A random integral calculus on generalized s-selfdecomposable probability measures*

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Abstract. The class $\mathcal{U}_\beta$ of generalized s-selfdecomposable probability distributions can be viewed as an image, via the random integral mapping $\mathcal{J}^\beta$, of the class ID of all infinitely divisible measures. We prove that a composition of the mappings $\mathcal{J}^{\beta_1}, \mathcal{J}^{\beta_2}, ..., \mathcal{J}^{\beta_n}$, $\beta_1 > 0, ..., \beta_n > 0$, is again a random integral but with a new deterministic inner time. Moreover, some elementary formulas concerning the distributions of products of powers of independent uniformly distributed random variables are established.

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Let a real β and infinitely divisible probability measures ν_j (on Euclidean space $\mathbb{R}^d$ or a Banach space E) be such that

$$T_{1/n}(\nu_1 * \nu_2 * ... * \nu_n)^{*n^{-\beta}} * \delta_{x_n} \Rightarrow \mu\,, \quad \text{as} \quad n \to \infty\,, \qquad (\star)$$

for some deterministic shifts x_n, where T_c denotes the dilation (multiplication) by $c > 0$. Then we write $\mu \in \mathcal{U}_\beta$ and call μ *a generalized s-selfdecomposable measure*. In a series of papers Jurek (1988, 1989), Jurek and Schreiber (1992) it was proved, among others, that for not degenerate μ in ($\star$) we must have $\beta \geq -2$ and that $\mathcal{U}_\beta$ ($\beta \geq -2$) form an increasing family of convolution semigroups that "almost" exhaust the whole class ID of all infinitely divisible measures.

In a recent paper by James and Zhang (2011) generalized s- selfdecomposable distributions from $\mathcal{U}_\beta$ ($\beta > 0$) were used for price models that exhibit volatility clustering.

For the purpose of this paper the most crucial is the fact that generalized s-selfdecomposable measures admit the *random integral representation* (1), (see below), which is a particular case of the following representation

$$\mu \equiv I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\big(\int_I h(t)dY_\nu(r(t))\big), \qquad (\star\star)$$

where $I = (a,b] \subset \mathbb{R}^+$, $h : \mathbb{R}^+ \to \mathbb{R}$, $Y_\nu(\cdot)$ *is a Lévy process and,*
$r : \mathbb{R}^+ \to \mathbb{R}^+$ *is a monotone function (deterministic time change in* Y_ν*)* .

In fact, it was shown that many classes of limit laws can be described as collections of probability distributions of random integrals of the form ($\star\star$) for suitable chosen parameters h, r and I (possibly a half-line). Later on, this led to the conjecture that *all classes of limit laws* should admit random integral representation; cf. a survey article Jurek (2011) and see the Conjecture on www.math.uni.wroc.pl/∼zjjurek [1].

One may hope that this note will lead to establishing of *a calculus on random integral mappings* $I^{h,r}_{(a,b]}$, and their domains of definition $\mathcal{D}^{h,r}_{(a,b]}$, analogous to that of the linear operator calculus in functional analysis.

[1] It might be of an interest to recall here that S. D. Chatterji's subsequence principle claiming that: *Given a limit theorem for independent identically distributed random variables under certain moment conditions, there exists an analogous theorem such that an arbitrary-dependent sequence (under the same moment conditions) always contains a subsequence satisfying this analogous theorem* was proved by David J. Aldous (1977). Although, we do not expect that the above Conjecture and Chatterji's subsequence principle are mathematically related, however, one may see a "philosophical" relation between those two.

In this paper we will prove that the class of the integral mappings (1), for $\beta > 0$, is closed under compositions, that is, their compositions are of the form $(\star\star)$ with the properly chosen time change r; (Theorem 1). As an auxiliary result we found a decomposition of number 1 as a sum of products of complex fractions; (Lemma 1). Also compositions of the mappings (1) are described in terms of the Lévy-Khintchine triples; (Theorem 2). Auxiliary Lemma 3 give probability distribution functions (p.d.f.) of products of powers of independent uniformly distributed random variables as linear combinations of other p.d.f.

1. Introduction and main results.

The results here are given for random vectors in $\mathbb{R}^d$ *. However, proofs are such that they are valid for infinite dimensional separable Banach spaces* E *when ones replaces a scalar product by the bilinear form on the product space* $E^{'} \times E$*, where* $E^{'}$ *denotes the dual space; see Araujo and Gine (1980) or Linde (1983) or Ledoux and Talagrand (1991). Of course,* $(\mathbb{R}^d)' = \mathbb{R}^d$.

Throughout the paper $\mathcal{L}(X)$ will denote the probability distribution of an $\mathbb{R}^d$-valued random vector X; (or a real separable Banach space E-valued random element X if the Reader is interested in that generality). Similarly, by $Y_\nu(t), t \geq 0$, we will denote an $\mathbb{R}^d$- valued (or an E-valued) Lévy process such that $\mathcal{L}(Y_\nu(1)) = \nu$. Recall that by a Lévy process we mean a process with stationary independent increments, starting from zero, and with paths that are continuous from the right and with finite left-hand limits (that is, cadlag paths). Of course, $\nu \in ID$, where ID stands for all *infinitely divisible* measures on $\mathbb{R}^d$ (or on a Banach space E).

For $\beta > 0$ and a Lévy process $Y_\nu(t), t \geq 0$, we define mappings

$$\mathcal{J}^{\{\beta\}}(\nu) \equiv \mathcal{J}^{\beta}(\nu) := \mathcal{L}\big(\int_{(0,1]} t^{1/\beta}\, dY_\nu(t)\big) = \mathcal{L}\big(\int_{(0,1]} t\, dY_\nu(t^\beta)\big) \qquad (1)$$

and the classes $\mathcal{U}_\beta := \mathcal{J}^{\beta}(ID)$. To the distributions from $\mathcal{U}_\beta$ we refer to as *generalized s-selfdecomposable distributions.*

These classes of probability measures were originally defined as limiting distributions in some schemes of summations; cf. Jurek (1988 and 1989). In particular, the class $\mathcal{U} \equiv \mathcal{U}_1$ of s-selfdecomposable was defined by the non-linear shrinking operations (in short: s-operation) $U_r, r > 0$, (for $x > 0$, $U_r(x) := \max(0, x-r)$); cf. Jurek (1981). [Terminology: "s-selfdecomposable" is abbreviation of that "shrinking-selfdecomposable".]

Remark 1. Since the process Y has values in a metric separable complete space we may and do assume that the paths of Y are cadlag; cf. Theorem A.1.1 in Jurek and Vervaat (1983), p. 260. Since the random integral in (1) is defined by a formal integration by parts formula, therefore the random integral in question does exist; cf. Jurek-Vervaat (1983), Lemma 1.1.

Furthermore, since Lévy processes are semi-martingales the random integral (1) can be defined as the Ito stochastic integral. However, for our purposes we do not need that generality.

For a positive natural m and a sequence of positive real $\beta_1, \beta_2, ..., \beta_m$ and a probability measure $\nu \in ID$, let us define the mappings

$$\mathcal{J}^{\{\beta_1,...,\beta_m\}}(\nu) := \mathcal{J}^{\beta_m}(\mathcal{J}^{\{\beta_1,...,\beta_{m-1}\}}(\nu)) = \mathcal{L}\Big(\int_{(0,1]} t\, dY_{\mathcal{J}^{\{\beta_1,...,\beta_{n-1}\}}(\nu)}(t^{\beta_m})\Big)\,.$$

Our main results say that the above composition can be written as a single integral of the form ($\star\star$) with a suitable chosen time change r. Furthermore, the composition is expressed in terms of the individual random integrals.

Theorem 1. *For positive reals $\beta_1, \beta_2, ..., \beta_m$ and an infinitely divisible probability measure ν we have*

$$\mathcal{J}^{\{\beta_1,...,\beta_m\}}(\nu) = \mathcal{L}\Big(\int_{(0,1]} t\, dY_\nu(r_{\{\beta_1,...,\beta_m\}}(t)\Big) = I_{(0,1]}^{t,\, r_{\{\beta_1,...,\beta_m\}}}(\nu) \qquad (2)$$

and the time scale change $r_{\{\beta_1,...,\beta_m\}}$ is given by

$$r_{\{\beta_1,...,\beta_m\}}(t) = \mathbb{P}\,[\,U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot ... \cdot U_m^{1/\beta_m} \le t\,], \quad 0 < t \le 1\,, \qquad (3)$$

where U_i's are mutually independent uniformly distributed, on the unit interval, random variables.

If all $\beta_1, ..., \beta_n$ are different then

$$r_{\{\beta_1,...,\beta_n\}}(t) := \sum_{j=1}^{n} C_{j,n}\, t^{\beta_j}, \quad C_{j,n}(\beta_1, ..., \beta_n) \equiv C_{j,n} := \prod_{k\neq j, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_j} \qquad (4)$$

and, in particular, we get the equality: $\sum_{j=1}^{n} C_{j,n} = 1$.

If $\beta_1 = \beta_2 = ... = \beta_m = \alpha$ $(m \ge 1)$ then

$$r_{\underbrace{\{\alpha, ..., \alpha\}}_{m-times}}(t) = t^\alpha \sum_{j=0}^{m-1} \frac{(-\alpha \log t)^j}{j!}, \quad \text{for } 0 < t \le 1\,. \qquad (5)$$

Remark 2. (a) Note that if $\mathcal{E}(\lambda)$ denotes the exponential random variable with the parameter λ then $e^{-\mathcal{E}(\lambda)} \stackrel{d}{=} U^{1/\lambda}$. Hence if $\mathcal{E}_i(\alpha), 1 \le i \le m$, independent and identically distributed exponential random variables then $r_{\{\alpha,...,\alpha\}}$ is

the cumulative distribution of $e^{-\mathcal{E}_1(\alpha)} \cdot e^{-\mathcal{E}_2(\alpha)} \cdot ... \cdot e^{-\mathcal{E}_m(\alpha)} \stackrel{d}{=} e^{-\gamma_{m,\alpha}}$, where $\gamma_{m,\alpha}$ is the gamma random variable with the shape parameter m and the scale parameter α, i.e., it has the density of the form $\alpha^m/(m-1)!\, x^{m-1}\, e^{-\alpha x}\, 1_{(0,\infty)}(x)$; in particular, for the case $\alpha = 1$ see Proposition 4 in Jurek (2004).
(b) For m=2, Theorem 1, and in particular the equality (4) were shown in Czyzewska-Jankowska and Jurek (2009).

Example. For $\beta_j := j\,\beta,\ j = 1, 2, ..., n$ with fixed $\beta > 0$, we get that

$$C_{j,n}(\beta, 2\beta, ..., n\beta) = (-1)^{j-1}\binom{n}{j} \text{ and } r_{\{\beta, 2\,\beta, ..., n\,\beta\}}(t) := \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j} t^{j\,\beta}.$$

In a proof of the above theorem the following identity, that might be also of an independent interest, is needed.

Lemma 1. *For distinct complex numbers $z_j, j = 1, 2, ..., n, n+1$ we have equality:*

$$\sum_{i=1}^{n} \frac{1}{z_i - z_{n+1}} \Big(\prod_{k=1; k\neq i}^{n} \frac{1}{z_k - z_i} \Big) = \prod_{i=1}^{n} \frac{1}{z_i - z_{n+1}}. \qquad (6)$$

Equivalently, for any distinct complex numbers $z_j, j = 1, 2, ..., n$, we have the identity

$$\sum_{i=1}^{n} \prod_{k=1; k\neq i}^{n} \frac{z_k - z}{z_k - z_i} \equiv 1, \quad \textit{for all} \quad z \in \mathbb{C}, \qquad (7)$$

that can be regarded as a decomposition of 1 as a sum of finite products of complex fractions.

(It would be interesting to have a geometric description of the above identity (7).)

Since the characteristic function of each $\nu \in ID$ is uniquely determined by the triple $[a, R, M]$ from its Lévy-Khintchine formula we will write formally that $\nu = [a, R, M]$; for details see the Section 2.1 below.

If $\nu = [a, R, M]$ and $\mathcal{J}^{\{\beta\}}(\nu) := [a^{\{\beta\}}, R^{\{\beta\}}, M^{\{\beta\}}]$ and

$$b_{M,\beta} := \int_{\{||x||>1\}} x\, ||x||^{-1-\beta}\, M(dx) \in \mathbb{R}^d \text{ (or a Banach space } E) \qquad (8)$$

then we have

$$a^{\{\beta\}} = \beta(\beta+1)^{-1}(a + b_{M,\beta}), \quad R^{\{\beta\}} = \beta(2+\beta)^{-1}R,$$
$$M^{\{\beta\}}(A) = \int_0^1 T_{t^{1/\beta}} M(A) dt = \beta \int_0^1 M(s^{-1}A)\, s^{\beta-1} ds, \text{ for } A \in \mathcal{B}_0, \quad (9)$$

where $\mathcal{B}_0$ stands for all Borel subsets of $\mathbb{R}^d \setminus \{0\}$ (or $E \setminus \{0\}$. The above identities follow from the Lévy-Khintchine formula (13) and Lemma 2; for more details cf. Jurek (1988).

With these notations Theorem 1 gives the description of random integrals (2) in terms of their corresponding triples.

Theorem 2. *For distinct positive reals $\beta_1, \beta_2, ..., \beta_n$, coefficients $C_{j,n}$ defined by (4), an infinitely divisible probability measure $\nu = [a, R, M]$ and $\mathcal{J}^{\{\beta_1,...,\beta_n\}}(\nu) = [a^{\{\beta_1,...,\beta_n\}}, R^{\{\beta_1,...,\beta_n\}}, M^{\{\beta_1,...,\beta_n\}}]$ we have*

$$a^{\{\beta_1,...,\beta_n\}} = a \prod_{j=1}^{n} \frac{\beta_j}{\beta_j + 1} + \sum_{j=1}^{n} \frac{\beta_j\, b_{M,\beta_j}}{\beta_j + 1} \prod_{k \neq j, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_j} = \sum_{j=1}^{n} C_{j,n}\, a^{\{\beta_j\}} \quad (10)$$

$$R^{\{\beta_1,...,\beta_n\}} = \prod_{j=1}^{n} \frac{\beta_j}{\beta_j + 2} R = \sum_{j=1}^{n} C_{j,n} R^{\{\beta_j\}}, \quad (11)$$

$$M^{\{\beta_1,...,\beta_n\}}(A) = \int_0^1 ... \int_0^1 T_{t_1^{1/\beta_1} ... t_n^{1/\beta_n}} M(A) dt_1 ... dt_n = \sum_{j=1}^{n} C_{j,n}\, M^{\{\beta_j\}}(A) \quad (12)$$

where b_{M,β_j}, $a^{\{\beta_j\}}$, $R^{\{\beta_j\}}$ and $M^{\{\beta\}}$ are given in (8) and (9).

Corollary 1. *For distinct positive reals $\beta_1, \beta_2, ..., \beta_n$ and the constants $C_{j,n}$ given in (4) we have*

$$\mathcal{J}^{\{\beta_1,...,\beta_n\}}(\nu) = (\mathcal{J}^{\{\beta_1\}}(\nu))^{*C_{1,n}} * ... * (\mathcal{J}^{\{\beta_n\}}(\nu))^{*C_{n,n}},$$

where for $C_{j,n} < 0$ the corresponding convolution power means the reciprocal of the corresponding infinitely divisible Fourier transform.

2. Auxiliary results and proofs.

2.1. Random integrals.

Let us recall that for a probability Borel measures μ on $\mathbb{R}^d$ (or on E), its *characteristic function* (Fourier transform) $\hat{\mu}$ is defined as

$$\hat{\mu}(y) := \int_{\mathbb{R}^d} e^{i<y,x>} \mu(dx), \ y \in \mathbb{R}^d, \quad (\text{or} \ \ y \in E')$$

where $< \cdot, \cdot >$ denotes the scalar product; (in case of Banach spaces, $< \cdot, \cdot >$ is the bilinear form on $E' \times E$). Further, the characteristic function of an infinitely divisible probability measure μ admits the following *Lévy-Khintchine representation:*

$$\hat{\mu}(y) = e^{\Phi(y)}, \ y \in \mathbb{R}^d, \quad \text{and the Lévy exponent} \ \ \Phi(y) = i < y, a > -$$
$$\frac{1}{2} < y, Ry > + \int_{\mathbb{R}^d \setminus \{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M(dx), \quad (13)$$

where a is a *shift vector*, R is a *covariance operator* corresponding to the Gaussian part of μ, $B := \{x : ||x|| \leq 1\}$ (the unit ball) and M is a *Lévy spectral measure*. (We add the term *spectral* to avoid the possible confusion with *Lévy measures* as sometimes are called the selfdecomposable (class L) measures.) Since there is a one-to-one correspondence between measures $\mu \in ID$ and triples a, R and M in its Lévy-Khintchine formula (13) we will formally write $\mu = [a, R, M]$.

Note that for $s \in \mathbb{R}$ we have

$$\Phi(sy) = i < y, s(a + \int_{E\setminus\{0\}} x(1_B(sx) - 1_B(x))M(dx)) > -\frac{1}{2}s^2 < y, Ry > + \int_{E\setminus\{0\}} [e^{i<y,z>} - 1 - i < y, z > 1_B(z)]M(s^{-1}dz) \quad (14)$$

Finally, let us recall that

$$M \text{ is Lévy spectral measure on } \mathbb{R}^d \text{ iff } \int_{\mathbb{R}^d} \min(1, ||x||^2)M(dx) < \infty \quad (15)$$

For infinity divisibility on Banach spaces we refer to the monograph by Araujo and Giné (1980) or Linde (1983) or Ledoux and Talagrand (1991). Let us emphasize here that the characterization (15) of Lévy spectral measures is NOT true on infinite dimensional Banach spaces ! However, it holds true on Hilbert spaces; cf. Parthasarathy (1967), Chapter VI.

For this note it is important to have the following technical result:

Lemma 2. *If the random integral* $A \equiv \int_{(a,b]} h(t)dY_\nu(r(t))$ *exists then we have*

$$\log \widehat{\mathcal{L}(A)}(y) = \int_{(a,b]} \log \widehat{\mathcal{L}(Y_\nu(1))}(h(s)y)dr(s) = \int_{(a,b]} \Phi(h(s)y)dr(s),$$

where $y \in \mathbb{R}^d$ *(or* E'*) and* Φ *is the Lévy exponent of* $\widehat{\mathcal{L}(Y_\nu(1))} = \hat{\nu}$. *In particular, if* r *is the cumulative probability distribution function of a random variable* T *concentrated of the interval* $(a, b]$ *then* $\log \widehat{\mathcal{L}(A)}(y) = \mathbb{E}[\Phi(h(T)y)]$.

The formula in Lemma 2 is a straightforward consequence of our definition (integration by parts) of the random integrals ($\star\star$). The proof is analogous to that in Jurek-Vervaat(1983), Lemma 1.1 or Jurek (1988), Lemma 3.2 (b).

Remark 3. Note that for bounded intervals $(a, b] \subset \mathbb{R}^+$, positive monotone functions r and real-valued continuous, bounded variation functions h, the integrals of the form A in Lemma 2 are well-defined.

2.2. Proof of Lemma 1. For each $1 \le i \le n$, the polynomials Q_i, of $n-1$ degree, given by the following

$$Q_i(z) := \prod_{k=1;k\neq i}^{n} \frac{z_k - z}{z_k - z_i} \tag{16}$$

satisfy the conditions

$$Q_i(z_i) = 1, \quad Q_i(z_j) = 0 \text{ for } \; 1 \le i \neq j \le n\,.$$

Consequently, the function

$$\mathbf{Q}(z) := \sum_{i=1}^{n} Q_i(z) - 1 \;\text{ is vanishing in n points } z_1, z_2, ..., z_n. \tag{17}$$

Since $\mathbf{Q}$ is a polynomial of n-1 degree we conclude that $\mathbf{Q}(z) \equiv 0$ which completes the proof of Lemma 1.

Remark 4. *It might be of an additional interest to recall here that in the interpolation theory for given set of points* (x_0, y_0), $(x_1, y_1), ..., (x_n, y_n)$ *in the plane* $\mathbb{R}^2$, *with distinct* $x_0, x_1, ..., x_n$,

$$P_n(x) := \sum_{j=0}^{n} y_j \prod_{k\neq j,k=0}^{n} \frac{x_k - x}{x_k - x_j},$$

is the unique Lagrange interpolating polynomial of degree less or equal $n-1$ *and such that*

$$P_n(x_i) = y_i \;\; \textit{for all} \;\; i = 0, 1, 2, ..., n;$$

cf. Kincaid and Cheney (1996), Chapter 6. Thus for the particular points $(x_0, 1), (x_1, 1), ..., (x_n, 1)$ *in* $\mathbb{R}^2$ *we get the line* $y = P_n(x) = 1$ *as the Lagrange interpolating polynomial.*

2.3. Products of independent uniformly distributed random variables.

Here are some elementary identities concerning the products of powers of independent uniformly distributed random variables. The main objective is to express the cumulative distribution function (c.d.f.) or probability density function (p.d.f.) of such products as a linear combinations (with not necessary positive coefficients) of other c.d.f. (or p.d.f.).

Lemma 3. *Let U_i, $1 \leq i \leq n$ be i.i.d. uniformly distributed over the interval $(0,1]$, $\alpha_i > 0$ and let $f_{\{\alpha_1,\alpha_2,...,\alpha_n\}}$ and $F_{\{\alpha_1,\alpha_2,...,\alpha_n\}}$ denote the probability density and cumulative distribution function of $U_1^{1/\alpha_1} \cdot U_2^{1/\alpha_2} \cdot ... \cdot U_n^{1/\alpha_n}$, respectively. Then*

(a) $f_{\{\alpha_1,\alpha_2,...,\alpha_n\}}(x_n) =$

$$\alpha_n\, x_n^{\alpha_n-1} \int_{x_n}^{1} \alpha_{n-1}\, x_{n-1}^{\alpha_{n-1}-\alpha_n-1} \int_{x_{n-1}}^{1} \alpha_{n-2}\, x_{n-2}^{\alpha_{n-2}-\alpha_{n-1}-1} \int_{x_{n-2}}^{1} \alpha_{n-3}\, x_{n-3}^{\alpha_{n-3}-\alpha_{n-2}-1} ...$$

$$... \int_{x_3}^{1} \alpha_2\, x_2^{\alpha_2-\alpha_3-1} \int_{x_2}^{1} \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1\, dx_2 ... dx_{n-2}\, dx_{n-1},\ \ 0 < x_n \leq 1. \quad (18)$$

(b) If $\alpha_1 = \alpha_2 = ... = \alpha_m = \alpha$ then

$$f_{\{\underbrace{\alpha, ..., \alpha}_{m-times}\}}(x) = \alpha\, x^{\alpha-1}\, \frac{(-\alpha\, \log x)^{m-1}}{(m-1)!} \quad \textit{for}\ \ 0 < x \leq 1, \quad (19)$$

$$F_{\{\underbrace{\alpha, ..., \alpha}_{m-times}\}}(s) = s^{\alpha} \sum_{j=0}^{m-1} \frac{(-\alpha \log s)^j}{j!} \quad \textit{for}\ \ 0 < s \leq 1, \quad (20)$$

and $F_{\{\underbrace{\alpha, ..., \alpha}_{m-times}\}}(s) = 1$ for $s \geq 1$ and zero for $s < 0$.

(c) If all positive reals α_i $(i = 1, 2, ..., n)$ are distinct and

$$C_{j,n}(\alpha_1, ..., \alpha_n) \equiv C_{j,n} := \prod_{k \neq j\,, k=1}^{n} \frac{\alpha_k}{\alpha_k - \alpha_j} \quad \textit{and}\ \ c_{j,n} := \prod_{k \neq j\,, k=1}^{n} \frac{1}{\alpha_k - \alpha_j} \quad (21)$$

then

$$f_{\{\alpha_1,\alpha_2,...,\alpha_n\}}(x) = \sum_{j=1}^{n} C_{j,n}\, \alpha_j x^{\alpha_j-1} = \alpha_1 ... \alpha_n \sum_{j=1}^{n} c_{j,n} x^{\alpha_j-1},\ \ 0 < x \leq 1,\ (22)$$

and

$$F_{\{\alpha_1,\alpha_2,...,\alpha_n\}}(s) = \sum_{j=1}^{n} C_{j,n}\ s^{\alpha_j},\ \textit{for}\ 0 < s \leq 1, \quad \textit{where} \sum_{j=1}^{n} C_{j,n} = 1. \quad (23)$$

Proof. For positive and independent rv X and Z with p.d.f. f_X and f_Z, respectively we have that $X \cdot Z$ has the p.d.f.

$$f_{X \cdot Z}(z) = \int_0^{\infty} f_X(\frac{z}{x}) \frac{1}{x} f_Z(x)\, dx\,. \quad (24)$$

Since $f_{\{\alpha\}}(x) = \alpha\, x^{\alpha-1}\, 1_{(0,1)}(x)$ is the p.d.f. of $U^{1/\alpha}$ therefore from (24) we get

$$f_{\{\alpha,\beta\}}(z) = f_{U^{1/\alpha}.U^{1/\beta}}(z) = \beta \int_z^1 f_{\{\alpha\}}(\frac{z}{x}) x^{\beta-2} dx = f_{\{\beta\}}(z) \int_z^1 f_{\{\alpha\}}(t)\, t^{-\beta}\, dt. \tag{25}$$

Hence for α_1 and α_2 we conclude that

$$f_{\{\alpha_1,\alpha_2\}}(x_2) = \alpha_2\, x_2^{\alpha_2-1} \int_{x_2}^1 \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1 \tag{26}$$

which is indeed of the form (18) for $n = 2$.

Assume, by the mathematical induction argument, that the formula (18) holds true for n. Then using (24) and (25) we obtain

$$f_{\{\alpha_1,\alpha_2,...,\alpha_n,\alpha_{n+1}\}}(x_{n+1}) = f_{\alpha_{n+1}}(x_{n+1}) \int_{x_{n+1}}^1 f_{\{\alpha_1,\alpha_2,...,\alpha_n\}}(x_n)\, x_n^{-\alpha_{n+1}} dx_n =$$

$$\alpha_{n+1} x_{n+1}^{\alpha_{n+1}-1} \int_{x_{n+1}}^1 \Big(\alpha_n\, x_n^{\alpha_n - 1} \int_{x_n}^1 \alpha_{n-1}\, x_{n-1}^{\alpha_{n-1}-\alpha_n-1} \int_{x_{n-1}}^1 \alpha_{n-2}\, x_{n-2}^{\alpha_{n-2}-\alpha_{n-1}-1}$$

$$... \int_{x_3}^1 \alpha_2\, x_2^{\alpha_2-\alpha_3-1} \int_{x_2}^1 \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1\, dx_2 \ldots dx_{n-2}\, dx_{n-1} \Big)\, x_n^{-\alpha_{n+1}} dx_n =$$

$$\alpha_{n+1} x_{n+1}^{\alpha_{n+1}-1} \int_{x_{n+1}}^1 \alpha_n\, x_n^{\alpha_n-\alpha_{n+1}-1} \int_{x_n}^1 \alpha_{n-1}\, x_{n-1}^{\alpha_{n-1}-\alpha_n-1} \int_{x_{n-1}}^1 \alpha_{n-2}\, x_{n-2}^{\alpha_{n-2}-\alpha_{n-1}-1}$$

$$... \int_{x_3}^1 \alpha_2\, x_2^{\alpha_2-\alpha_3-1} \int_{x_2}^1 \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1\, dx_2 \ldots dx_{n-1}\, dx_n,$$

which is the equality (18) for $n+1$. Thus the proof of the part (a) is complete.

Taking in (18), $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ and performing the successive integrations, we get the formula (19). (Or simply prove (19) by the induction argument utilizing (24)). Integrating p.d.f. (19) we get c.d.f. (20) and this establishes the part (b).

In the part(c), formulae (22) and (23) are obvious for n=1. Assume that (22) holds true for n. First, from Lemma 1 formula (6) we infer that for $1 \le j \le n$ we get

$$c_{j,n}(\alpha_{n+1} - \alpha_j)^{-1} = c_{j,n+1}, \qquad \sum_{j=1}^{n} (\alpha_j - \alpha_{n+1})^{-1} c_{j,n} = c_{n+1,n+1}\ . \tag{27}$$

Then from (25), (22) and (27) and again (6) from Lemma 1 we get

$$f_{\{\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_{n+1}\}}(x) = \alpha_{n+1}x^{\alpha_{n+1}-1}\int_x^1 f_{\{\alpha_1,\alpha_2,\ldots,\alpha_n\}}(t)\,t^{-\alpha_{n+1}}dt$$

$$= \alpha_1\ldots\alpha_{n+1}\sum_{j=1}^{n} c_{j,n}\; x^{\alpha_{n+1}-1}\int_x^1 t^{\alpha_j-\alpha_{n+1}-1}dt$$

$$= \alpha_1\ldots\alpha_{n+1}\sum_{j=1}^{n} c_{j,n}\frac{1}{\alpha_j-\alpha_{n+1}}(x^{\alpha_{n+1}-1}-x^{\alpha_j-1})$$

$$= \alpha_1\ldots\alpha_{n+1}\sum_{j=1}^{n} c_{j,n+1}x^{\alpha_j-1} + \alpha_1\ldots\alpha_{n+1}\big(\sum_{j=1}^{n} c_{j,n}\frac{1}{\alpha_j-\alpha_{n+1}}\big)x^{\alpha_{n+1}-1}$$

$$= \alpha_1\ldots\alpha_{n+1}(\sum_{j=1}^{n} c_{j,n+1}x^{\alpha_j-1}+c_{n+1,n+1}\; x^{\alpha_{n+1}-1}) = \alpha_1\ldots\alpha_{n+1}\sum_{j=1}^{n+1} c_{j,n+1}x^{\alpha_j-1},$$

which completes the proof of (22) and consequently of (23). This completes the proof of Lemma 3.

2.4. Proof of Theorem 1.

In view of Lemma 2, to prove formula (2) it is necessary and sufficient to show the equality

$$\log\big((\mathcal{J}^{\{\beta_1,\ldots,\beta_n\}}(\nu))\widehat{\ }\big)(y) = \mathbb{E}[\log\hat{\nu}(U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot\ldots\cdot U_n^{1/\beta_n}y)]$$

$$= \int_0^1 \log\hat{\nu}(ty)dr_{\{\beta_1,\ldots,\beta_n\}}(t), \quad y\in E'. \tag{28}$$

Of course, (28) holds for $n=1$. Assume it is true for $n-1$. Then from Lemma 2 and the definition of the mapping $\mathcal{J}^{\{\beta_1,\ldots,\beta_n\}}$ (given before Theorem 1) we get

$$\log(\mathcal{J}^{\{\beta_1,\ldots,\beta_{n-1}\beta_n\}}(\nu))\widehat{\ }(y) = \int_0^1 \log(\mathcal{J}^{\{\beta_1,\ldots,\beta_{n-1}\}}(\nu))\widehat{\ }(s\,y)ds^{\beta_n} =$$

$$\int_0^1 \mathbb{E}[\log\hat{\nu}(U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot\ldots\cdot U_{n-1}^{1/\beta_{n-1}}sy)]ds^{\beta_n} = \mathbb{E}[\log\hat{\nu}(U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot\ldots\cdot U_n^{1/\beta_n}y)]$$

$$= \int_0^1 \log\hat{\nu}(sy)dr_{\{\beta_1,\ldots,\beta_{n-1},\beta_n\}}(t), \tag{29}$$

which completes proof of (28) and consequently the formulae (2) and (3). Explicit expressions for time changes $r_{\{\beta_1,\ldots,\beta_{n-1},\beta_n\}}(t)$ are given in Lemma 3, part (c).

2.5. Proof of Theorem 2.

First, putting $\Phi(y) = \log \hat{\nu}(y)$ (the Lévy exponents of ν) into (28) we get

$$i < y, a^{\{\beta_1,\ldots,\beta_n\}} > -\frac{1}{2} < y, R^{\{\beta_1,\ldots,\beta_n\}}y >$$
$$+ \int_{\mathbb{R}^d\setminus\{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M^{\{\beta_1,\ldots,\beta_n\}}(dx)$$
$$= \mathbb{E}[\Phi(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n} y)]$$

Since for $0 < s \leq 1$, we have that $1_B(sx) - 1_B(x) = 1_{\{1<||x||\leq s^{-1}\}}(x)$ therefore from the above and (14) we get

$$a^{\{\beta_1,\ldots,\beta_n\}} = \mathbb{E}[U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot\ldots\cdot U_n^{1/\beta_n}(a+\int_{1<||x||\leq(U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot\ldots\cdot U_n^{1/\beta_n})^{-1}} xM(dx))]$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j}\, a + \int_0^1 \int_{1<||x||\leq s^{-1}} sx\, M(dx) dr_{\{\beta_1,\ldots,\beta_n\}}(s) \quad \text{(by (3) and (4))}$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j}\, a + \beta_1\beta_2\ldots\beta_n \sum_{j=1}^{n} c_{j,n} \int_0^1 \int_{1<||x||\leq s^{-1}} sx\, M(dx) s^{\beta_j -1} ds \quad \text{(by (8))}$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j}\, a + \sum_{j=1}^{n} \frac{\beta_j}{\beta_j+1} (\prod_{k\neq j,k=1}^{n} \frac{\beta_k}{\beta_k-\beta_j})\, b_{M,\beta_j} \quad \text{(by (7) with z = -1)}$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j} [\sum_{j=1}^{n} (a + b_{M,\beta_j}) \prod_{k\neq j,k=1}^{n} \frac{\beta_k+1}{\beta_k-\beta_j}] = \sum_{j=1}^{n} C_{j,n}\, a^{\{\beta_j\}},$$

which proves the formula for the shift vector.

Similarly, for the Gaussian part, using again the identity (7) (with $z_j = \beta_j$ and $z = -2$) we get

$$R^{\{\beta_1,\ldots,\beta_n\}} = \mathbb{E}[(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n})^2]\, R = \prod_{j=1}^{n} \frac{\beta_j}{\beta_j+2}\, R$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{\beta_j+2} (\sum_{l=1}^{n} \prod_{k\neq l,k=1}^{n} \frac{\beta_k+2}{\beta_k-\beta_l})\, R = \sum_{l=1}^{n} (\prod_{k\neq l,k=1}^{n} \frac{\beta_k}{\beta_k-\beta_l})\, R^{\{\beta_l\}} = \sum_{l=1}^{n} C_{l,n}\, R^{\{\beta_l\}},$$

which gives the formula for Gaussian covariance.

Finally for the Lévy spectral measure using (28), (4) and (9) we have

$$M^{\{\beta_1,\ldots,\beta_n\}}(A) = \mathbb{E}[M((U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n})^{-1}A)]$$

$$= \int_0^1 T_s M(A) dr_{\{\beta_1,\ldots,\beta_n\}}(s) = \beta_1\beta_2 \ldots \beta_n \sum_{j=1}^{n} c_{j,n} \int_0^1 M(s^{-1}A)\, s^{\beta_j - 1} ds$$

$$= \sum_{j=1}^{n} (\prod_{k \neq j, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_j}) \int_0^1 M(s^{-1/\beta_j}A)\, ds = \sum_{j=1}^{n} C_{j,n}\, M^{\{\beta_j\}}(A),$$

which completes the proof of Theorem 2.

3. Concluding remarks. Although generalized s-selfdecomposable distributions are known for $\beta < 0$ analogous calculus on the corresponding random integrals seems to be more complicated. Similarly, Theorem 1 for not necessarily all different $\beta' s$ might be much more involved and is not discussed in this note.

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