M(\{x \in E :< y, x >\in C\})$, for Borel subsets C in $\mathbb{R}$, are Lévy measure on real line therefore

$$\int_{\{||x|| \leq 1\}} | < y, x > |^2 \, M(dx) \leq \int_{\{x \in E: \, |<y,x>| \leq 1\}} | < y, x > |^2 M(dx)$$
$$= \int_{\{|t| \leq 1\}} t^2 \, (\pi_y \, M)(dt) < \infty \ .$$

On the other hand, using (28) the integral

$$\int_{\{0<||x|| \leq 1\}} | < y, x > |^2 \, M(dx) = \int_{(0,1] \cdot S} | < y, u > |^2 \, t^2 \, M(du \cdot dt)$$
$$= \int_S \int_0^1 | < y, u > |^2 \, t^2 \int_1^\infty \frac{z-1}{t^z} \, \sigma^{\sim}(du, dz) \, dt$$
$$= \int_S \int_1^\infty | < y, u > |^2 (z-1) \, [\int_0^1 t^{-(z-2)} dt \,] \, \sigma^{\sim}(du, dz),$$

is finite only if $z < 3$, because $\sigma^{\sim}(D, \cdot)$ are finite measures. Changing the variable and putting $\sigma(D, dz) := z\sigma^{\sim}(D, dz+1), \quad 0 < z < 2$, we obtain the formula (27) together with the integrability condition. Thus the necessity part of the proposition is proved.

For the converse, let $\rho = [a, R, M]$ with the spectral measure M of the form in (27). Hence, by (12), $\mathcal{J}(\rho)$ has spectral measure

$$M'(A \cdot D) = \int_{(0,1)} \int_{(0,2)} \int_{s^{-1}A} \frac{1}{w^{z+1}} \, dw \, \sigma(D, dz) ds = \int_{(0,2)} \int_A \frac{1}{x^{z+1}} dx \, \sigma_1(D, dz),$$

where $\sigma_1(D, dz) := (z+1)^{-1} \sigma(D, dz)$ is another finite measure on the interval $(0, 2)$ and therefore $\mathcal{J}(\rho)$ has the Lévy spectral measure of the form (27) again. Consequently, $\rho \in \mathcal{U}^{<m>}$ for all m, and thus the sufficiency of (27) is completed.

Let put

$$\Sigma := \{\rho = [a, R, M_\sigma] : M_\sigma \quad \textit{is of the form (27)}\},$$

i.,e., Σ is the more explicit description of $\mathcal{U}^{<\infty>}$. Further, let us recall that

$$\int_E \log(1 + ||x||) \, \rho(dx) \quad < \infty \ \text{ iff } \int_{\{||x||>1\}} \log ||x|| \, M_\sigma(dx) < \infty,$$

By the formula (27), the last integral is equal

$$-\int_1^\infty \log t \, dL_{M_\sigma}(S,t) = \int_1^\infty L_{M_\sigma}(S,t)\, t^{-1}\, dt$$
$$= \int_1^\infty \int_{(0,2)} t^{-z-1}\, \sigma(S,dz)\, dt = \int_{(0,2)} z^{-1}\, \sigma(S,dz) \; < \infty.$$

Similar integrability formulas hold for functions $g_k(x) := \log^k(1+||x||)$ and Lévy measures M. Recall that the integrability condition of g_k appears in the random integral representation for the class L_k.

COROLLARY 7. *The class of completely s-selfdecomposable probability measures coincides with the class of completely selfdecomposable ones, i.e.,* $\mathcal{U}^{<\infty>} = L_\infty$.

Proof. If $\rho = [a, R, M]$ and $\mathcal{I}(\rho) := \mathcal{L}\big(\int_0^\infty e^{-t}\, dY_\rho(t)\big) = [A^0, R^0, M^0]$, then

$$\int_{||x||>1} \log ||x|| M(dx) < \infty \;\; \text{and} \;\; M^0(A) := \int_0^\infty M(e^s A) ds,$$

for all sets $A \in \mathcal{B}_0$ cf. Jurek (1985), p.603 or Jurek and Mason (1993), p.120. Simple calculation show that

$$(M_\sigma)^0(A \cdot D) = \int_0^\infty \int_{(0,2)} \int_{e^t A} \frac{1}{w^{z+1}} dw\, \sigma(D,dz) dt$$
$$= \int_{(0,2)} \int_A v^{-(z+1)} \int_0^\infty e^{-tz} dt\, dv \sigma(D,dz) = \int_{(0,2)} \int_A \frac{1}{v^{z+1}} dv\, \sigma_2(D,dz),$$

where $\sigma_2(D,dz) := z^{-1}\sigma(D,dz)$ is another finite measure on $(0,2)$ because of the logarithmic moment assumption. This shows that $\mathcal{I}(\Sigma) \subset \Sigma$. Since $L_k = \mathcal{I}((...(\mathcal{I}(ID_{log^k})))),$ (k-times composition of $\mathcal{I}$ and ID_{log^k} denotes the class of infinitely divisible measures with finite $\log^k$-moments) we infer that $\Sigma \subset L_k$, for $k = 1, 2 \ldots$. Consequently, $\Sigma \subset L_\infty := \cap_{k=1}^\infty L_k \subset \mathcal{U}^{<\infty>} = \Sigma$, which completes the proof.

REMARK 5. Measures M_σ are mixtures of Lévy measures of stable laws. The mixture is done with respect to the exponents $p \in (0,2)$. Since Fourier transforms of p-stable measures are known explicitly we can have analogous formulas for completely s-selfdecomposable measures; cf. a similar result (on the real line) for L_∞ in Urbanik (1973), or Thu (1986), or Sato (1980) or Jurek (1983),Theorem 7.2.

3. Concluding remarks and two examples.

A). The classes $\mathcal{U}^{<m>}$ were introduced by an inductive procedure and thus we have the natural index m. For a positive non-integer α one may proceeds as in Thu (1986) using the fractional calculus. However, we may utilize our random integral approach and define

$$\mathcal{U}^{<\alpha>} = \{\mathcal{L}\big(\int_{(0,1)} t\, dY_\rho(\tau_\alpha(t))\big) : \rho \in ID\}, \tag{29}$$

where $Y_\rho(\cdot)$ is a Lévy process with $\mathcal{L}(Y_\rho(1)) = \rho$. Equivalently, we have

$$[a^{<\alpha>}, R^{<\alpha>}, M^{<\alpha>}] = \mathcal{J}^\alpha(\rho) = \mathcal{L}\big(\int_{(0,1)} t\, dY_\rho(\tau_\alpha(t))\big),$$

where

$$R^{<\alpha>} = 3^{-\alpha} R, \quad M^{<\alpha>}(A) = \int_{(0,1)} M(t^{-1}A) d\tau_\alpha(t), \quad A \in \mathcal{B}_0,$$
$$a^{<\alpha>} = 2^{-\alpha}\, [\, a + \frac{1}{\Gamma(\alpha)} \int_{\{||x||>1\}} x\, \Gamma(\alpha, 2\ln ||x||)\, M(dx)], \tag{30}$$

cf. (14), (15) and for the shift vector (16) with (21),(22) and (24).

Furthermore, for any continuous and bounded f on $(0,\infty)$ and gamma random variables G_α and G_β we have

$$\int_0^\infty \int_0^\infty f(x+y)\mathcal{L}(G_\alpha)(dx)\mathcal{L}(G_\beta)(dy) = \int_0^\infty f(z)\mathcal{L}(G_{\alpha+\beta})(dz),$$

i.e., gamma distributions form an one-parameter convolution semigroup of measures on $(0,\infty)$ with the addition . Consequently, for any continuous bounded h on interval $(0,1)$

$$\int_{(0,1)} \int_{(0,1)} h(st) d\tau_\alpha(t) d\tau_\beta(s) = \int_{(0,1)} h(u) d\tau_{\alpha+\beta}(u),$$

thus τ_α form one parameter semigroup of measures on $(0,1)$ with the multiplication. Hence we infer that

COROLLARY 8. *For any positive α and β we have*

(a) $\mathcal{J}^{<\alpha+\beta>} = \mathcal{J}^\alpha(\mathcal{J}^\beta)$,

(b) *if* $\alpha < \beta$ *then* $\mathcal{U}^{<\beta>} \subset \mathcal{U}^{<\alpha>}$.

B). In this subsection we consider only $\mathbb{R}$-valued random variables or Borel measures on the real line.
Because of the inclusion $L \subset \mathcal{U}$ each selfdecomposable distribution is an example of s-selfdecomposable one. On the other hand, by Proposition 3 in Iksanow, Jurek and Schreiber (2002), selfdecomposable distributions of random variables of the form $X := \sum_{k=1}^{\infty} a_k \eta_k$, where η_k's are independent identically distributed Laplace (double exponential) random variables and $\sum_k a_k^2 < \infty$, have the background driving probability measures $\nu \in \mathcal{U}$. Furthermore, by Proposition 3 in Jurek (2001) we have that

$$\hat{\nu}(t) = \exp\left[t\phi'_X(t)/\phi_X(t)\right], \qquad t \in \mathbb{R}. \tag{31}$$

In Jurek (1996) it was noticed that $\phi_S(t) = t/(\sinh t)$ ("S" stands for the hyperbolic 'sine') and $\phi_C(t) := 1/(\cosh t)$ ("C" stands for the hyperbolic 'cosine') are the characteristic functions of random variables of the above series form X. Using (31) we conclude

$$\psi_S(t) := \exp(1 - t\coth t),\ \psi_C(t) := \exp(-t\tanh t) \ \text{ are class } \mathcal{U} \text{ char. f.}$$

Thus both are characteristic functions of integrals (3). Furthermore from Corollary 6 we have that

$$\begin{aligned} &\mathcal{D}(\log\psi_S(t)) = 1 - 2\coth t + t^2/(\sinh^2 t) \ \text{ and} \\ &\qquad \mathcal{D}(\log\psi_C(t)) = -2t\tanh t - t^2/(\cosh^2 t) \ \text{ are Lévy exponents.} \end{aligned} \tag{32}$$

It might be worthy to mention here that $\phi_S(t) \cdot \psi_S(t)$ is a characteristic function of *a conditional Lévy's random area integral*; cf. Lévy (1951) or Yor (1992) and Jurek (2001). Similarly, $(\phi_C(t) \cdot \psi_C(t))^{1/2}$ is a characteristic function of *an integral functional of Brownian motion*; cf. Wenocur (1986) and Jurek (2001), p. 248.

Recently in Jurek and Yor (2002) the probability distributions corresponding to both ψ_S and ψ_C were expressed in terms of squared Bessel bridges. Also both functions viewed as the Laplace transform in $t^2/2$ can be interpreted as the hitting time of 1 by the Bessel process starting from zero; cf. Yor (1997), p. 132. At present we are not aware of any stochastic representation for the analytic expressions in (32). Finally, it seems that the operators $\mathcal{A}^m$ may be related to some Markov processes.

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