A NOTE ON GAMMA RANDOM VARIABLES AND DIRICHLET SERIES*

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ABSTRACT. We give necessary and sufficient conditions for convergence of series of centered gamma random variables. Those series provide distributions from Lévy class L of the selfdecomposable probability mesures. Relations to Dirichlet series and the background driving Lévy processes (BDLP's) are investigated.

Keywords: Gamma distributions; infinite divisibility; selfdecomposability or class L distributions; Lévy processes; random integrals; Dirichlet series, Thorin distributions.

1. The gamma random variable and its selfdecomposability property.

The well-known *gamma random variables* (rv) $\gamma_{\alpha,\lambda}$ are defined by their densities

(1) $$\lambda^\alpha/\Gamma(\alpha)x^{\alpha-1}\exp(-\lambda x)1_{(0,\infty)}(x),$$

where $\alpha > 0$, $\lambda > 0$ are called *shape* and *scale parameters*, respectively, and $\Gamma(z)$ is the Euler gamma function $\Gamma(z) := \int_0^\infty x^{z-1}\exp(-x)dx, \quad \Re z > 0$, satisfying the equation $z\Gamma(z) = \Gamma(z+1)$. Of course, $\gamma_{1,\lambda}$ is the exponential rv. (Note: in sequel we use the same notation for gamma rv and gamma probability density.) Furthermore, it can be easily seen that

(2) $$\mathbb{E}[\gamma_{\alpha,\lambda}] = \alpha/\lambda; \quad \mathbb{E}[\gamma_{\alpha,\lambda}^2] = \alpha(\alpha+1)/\lambda^2; \quad \gamma_{\alpha,\lambda} \stackrel{d}{=} \lambda^{-1}\gamma_{\alpha,1};$$

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and also that its charateristic function is

$$(3) \quad \mathbb{E}[\exp(it\gamma_{\alpha,\lambda})] = (1-it/\lambda)^{-\alpha} = \exp\left[\int_0^\infty (e^{itx}-1)\alpha e^{-\lambda x}/x dx\right], t \in \mathbb{R},$$

where $\mathbb{E}[\cdot]$ denotes the expected value with respect to probability $\mathcal{P}$, and $\stackrel{d}{=}$ menas equality in distribution. From (3) we see that

$$(4) \quad \gamma_{\alpha,\lambda} \quad \textit{is infinitely divisible and its Lévy spectral measure} M$$
$$(\textit{in the Lévy-Khintchine formula}) \textit{ has density } \alpha e^{-\lambda x}/x 1_{(0,\infty)}(x).$$

Recall also here that the class ID of all infinitely divisible distributions is a closed convolution semigroup (in weak topology) and it coincides with the class of all processes Y with stationary and independent increments starting from 0, with probability 1. If additionally Y has cadlag paths we refer to it as *a Lévy process.*

For $0 < c < 1$ let us define

$$\psi_c(t) := (1-ict)/(1-it) = c1 + (1-c)/(1-it), \quad t \in \mathbb{R},$$

which by (3) is a charateristic function (convex combination of 1 and characteristic function of $\gamma_{1,1}$). More precisely, ψ_c is ID with Lévy spectral measure $x^{-1}(\exp(-x) - \exp\{-c^{-1}x\})1_{(0,\infty)}(x)dx$. Consequently, for each $0 < c < 1$ we have a factorization $\hat{\gamma}_{\alpha,\lambda}(t) = \hat{\gamma}_{\alpha,\gamma}(ct)\psi_c^\alpha(t/\lambda)$ (the selfdecomposability property). Consequently $\hat{\gamma}_{\alpha,\lambda}$ belongs to class L of *all selfdecomposable characteristic functions* defined as follows

$$(5) \quad \phi \in L \quad \text{iff} \quad \forall (0 < c < 1) \exists (\text{char.f. } \phi_c) \ \phi(t) = \phi(ct)\phi_c(t).$$

The class L is a closed subsemigroup in ID and coincides with the limiting distributions of normalized partial sums of independent but not necessarily identically distributed rv's. In terms of random rv's selfdecomposability means that rv X has probability distribution (or char. f.) in class L iff $X \stackrel{d}{=} cX + X_c$, for each $0 < c < 1$, where X_c is independent of X; cf. [7] Chapter 3. Stable laws are the primary examples of class L distributions but it is much larger and includes many classical distributions from mathematical statistics like t-Student, F-Fisher, log-normal, etc.; Cf.[5] and [6] for other examples. The fundamental characterization of the class L is via *the integral functionals of some Lévy processes.* That fact is crucial for the consideration in this paper. Namely, for class L distribution or rv's the following *random integral representation* (RIR) holds true:

$$(6) \quad X \in L \quad \text{iff} \quad X \stackrel{d}{=} \int_0^\infty e^{-s} dY(s),$$

where Y is a Lévy process unique (in distribution) with finite logarithmic moment $\mathbb{E}[\log(1 + |Y(1)|)] < \infty$. One referes to Y as the BDLP for X (background driving Lévy process). If ϕ is char. f. of X and ψ is char. f. of $Y(1)$ then (6) equivalently reads as follows

$$(7) \quad \log \phi(t) \in L \quad \text{iff} \quad \log \phi(t) = \int_0^1 \log \psi(tu) u^{-1} du,$$

cf. [5], [6], [7]. For the gamma rv we have that its BDLP Y is the compound Poisson process $Y(t) = \sum_{n=1}^{\overline{N}_\alpha(t)} \gamma_{1,\lambda}^{(n)}$, where $\gamma_{1,\lambda}^{(n)}$, $n = 1, 2, \ldots$ are independent copies of gamma rv $\gamma_{1,\lambda}$ and $\overline{N}_\alpha(t)$ is Poisson process with intensity α; cf. [5], formula (11) and [6] formula (2.2) (note the misprints!). Consequently $Y(1)$ has compound Poisson distribution with (finite) Lévy spectral measure N with density

$$(8) \quad dN(x) = \alpha\lambda \exp(-\lambda x) 1_{(0,\infty)}(x) dx$$

[do not confuse here a Lévy spectral measure N with Poisson process $\overline{N}(t)$!].

All in all we have the following:

PROPOSITION 0. *Gamma rv's $\gamma_{\alpha,\lambda}$ are are selfdecomposable without Gaussian part and admit representations:*

$$(9) \quad \hat{\gamma}_{\alpha,\lambda}(t) = (1 - it/\lambda)^{-\alpha} = \exp\left\{ \int_0^\infty (e^{itx} - 1) \alpha e^{-\lambda x}/x dx \right\}$$
$$= \exp\left\{ \int_0^1 \left[\int_0^\infty (e^{itux} - 1) \alpha\lambda e^{-\lambda x} dx \right] du/u \right\}.$$

This summurizes the study of a single gamma variable. In next section we will consider sequences of independent gamma random variables.

2. Series of independent gamma rv's.

In the sequel $\gamma_{\alpha_n,\lambda_n}$ denotes a sequence of *independent* gamma rv's. By (2) they belong to $L^2(\Omega, \mathcal{F}, \mathcal{P})$, i.e., to the Hilbert space of square integrable rv's with the usual L^2 norm. Thus $X_n \to X$ in L^2 if $\mathbb{E}[|X_n - X|^2] \to 0$. Consequently we get:

PROPOSITION 1. *A series $\sum_{n=1}^{\infty}(\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n)$ of centered gamma rv's converges in $L^2(\Omega, \mathcal{F}, \mathcal{P})$ iff $\sum_{n=1}^{\infty} \alpha_n/\lambda_n^2 < \infty$.*

Proof. From the stochastic independence and (2) we get that

$$(10)\quad \mathbb{E}\Big(\sum_{n=k}^{m}(\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n)\Big)^2 = \sum_{n=k}^{m} \alpha_n/\lambda_n^2$$

and therefore the completness of $L^2(\Omega, \mathcal{F}, \mathcal{P})$ gives the proof. □

PROPOSITION 2. (1) *If $\sum_n \alpha_n/\lambda_n^2 < \infty$ then $\sum_n(\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n)$ converges in distribution.*

(2) *Let denote $\lambda_\infty := \lim_{n\to\infty} \lambda_n$ and assume that $\sum_n(\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n)$ converges in distribution. Then $\sum_n \alpha_n/\lambda_n^2 < \infty$, if $\lambda_\infty > 0$; and $\sum_n \alpha_n < \infty$, if $\lambda_\infty = 0$.*

Proof. Since L^2 convergence implies convergence in probability which in turn implies convergence in distribution therefore part (1) follows from Proposition 1. For part (2), let us observe that by Proposition 0 and formula (9), if S is the sum of series in question, then it has Lévy spectral measure

$$(11)\quad dM(x) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n x}/x 1_{(0,\infty}(x)dx.$$

(in particular the series converges for all $x > 0$). Since Lévy spectral measures integrate x^2 in each finite neighbourhood of zero, we have

$$(12)\quad \int_{|x|<1} x^2 dM(x) = \sum_n \alpha_n/\lambda_n^2 \int_0^1 \lambda_n x e^{-\lambda_n x}\lambda_n dx =$$

$$\sum_n \alpha_n/\lambda_n^2 \int_0^{\lambda_n} u e^{-u} du < \infty.$$

Last integral has a limit 2 when $\lambda_n \to \infty$ which allows to conclude the first case in part (2). Similarly, when $\lambda_n \to 0$ then the integral divided by λ_n^2 tends to $1/2$ which justifies the second case. The proof is complete. $\square$

REMARK 1. One may try to prove the previous result by the Kolmogorov Three Series Theorem. However, it leads to series of incomplete gamma functions.

COROLLARY 1. *Let $\alpha_n > 0$, $\lambda_n > 0$ be such that $\inf_n \lambda_n > 0$. Then for*

$$S_m := \sum_{n=1}^{m} (\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n)$$

the following are equivalent:

(a) $\sum_n \alpha_n/\lambda_n^2$ *converges;*

(b) (S_m) *converges in* $L^2(\Omega, \mathcal{F}, \mathcal{P})$*;*

(c) (S_m) *converges in distribution;*

(d) *series* $h(x) := \sum_n \alpha_n e^{-\lambda_n x}$ *converges for all* $x > 0$*, and the integral* $\int_0^1 x h(x) dx < \infty$.

Proof. First of all note that for the claim in Proposition 2 (ii), it is enough to know that (λ_n) are bounded away from zero. Further, (a) implies (b) by Proposition 1. (b) always implies (c). If S is the limit of S_m in distribution then, by Proposition 0, $h(x)/x 1_{(0,\infty)}(x)$ is the density of Lévy spectral measure of S. Thus it integrates x^2 in any finite neighourhood of zero. Thus (c) implies (d). The integrability condition in (d) via (12) implies (a). $\square$

COROLLARY 2. *For the sequence of partial sums*

$$S_m := \sum_{n=1}^{m} (\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n),$$

of independent and centered gamma rv with $\inf_n \lambda_n > 0$ *the following conditions are equivalent:*

(1) (S_m) *converges a.s.;*

(2) (S_m) *converges in probability;*

(3) (S_m) *converges in distribution;*

(4) (S_m) *converges in* $L^2(\Omega, \mathcal{F}, \mathcal{P})$*;*

(5) $M := sup_m|S_m| \in L^2(\Omega, \mathcal{F}, \mathcal{P})$*;*

(6) $K := sup_n|\gamma_{\alpha_n,\lambda_n} - \alpha_n/\lambda_n| \in L^2(\Omega, \mathcal{F}, \mathcal{P})$.

Proof. Of course , the first three conditions are equivalent for *any* series with independent summands, due to Lévy Theorem; cf. for instance [2], Theorem.2.10. (3) is equivalent to (4) by Corollary 1. Finally (4),(5) and (6) are equivalent by Hoffmann-Jorgensen Theorem; cf. Thm 2.11 in [2]. □

3. Dirichlet series.

In complex analysis and notably in the analytic number theory an important role is played by the so called *generalized Dirichlet series.* These are series of the following form:

$$(13)\quad w(z) := \sum_n a_n e^{-\lambda_n z}, \quad \Re z > s_c,$$

where the coefiicients a_n are complex numbers and the exponents

$$0 < \lambda_1 < \lambda_2 < \ldots \to \infty.$$

Of course, Dirichlet series can be viewed as a generalization of power series and as a special case of the Laplace transform. For $\lambda_n = \log n$ we obtain *ordinary Dirichlet series* and in particular Riemann zeta function. For our purpose here let us recall that (13) converges in half planes, i.e., there exist s_c (possible $\pm\infty$) such that (13) converges for all $z \in \mathbb{C}$ with $\Re z > s_c$ and diverges for all z with $\Re z < s_c$. Line $\Re z = s_c$ is referred to as *the abscisa of convergence* of Dirichlet series. Functions $w(z)$ are holomorphic in the half plane of convergence. Analogously one defines the line $\Re z = s_a$ as *the abscissa of absolute convergence*; cf. for more details [1] Chapter 8, [8] Chapter 9 or the original work of Harald Bohr in [3] on Dirichlet series and almost periodic functions.

COROLLARY 3. *If series* $\sum_n |a_n|/\lambda_n^2 < \infty$ *converges then generalized Dirichlet series* $\sum_n a_n e^{-\lambda_n z}$ *converges absolutely for* $\Re z > 0$*. If additionally series* $\sum_n |a_n|$ *diverges then* $\lim_{m\to\infty} \left(\sum_{n=1}^m |a_n|\right)^{1/\lambda_m} = 1$.

Proof. First part follows from Corollary 1 (a) and (d). The second we get from [1] Thm 8.3. □

Of course, we have similar statements for ordinary Dirichlet series, i.e. when $\lambda_n = \log n$.

REMARK 2. Note that having absolutely converging Dirichlet series $w(z)$ we have class L distributions with $w(x)/x1_{(0,\infty)}(x)$ as the density of the Lévy spectral measure M provided the integrabilty condition in Corollary 1(d) is satisfied. Moreover, $-w'(x)$ is the density of the corresponding measure N from BDLP Y (N is the Lévy spectral measure of $Y(1)$). Conversely, converging series of centered gamma rv's provide examples of summable Dirichlet series in $\Re z > 0$. Cf.[5] for some examples.

4. Thorin's class $\mathcal{T}$. *Generalized gamma distributions* (called also Thorin's class $\mathcal{T}$-distribution) are defined via a formula for the moment generating function; see [4], formula (3.1.1) on page 29. For our purpose here it is enough to quote the following fact:

(14) *$\mathcal{T}$ is the smallest class of distributions on $\mathbb{R}^+$ that contains (shifted) gamma distributions and is closed with respect to convolutions and weak limits* ;

cf.[4],Theorem 3.1.5. Furthermore, so called *Pick functions* (complex analysis) are the main analytic tool. These are functions $\psi(z)$ that are analytic in the upper complex half-plane $\Im z > 0$ and have nonnegative imaginary part there; cf.[4], page 20 for more details. From our point of view we obviously have that:

(15) *converging series of centered gamma variables have distributions in $\mathcal{T}$*;

their Lévy spectral measures have densities given by Dirichlet series of the form (13) or their BDLP Y are infinite series of compound Poisson processes. To conclude what part of class $\mathcal{T}$ we get from week limits of infinite series of centered gamma random variables we need to describe limits of sequences of Dirichlet series of the from (13). One can also ask when densities $h(x)$, $x > 0$, of Lévy spectral measures M (in class L) admit expension in form of a Dirichlet series (13), with coefficients λ_n not neccesarily increasing to infinity. Equivalently, one may ask which holomorphic functions $f(z)$, in half plane

$\Re z > 0$ with $h(x) > 0$, for $x > 0$, and satisfying the integrability condition of Lévy spectral mesures, have representation in terms of Dirichlet series. It seems that the original works of Harald Bohr on almost periodic functions and Dirichlet series are still the best reference when one tries to answer those questions.

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