Proc. 3rd Internatl Sympos. on Multivar. Analysis, Wright State Univ., Dayton, June 19–24, 1972

Reprinted from:
MULTIVARIATE ANALYSIS – III

ACADEMIC PRESS, INC.

Limit Laws for Sequences of Normed Sums Satisfying Some Stability Conditions

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In this paper we discuss the limit laws arising from normed sums of independent random variables satisfying some stability conditions. These are, roughly speaking, sequences for which the limit properties of suitably normed sums are similar to those for sequences of identically distributed random variables. The result we obtain is an analogue of the Lévy–Khinchin representation of infinitely divisible laws. The present investigation arose from a study of self-decomposable probability measures.

Throughout this paper we denote by P the set of all probability measures on the real line. With the topology of weak convergence and multiplication defined by the convolution, P becomes a topological semigroup. We denote the convolution of two measures λ and μ by $\lambda * \mu$. Moreover, by δ_a we denote the probability measure concentrated at the point a. Further, for any real number a $(a \neq 0)$ and any measure μ from P we denote by $a\mu$ the measure defined by the formula $a\mu(E) = \mu(a^{-1}E)$ for all Borel subsets E of the real line. The characteristic function $\hat{\mu}$ of a measure $\mu \in P$ is defined by the formula

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx}\mu(dx).$$

It is easy to check the equation $\widehat{a\mu}(t) = \hat{\mu}(at)$.

By a triangular array we shall understand a system

$$\begin{array}{llll} X_{11} & & & \\ X_{12}, & X_{22} & & \\ \vdots & \vdots & & \\ X_{1n}, & X_{2n}, & \ldots, & X_{nn} \\ \vdots & \vdots & & \vdots \quad \ldots \end{array}$$

of random variables such that $X_{1n}, X_{2n}, \ldots, X_{nn}$ are independent and for every $c > 0$

$$\lim_{n\to\infty} \max_{1\le k\le n} P(|x_{kn}| > cn) = 0.$$

Further, we say that a triangular array $\{X_{kn}\}$ is generated by a sequence $\{X_n\}$ of random variables if $X_{kn} = X_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$.

A probability measure is said to be a limit distribution of the triangular array $\{X_{kn}\}$ if it is the weak limit of the sequence of probability distributions of normed sums $(1/n)\sum_{k=1}^{n} X_{kn} - a_n$ for suitably chosen constants a_n. It is obvious that the limit distribution of $\{X_{kn}\}$ is defined uniquely up to a shift transformation. Moreover, this limit distribution is infinitely divisible (see [2], p. 309). Further, we call two triangular arrays equivalent if they have the same limit distribution. In particular, $\{X_{kn}\}$ is equivalent to an array generated by a sequence $\{X_n\}$ if and only if for suitably chosen a_n and b_n the sequences of normed sums

$$\frac{1}{n}\sum_{k=1}^{n} X_{kn} - a_n \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} X_k - b_n$$

have the same limit distribution.

We define classes S_m $(m = 0, 1, \ldots)$ of sequences $\{X_n\}$ of independent random variables recursively. Let S_0 be the class of all sequences $\{X_n\}$ generating convergent triangular arrays, i.e., the class of all sequences for which the sequence $(1/n)\sum_{k=1}^{n} X_k - a_n$ with suitably chosen constants a_n has a limit distribution. Further, $\{X_n\} \in S_m$ $(m \ge 1)$ whenever $\{X_n\} \in S_0$ and the following stability condition is fulfilled: for every positive number c the triangular array $X_{kn} = X_{[cn]+k}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ is equivalent to an array generated by a sequence from S_{m-1}. The square brackets here denote the integral part of the real number. It is clear that the classes S_m form a contracting sequence. Put $S_\infty = \bigcap_{m=1}^{\infty} S_m$. The sequences belonging to S_∞ will be called slowly varying. For instance, each sequence of independent identically distributed random variables generating a convergent triangular array belongs to all classes S_m and, consequently, is slowly varying. We get a less trivial example of slowly varying sequences from results of Koroljuk and Zolotarev [1] (see also [5]), namely, each sequence of independent random variables generating a convergent triangular array and such that there are no more than two different distribution laws among the laws of the random variables $X_1, X_2, \ldots$ is slowly varying.

Let L_m $(m = 0, 1, \ldots, \infty)$ be the set of all possible limit distributions of normed sums $(1/n)\sum_{k=1}^{n} X_k - a_n$ where $\{X_n\} \in S_m$ and a_n are constants. The problem of a description of probability measures from L_0 was solved by Lévy, who obtained an explicit representation of the characteristic function of those measures [2, p. 324]. Another characterization of the set L_0 was given [3]. The aim of the present paper is to give a characterization of all sets L_m $(m = 1, 2, \ldots, \infty)$.

It is easy to see that the set L_m is invariant under the shift transformations $\mu \to \mu * \delta_c$ and under the transformations $\mu \to a\mu$. Moreover, it is closed under convolution. We characterize L_m by a decomposability property, and then we characterize the corresponding characteristic functions.

Proposition 1. *A probability measure μ belongs to L_m $(m = 0, 1, \ldots)$ if and only if for every number $a \in (0, 1)$ there exists a measure $\mu_a \in L_{m-1}$ such that $\mu = a\mu * \mu_a$. L_{-1} denotes here the set P of all probability measures.*

Proof. We shall prove Proposition 1 by induction with respect to m. Owing to Lévy's results, it is true for $m = 0$ [2, p. 323]. Suppose that $m \ge 1$ and for indices less than m the statement is true. Consider a measure μ from L_m. Suppose that it is a limit distribution of normed sums $(1/n)\sum_{k=1}^{n} X_k - a_n$ where $\{X_n\} \in S_m$. Given $a \in (0, 1)$, we put

$$Y_n = \frac{1}{n}\sum_{k=1}^{[an]} X_k - \frac{[an]}{n} a_{[an]}, \qquad Z_n = \frac{1}{n}\sum_{k=[an]+1}^{n} X_k + \frac{[an]}{n} a_{[an]} - a_n.$$

It is evident that Y_n, Z_n are independent, and that μ and $a\mu$ are limit distributions of $\{Y_n + Z_n\}$ and $\{Y_n\}$, respectively. Further, taking into account that $a\mu$, being infinitely divisible, has a nonvanishing characteristic function, we infer that the sequence $\{Z_n\}$ also has a limit distribution, say, μ_a. Of course, $\mu = a\mu * \mu_a$. By the assumption the triangular array $X_{kn} = X_{[an]+k}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ is equivalent to an array generated by a sequence from S_{m-1}. Since

$$Z_n = \frac{1}{n}\sum_{k=1}^{r_n} X_{kr_n} - c_n,$$

where $r_n = n - [an]$ and c_n are constants, we infer that its limit distribution μ_a belongs to L_{m-1}, which completes the proof of the necessity of the condition.

Suppose now that μ is a probability distribution satisfying, for every $a \in (0, 1)$, the condition $\mu = a\mu * \mu_a$ where $\mu_a \in L_{m-1}$. We have to prove that $\mu \in L_m$. It is clear that $\mu \in L_0$ and consequently, being infinitely divisible, has a nonvanishing characteristic function. Setting $\nu_1 = \mu$, $\nu_n = n\mu_{(n-1)/n}$ $(n = 2, 3, \ldots)$, we have the formula

$$\hat{\nu}_n(t) = \frac{\hat{\mu}_{(nt)}}{\hat{\mu}((n-1)t)}$$

whence the relation

$$\lim_{n\to\infty} \max_{1\le k\le n} \left|\hat{\nu}_k\left(\frac{t}{n}\right) - 1\right| = 0 \tag{1}$$

follows. Let $\{X_n\}$ be a sequence of independent random variables with probability distributions $\nu_1, \nu_2, \ldots$. By (1) we have the convergence

$$\lim_{n\to\infty} \max_{1\le k\le n} P(|X_k| > cn) = 0$$

for every positive number c. Thus $\{X_n\}$ generates a triangular array. Given a positive number c, we put $X_{kn} = X_{[cn]+k}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$. Let λ_n be the probability distribution of the sum $(1/n)\sum_{k=1}^{n} X_{kn}$. Then

$$\hat{\lambda}_n(t) = \frac{\hat{\mu}(n^{-1}([cn]+n)t)}{\hat{\mu}(n^{-1}[cn]t)}$$

which yields the relation

$$\hat{\lambda}_n(t) \to \frac{\hat{\mu}((1+c)t)}{\hat{\mu}(ct)}.$$

Now taking into account the equation $\hat{\mu}((1+c)t) = \hat{\mu}(ct)\hat{\mu}_{c/(1+c)}((1+c)t)$, we infer that the probability measure $(1+c)\mu_{c/(1+c)}$ is a limit distribution of the array $\{X_{kn}\}$. This probability measure, being an element of L_{m-1}, is, by the induction assumption, a limit distribution of an array generated by a sequence from S_{m-1}. Thus $\{X_{kn}\}$ is equivalent to an array generated by a sequence from S_{m-1} and, consequently, $\{X_n\} \in S_m$. Since μ is the probability distribution of the normed sums $(1/n)\sum_{k=1}^{n} X_k$ $(n = 1, 2, \ldots)$, we have $\mu \in L_m$, which completes the proof.

As a direct consequence of Proposition 1 we get the following

Corollary. *A probability measure μ belongs to L_∞ if and only if every number $a \in (0, 1)$ there exists a probability measure $\mu_a \in L_\infty$ such that $\mu = a\mu * \mu_a$.*

Our next aim is to give a representation of the characteristic functions of probability measures from L_m. The following representation formula was established in [3]: a measure μ is self-decomposable, i.e., belongs to L_0, if and only if its characteristic function is of the form

$$\hat{\mu}(t) = \exp\left\{iqt + \int_{-\infty}^{\infty}\left(\int_0^{tu}\frac{\exp(iv)-1}{v}\,dv - it\arctan u\right)\frac{Q(du)}{\log(1+u^2)}\right\}, \tag{2}$$

where q is a real constant, Q is a finite Borel measure on the real line, and the integrand is defined as its limiting value $-\frac{1}{4}t^2$ when $u = 0$. Moreover, the measure μ determines the couple q, Q uniquely. We shall call Q the spectral measure for μ. It is evident that the convolution of measures there corresponds to the sum of spectral measures of factors. Further, it is easy to check that if Q is the spectral measure for μ, then

$$Q_a = \int_E \frac{\log(1+u^2)}{\log(1+(u/a)^2)}\,Q(a^{-1}\,du) \tag{3}$$

is the spectral measure for a.

$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\,dt$

By L_m^+ $(m = 0, 1, \ldots, \infty)$ we shall denote the subset of L_m consisting of probability measures with vanishing constant q in (2) and whose spectral measure is concentrated on the open right half-line $(0, \infty)$. If $\upsilon \in L_m^+$, then, of course, $(-1)\upsilon \in L_m$ and its spectral measure is concentrated on the half-line $(-\infty, 0)$. Let us introduce the operation $\mu \to \mu^+$, which associates with every measure μ from L_0 having the spectral measure Q a probability measure μ^+ from L_0^+ with the spectral measure $Q^+(E) = Q(E \cap (0, \infty))$. It is easy to verify the equation $(a\mu)^+ = a\mu^+ * \delta_b$ for $a > 0$ where

$$b = \int_0^\infty (\arctan u - \arctan au)\,\frac{Q(du)}{\log(1+u^2)}.$$

Further, it is clear that $(\mu * \lambda)^+ = \mu^+ * \lambda^+$. Thus the relation $\mu = a\mu * \mu_a$ yields the following one $\mu^+ = a\mu^+ * (\mu_a)^+ * \delta_b$. Hence, by a simple induction, we get the relation $\mu^+ \in L_m$ whenever $\mu * L_m$. Suppose that, $\mu \in L_m$ and $\hat{\mu}$ is given by (2). Denoting by μ^0 the Gaussian probability measure with the characteristic function

$$\hat{\mu}^0(t) = \exp(iqt - \tfrac{1}{4}Q(\{0\})t^2),$$

we have the equation

$$\mu = \mu^0 * \mu^+ * (-1)((-1)\mu)^+.$$

It is evident that Gaussian measures belong to L_∞. Consequently, formula (4) reduces the investigation of L_m to that of L_m^+. Moreover, we have the following criterion: $\lambda \in L_m^+$ if and only if for every $a \in (0, 1)$, $\lambda = a\lambda * \gamma_a * \delta_c$ where c is a constant and $\lambda_a \in L_{m-1}^+$ $(m = 1, 2, \ldots, \infty)$.

With every probability measure λ from L_0^+ we associate a function F_λ defined on the real line by means of the formula

$$F_\lambda(x) = \int_{e^x}^\infty \frac{Q(du)}{\log(1+u^2)}. \tag{5}$$

Here Q denotes the spectral measure for λ. It is clear that this correspondence is one-to-one and

$$Q(E) = -\int_E \log(1+x^2)\,dF_\lambda(\log x) \tag{6}$$

for any Borel subset E of $(0, \infty)$. Moreover

$$F_{\lambda * \nu} = F_\lambda + F_\nu \tag{7}$$

and, by (3), for $a > 0$

$$F_{a\lambda}(x) = F_\lambda(x - \log a). \tag{8}$$

First we characterize L_m^+ in terms of the function F_λ.

Proposition 2. *Let $m = 0, 1, \ldots$. A function F is associated with a probability measure λ from L_m^+, i.e., $F = F_\lambda$, if and only if*

$$F(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^m H(dy)}{\log^{m+1}(1 + y^{2/(m+1)})}, \tag{9}$$

where H is a finite Borel measure on $(0, \infty)$.

Proof. We shall prove Proposition 2 by induction with respect to m. For $m = 0$ it is a direct consequence of formula (5). Suppose now that $m > 0$ and that for indices less than m the statement is true. Let λ be a probability measure from L_m^+. Since it belongs to L_{m-1}^+, we have, by the induction assumption,

$$F_\lambda(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} H_0(dy)}{\log^m(1 + y^{2/m})}, \tag{10}$$

where H_0 is a finite Borel measure on $(0, \infty)$. Moreover, for any $a \in (0, 1)$ we have the equation $\lambda = a\lambda * \lambda_a * \delta_c$ where $\lambda_a \in L_{m-1}^+$. Consequently, by the induction assumption,

$$F_{\lambda a}(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} G_a(dy)}{\log^m(1 + y^{2/m})}$$

where G_a are finite Borel measures on $(0, \infty)$. On the other hand, by (7), (8), and (10),

$$F_a(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1}}{\log^m(1 + y^{2/m})} \left(H_0(dy) - \frac{\log^m(1 + y^{2/m})}{\log^m(1 + (y/a)^{2/m})} H_0(a^{-1}\, dy) \right).$$

Hence we get the equation

$$G_a(E) = H_0(E) - \int_E \frac{\log^m(1 + y^{2/m}) H_0(a^{-1}\, dy)}{\log^m(1 + (y/a)^{2/m})}$$

for any Borel subset E of $(0, \infty)$. Consequently,

$$\int_E \frac{H_0(dy)}{\log^m(1 + y^{2/m})} - \int_E \frac{H_0(a^{-1}\, dy)}{\log^m(1 + (y/a)^{2/m})} \geq 0. \tag{11}$$

Put

$$g(x) = \int_{e^x}^{\infty} \frac{H_0(dy)}{\log^m(1 + \lambda^{2/m})}. \tag{12}$$

By (11) for any $a \in (0, 1)$ and $u < v$ we have the inequality

$$g(u) - g(v) - g(u - \log a) + g(v - \log a) \geq 0,$$

which for $u = v + \log a$ yields

$$g(v) \leq \tfrac{1}{2}(g(v - \log a) + g(v + \log a)).$$

Thus the function g is convex and, consequently, can be represented in the form $g(x) = \int_x^\infty h(u)\, du$, where h is a nonnegative monotone nonincreasing function. Further, by (12) we have the formula

$$H_0(E) = \int_E \log^m(1 + y^{2/m}) h(\log y) \frac{dy}{y}.$$

Consequently,

$$\int_0^\infty \log^m(1 + y^{2/m}) h(\log y) \frac{dy}{y} < \infty. \tag{13}$$

Moreover, by (10),

$$F_\lambda(x) = \int_{e^x}^{\infty} (\log y - x)^{m-1} h(\log y) \frac{dy}{y}. \tag{14}$$

By (13) the limit inferior of the function $h(\log y) \log^{m+1}(1 + y^{2/(m+1)})$ at 0 and ∞ is equal to 0. Consequently, integration of (14) by parts gives

$$F_\lambda(x) = -\frac{1}{m} \int_{e^x}^{\infty} (\log y - x)^m\, dh(\log y). \tag{15}$$

Moreover, by (13),

$$-\int_0^\infty \log^{m+1}(1 + y^{2/(m+1)})\, dh(\log y)$$

$$= 2 \int_0^\infty \frac{y^{2/(m+1)}}{1 + y^{2/(m+1)}} \log^m(1 + y^{2/(m+1)}) h(\log y) \frac{dy}{y} < \infty.$$

Thus the measure H defined by means of the formula

$$H(E) = -\frac{1}{m} \int_E \log^{m+1}(1 + y^{2/(m+1)})\, dh(\log y)$$

is finite on $(0, \infty)$. Setting it into (15), we get desired representation (9).

Suppose now that the function F is given by formula (9). We note that F can be written in the form

$$F(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} G(dy)}{\log^m(1 + y^{2/m})}$$

where the measure

$$G(E) = m \int_E \log^m(1 + y^{2/m}) \int_y^\infty \log^{-m-1}(1 + x^{2/(m+1)}) H(dx) \frac{dy}{y}$$

is also finite on $(0, \infty)$. Consequently, by the induction assumption, $F = F_\lambda$ for a measure λ from L_{m-1}^+. In order to prove that $\lambda \in L_m^+$, consider for any

$a \in (0, 1)$ the decomposition $\lambda = a\lambda * \lambda_a$ with $\lambda_a \in L_{m-2}$. Here L_{-1} denotes the set of all probability measures on the real line. By virtue of (8) we have the formula

$$F_\lambda(x) - F_{a\lambda}(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} H_a(dy)}{\log^m(1 + y^{2/m})}$$

where the measure

$$H_a(E) = m \int_E \log^m(1 + y^{2/m}) \int_y^{y/a} \log^{-m-1}(1 + x^{2/(m+1)}) H(dx) \frac{dy}{y}$$

is finite on $(0, \infty)$. Hence, by (7) and the induction assumption, we infer that $\lambda_a = \lambda_a^+ * \delta_c$ where $\lambda_a^+ \in L_{m-1}^+$ and c is a constant. Thus $\lambda_a \in L_{m-1}$, which, by virtue of Proposition 1, shows that $\lambda \in L_m^+$. This completes the proof of Proposition 2.

As an immediate consequence of formula (6) and Proposition 2 we get the following characterization of L_m^+ in terms of the spectral measures.

Proposition 3. *Let $m = 1, 2, \ldots$. A measure Q defined on $(0, \infty)$ is the spectral measure for a probability distribution from L_m^+ if and only if it is of the form*

$$Q(E) = \int_E \log(1 + x^2) \int_x^{\infty} \left(\log \frac{y}{x}\right)^{m-1} \log^{-m-1}(1 + y^{2/(m+1)}) N(dy) \frac{dx}{x} \quad (16)$$

where N is a finite Borel measure on $(0, \infty)$.

Setting this representation of the spectral measure into (2), we obtain the following result.

Proposition 4. *Let $m = 1, 2, \ldots$. A function φ is the characteristic function of a probability measure from L_m^+ if and only if it is of the form*

$$\varphi(t) = \exp \int_0^{\infty} k_m(t, y) N(dy) \quad (17)$$

where N is an arbitrary finite Borel measure on $(0, \infty)$ and the kernel k_m is defined by the formula

$$k_m(t, y) = \frac{1}{(m-1)!} \int_0^y \left(\int_0^{tx} \frac{e^{iu} - 1}{u} du - it \arctan x \right) \times \left(\log \frac{y}{x}\right)^{m-1} \frac{dx}{x} \log^{-m-1}(1 + y^{2/(m+1)}).$$

We note that the function φ determines the measure N in (17). Indeed, by a simple calculation (17) can be transformed into formula (2), whence, by the uniqueness of the spectral measure, Eq. (16) follows. Now it is obvious that the spectral measure, and consequently the function φ, determines the measure N.

Integration by parts shows that

$$k_m(t, y) = \left(\frac{it}{(m+1)!} \int_0^y e^{itx} \left(\log \frac{y}{x}\right)^{m+1} dx - it \arctan y \right) \times \log^{-m-1}(1 + y^{2/(m+1)}) + itr_m(y) \quad (18)$$

where r_m is a bounded continuous function on the right half-line tending to 0 as $y \to 0$. Moreover, it is easy to check that

$$\lim_{y \to 0} k_m(t, y) = -\frac{t^2}{2^{m+2}}. \quad (19)$$

We turn now to probability measures from L_m $(m = 1, 2, \ldots)$. Given $\mu \in L_m$, we denote by N_+ and N_- the Borel measures corresponding by Proposition 4 to μ^+ and $((-1)\mu)^+$, respectively. Further, by d^2 we denote the variance of the Gaussian component μ_0 of μ. Put for any Borel subset E of the real line

$$M(E) = N_+(E \cap (0, \infty)) + N_-((-E) \cap (0, \infty)) + 2^{m+1} d^2 \delta_0(E)$$

where $-E = \{-x: x \in E\}$. From decomposition formula (4) and Proposition 4 we get a characterization of L_m in terms of the measures M. Using relations (18) and (19), we finally obtain the following theorem.

Theorem 1. *Let $m = 1, 2, \ldots$. A function φ is the characteristic function of a probability measure from L_m if and only if*

$$\phi(t) = \exp\left\{ ict + it \left(\frac{1}{(m+1)!} \int_0^y \exp(itx) \left(\log \frac{y}{x}\right)^{m+1} dx - \arctan y \right) \times \frac{M(dy)}{\log^{m+1}(1 + |y|^{2/(m+1)})} \right\}$$

where c is a constant, M is a finite Borel measure on the real line, and the integrand is defined as its limiting value $it/2^{m+2}$ when $y = 0$. Moreover, the function φ determines c and M uniquely.

We proceed now to a characterization of the set L_∞. We begin with that of L_∞^+.

Proposition 5. *A function F is associated with a probability measure λ from $L_\infty{}^+$, i.e., $F = F_\lambda$, if and only if*

$$F(x) = \int_0^2 \sin[(\pi/2)y]e^{-xy}N(dy), \tag{20}$$

where N is a finite Borel measure on $(0, 2)$.

Proof. Suppose that $\lambda \in L_\infty{}^+$. Then, by Proposition 2, for every m we have the formula

$$F_\lambda(x) = \int_{e^x}^\infty \frac{(\log y - x)^m H_m(dy)}{\log^{m+1}(1 + y^{2/(m+1)})}$$

for some finite measures H_m on $(0, \infty)$. Hence it follows that F_λ is infinitely differentiable and

$$F^m_{(\lambda)}(x) = (-1)^m m! \int_{e^x}^\infty \frac{H_m(dy)}{\log^{m+1}(1 + y^{2/(m+1)})} \qquad (m = 0, 1, \ldots).$$

Thus the function F_λ is completely monotonic on the real line, and consequently, by Bernstein's theorem [4, p. 155], has a representation

$$F_\lambda(x) = \int_0^\infty e^{-xy}R(dy) \tag{21}$$

where R is a finite Borel measure on the right half-line. By (6) for the spectral measure Q corresponding to λ we have the equation

$$Q((0, \infty)) = \int_0^\infty \left(\int_{-\infty}^\infty \log(1 + x^2)e^{-xy}\,dx\right) yR(dy).$$

Since the integrand is equal to $\pi/(\sin(\pi/2)y)$ in the interval $(0, 2)$ and is infinite outside it, we infer that the measure R is concentrated on $(0, 2)$ and

$$\int_0^2 \frac{R(dy)}{\sin(\pi/2)y}$$

is finite. Setting the measure

$$N(E) = \int_E \frac{R(dy)}{\sin(\pi/2)y}$$

into (21), we get representation (20). The necessity of the condition is thus proved.

Suppose now that the function F is given by formula (20). Note first that

$$\int_0^\infty \log^{m+1}(1 + x^{2/(m+1)}) \frac{dx}{x^{1+y}} \le \frac{b}{y^{m+2}} + \frac{c}{2-y}$$

for $y \in (0, 2)$, where b and c are constants. Hence it follows that the measure H_m defined on $(0, \infty)$ by means of the formula

$$H_m(E) = \frac{1}{m!}\int_0^2 y^{m+1} \sin\frac{\pi}{2}y \int_E \log^{m+1}(1 + x^{2/(m+1)}) \frac{dx}{x^{1+y}} N(dy)$$

is finite. Consequently, by Proposition 2, the function

$$G_m(x) = \int_{e^x}^\infty \frac{(\log y - x)^m H_m(dy)}{\log^{m+1}(1 + y^{2/(m+1)})}$$

is associated with a probability measure, say, λ_m, from $L_m{}^+$. By a simple calculation we get the formula

$$G_m^{(m)}(x) = (-1)^m \int_0^2 y^m \sin\frac{\pi}{2}y \exp(-xy)N(dy)$$

which, by (20), yields the equation $G_m^{(m)} = F^{(m)}$. Since both functions F and G_m approach 0 at infinity, the last equation implies $G_m = F$. Thus $F = F_{\lambda_m}$ $(m = 1, 2, \ldots)$. Since the correspondence between measures λ and functions F_λ is one-to-one, we have the equations $\lambda_1 = \lambda_2 = \cdots$, which show that F is associated with a measure belonging to all sets $L_m{}^+$ and, consequently, to $L_\infty{}^+$. The sufficiency of the condition is thus proved.

Proposition 5 and formula (6) yield the following characterization of $L_\infty{}^+$ in terms of the spectral measures.

Proposition 6. *A measure Q defined on $(0, \infty)$ is the spectral measure for a probability distribution from $L_\infty{}^+$ if and only if it is of the form*

$$Q(E) = \int_E \log(1 + x^2) \int_0^2 x^{-y-1} y \sin[(\pi/2)y]N(dy)\,dx$$

where N is a finite Borel measure on $(0, 2)$.

Setting this expression into (2) we get the following result.

Proposition 7. *A function φ is the characteristic function of a probability measure from $L_\infty{}^+$ if and only if*

$$\varphi(t) = \exp\int_0^2 k_\infty(t, y)N(dy) \tag{22}$$

where N is an arbitrary finite Borel measure on $(0, 2)$ and the kernel k_∞ is defined by the formula

$$k_\infty(t, y) = y \sin\frac{\pi}{2}y \int_0^\infty \left(\int_0^{tx} \frac{e^{iu} - 1}{u}\,du - it \arctan x\right) \frac{dx}{x^{1+y}}.$$

In the same way as in Proposition 4 we conclude that the function φ determines the measure N uniquely. Moreover, integrating by parts, we have

$$k_\infty(t, y) = \sin\frac{\pi}{2}y \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{dx}{x^{1+y}}.$$

This integral occurs in the investigation of stable laws [2, p. 329]. For $y \in (0, 1) \cup (1, 2)$ we have the formula

$$k_\infty(t, y) = -it\frac{\pi}{2}\tan\frac{\pi}{2}y - \frac{\Gamma(2-y)\sin(\pi/2)y}{y(1-y)}|t|^y\left(\cos\frac{\pi}{2}y - i\frac{t}{|t|}\sin\frac{\pi}{2}y\right),$$

which by the continuity of the kernel yields

$$k_\infty(t, 1) = it(1-C) - it\log|t| - \frac{\pi}{2}|t|$$

where C is Euler's constant. Further, by a simple calculation we get the formula

$$k_\infty(t, y) = -\frac{\Gamma(2-y)\sin(\pi/2)y}{y(1-y)}\left[|t|^y\left(\cos\frac{\pi}{2}y - i\frac{t}{|t|}\sin\frac{\pi}{2}y\right) + ity\right] + itr(y) \tag{23}$$

where r is a bounded continuous function on $(0, 2)$ tending to $-\pi/2$ as $y \to 2$. For $y = 1$ we take the limiting value of the kernel. It is also easy to verify the formula

$$\lim_{y\to 2} k_\infty(t, y) = -\frac{\pi t^2}{4}. \tag{24}$$

We note that the function $y^{-1}\Gamma(2-y)\sin(\pi/2)y$ is bounded and has a positive greatest lower bound in the interval $(0, 2)$. Consequently, we may replace the measure N in representation formula (22) by the measure I defined as follows.

$$I(E) = \int_E y^{-1}\Gamma(2-y)\sin\frac{\pi}{2}yN(dy).$$

Taking into account (23), we get an equivalent form of representation (22)

$$\varphi(t) = \exp\left\{ibt - \int_0^2 \left[|t|^y\left(\cos\frac{\pi}{2}y - i\frac{t}{|t|}\sin\frac{\pi}{2}y\right) + ity\right]\frac{I(dy)}{1-y}\right\} \tag{25}$$

where I is an arbitrary finite Borel measure on $(0, 2)$ and b a suitably chosen constant.

Consider a probability measure μ from L_∞. Let I_+ and I_- be the measures corresponding in representation (25) to μ^+ and $((-1)\mu)^+$, respectively. Suppose that the characteristic function of the Gaussian component μ^0 of μ is given by the formula

$$\widehat{\mu^0}(t) = \exp(iat - d^2t^2).$$

For any subset E of $(-2, 0) \cup (0, 2]$ we put

$$M(E) = I_+(E \cap (0, 2)) + I_-((-E) \cap (0, 2)) + d^2\delta_2(E).$$

From decomposition formula (4) and representation (25) we get a characterization of L_∞ in terms of the measures M. Namely, we have the following theorem.

Theorem 2. *A function φ is the characteristic function of a probability measure from L_∞ if and only if*

$$\varphi(t) = \exp\left\{ict - \int_{-2}^2 \left[|t|^{|y|}\left(\cos\frac{\pi}{2}y - i\frac{t}{|t|}\sin\frac{\pi}{2}y\right) + ity\right]\frac{M(dy)}{1-|y|}\right\},$$

where c is a real constant, M is a finite Borel measure on $(-2, 0) \cup (0, 2]$, and the integrand is defined as its limiting values $(\pi/2)|t| + it\log|t| - it$ and $(\pi/2)|t| - it\log|t| + it$ when $y = 1$ and $y = -1$, respectively.

Corollary 1. *The set L_∞ is the smallest set containing all stable probability measures and closed under convolutions and passages to the limit.*

Corollary 2. *Each sequence of independent random variables with stable probability distributions (not necessarily with the same exponent) generating a convergent triangular array is slowly varying.*

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