Limit Laws for Sequences of Normed Sums
Satisfying Some Stability Conditions

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In this paper we discuss the limit laws arising from normed sums of independent random variables satisfying some stability conditions. These are, roughly speaking, sequences for which the limit properties of suitably normed sums are similar to those for sequences of identically distributed random variables. The result we obtain is an analogue of the Lévy–Khinchin representation of infinitely divisible laws. The present investigation arose from a study of self-decomposable probability measures.

Throughout this paper we denote by $P$ the set of all probability measures on the real line. With the topology of weak convergence and multiplication defined by the convolution, $P$ becomes a topological semigroup. We denote the convolution of two measures $\lambda$ and $\mu$ by $\lambda \ast \mu$. Moreover, by $\delta_a$, we denote the probability measure concentrated at the point $a$. Further, for any real number $a$ ($a \neq 0$) and any measure $\mu$ from $P$ we denote by $a\mu$ the measure defined by the formula $a\mu(E) = \mu(a^{-1}E)$ for all Borel subsets $E$ of the real line. The characteristic function $\hat{\mu}$ of a measure $\mu \in P$ is defined by the formula

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx} \mu(dx).$$

It is easy to check the equation $\hat{a\mu}(t) = \hat{\mu}(at)$.

By a triangular array we shall understand a system

$$X_{11}, X_{12}, X_{122}, \ldots,$$

$$X_{1n}, X_{2n}, \ldots, X_{nn}, \ldots$$
of random variables such that \( X_{1n}, X_{2n}, \ldots, X_{nn} \) are independent and for every \( c > 0 \)

\[
\lim_{n \to \infty} \max_{1 \leq k \leq n} P(|X_{kn}| > cn) = 0.
\]

Further, we say that a triangular array \( \{X_{kn}\} \) is generated by a sequence \( \{X_n\} \) of random variables if \( X_{kn} = X_k (k = 1, 2, \ldots, n; n = 1, 2, \ldots) \).

A probability measure is said to be a limit distribution of the triangular array \( \{X_{kn}\} \) if it is the weak limit of the sequence of probability distributions of normed sums \((1/n) \sum_{k=1}^{n} X_{kn} - a_n\) for suitably chosen constants \(a_n\). It is obvious that the limit distribution of \( \{X_n\} \) is defined uniquely up to a shift transformation. Moreover, this limit distribution is infinitely divisible (see [2], p. 309). Further, we call two triangular arrays equivalent if they have the same limit distribution. In particular, \( \{X_{kn}\} \) is equivalent to an array generated by a sequence \( \{X_n\} \) if and only if for suitably chosen \(a_n\) and \(b_n\), the sequences of normed sums

\[
\frac{1}{n} \sum_{k=1}^{n} X_{kn} - a_n \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} X_k - b_n
\]

have the same limit distribution.

We define classes \( S_m (m = 0, 1, \ldots) \) of sequences \( \{X_n\} \) of independent random variables recursively. Let \( S_0 \) be the class of all sequences \( \{X_{kn}\} \) generating convergent triangular arrays, i.e., the class of all sequences for which the sequence \((1/n) \sum_{k=1}^{n} X_k - a_n\) with suitably chosen constants \(a_n\) has a limit distribution. Further, \( \{X_n\} \in S_m \) whenever \( \{X_{kn}\} \in S_0 \) and the following stability condition is fulfilled: for every positive number \(c\) the triangular array \( X_{kn} = X_{(m+1)k} \in S_m \) (for \( k = 1, 2, \ldots, n; n = 1, 2, \ldots \)) is equivalent to an array generated by a sequence from \( S_{m-1} \). The square brackets here denote the integral part of the real number. It is clear that the classes \( S_m \) form a contracting sequence. Put \( S_\infty = \bigcap_{m=1}^{\infty} S_m \). The sequences belonging to \( S_\infty \) will be called slowly varying. For instance, each sequence of independent identically distributed random variables generating a convergent triangular array belongs to all classes \( S_m \) and, consequently, is slowly varying. We get a less trivial example of slowly varying sequences from results of Koroljuk and Zolotarev [1] (see also [5]), namely, each sequence of independent random variables generating a convergent triangular array and such that there are no more than two different distribution laws among the laws of the random variables \( X_1, X_2, \ldots \) is slowly varying.

Let \( L_m (m = 0, 1, \ldots, \infty) \) be the set of all possible limit distributions of normed sums \((1/n) \sum_{k=1}^{n} X_k - a_n\) where \( \{X_n\} \in S_m \) and \(a_n\) are constants. The problem of a description of probability measures from \( L_0 \) was solved by Lévy, who obtained an explicit representation of the characteristic function of those measures [2, p. 324]. Another characterization of the set \( L_0 \) was given [3]. The aim of the present paper is to give a characterization of all sets \( L_m \) (\( m = 1, 2, \ldots, \infty \)).

It is easy to see that the set \( L_m \) is invariant under the shift transformations \( \mu \to \mu + \delta \), and under the transformations \( \mu \to a \mu \). Moreover, it is closed under convolution. We characterize \( L_m \) by a decomposability property, and then we characterize the corresponding characteristic functions.

**Proposition 1.** A probability measure \( \mu \) belongs to \( L_m (m = 0, 1, \ldots) \) if and only if for every number \( a \in (0, 1) \) there exists a measure \( \mu_a \in L_{m-1} \) such that \( \mu = a \mu \ast \mu_a \). Let \( \mu \) denote here the set \( P \) of all probability measures.

**Proof.** We shall prove Proposition 1 by induction with respect to \( m \). Owing to Lévy’s results, it is true for \( m = 0 \) [2, p. 323]. Suppose that \( m \geq 1 \) and for indices less than \( m \) the statement is true. Consider a measure \( \mu \) from \( L_m \). Suppose that it is a limit distribution of normed sums \((1/n) \sum_{k=1}^{n} X_k - a_n\) where \( \{X_n\} \in S_m \). Given \( a \in (0, 1) \), we put

\[
Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k - \frac{a_n}{n}, \quad Z_n = \frac{1}{n} \sum_{k=1}^{[an]} X_k + \frac{an}{n} - a_n.
\]

It is evident that \( Y_n, Z_n \) are independent, and that \( \mu \) and \( a \mu \) are limit distributions of \( \{Y_n + Z_n\} \) and \( \{Y_n\} \), respectively. Further, taking into account that \( a \mu \), being infinitely divisible, has a nonvanishing characteristic function, we infer that the sequence \( \{Z_n\} \) has also a limit distribution, say, \( \mu_a \). Of course, \( \mu = a \mu \ast \mu_a \). By the assumption the triangular array \( X_{kn} = X_{[an]+k} (k = 1, 2, \ldots, n; n = 1, 2, \ldots) \) is equivalent to an array generated by a sequence from \( S_{m-1} \).

\[
Z_n = \frac{1}{n} \sum_{k=1}^{[an]} X_{kn} - c_n,
\]

where \( r_n = n - [an] \) and \( c_n \) are constants, we infer that its limit distribution \( \mu_a \) belongs to \( L_{m-1} \), which completes the proof of the necessity of the condition.

Suppose now that \( \mu \) is a probability distribution satisfying, for every \( a \in (0, 1) \), the condition \( \mu = a \mu \ast \mu_a \) where \( \mu_a \in L_{m-1} \). We have to prove that \( \mu \in L_m \). It is clear that \( \mu \in L_0 \) and consequently, being infinitely divisible, has a nonvanishing characteristic function. Setting \( v_1 = \mu \), \( v_n = n \mu_n (n-1)/n \) (\( n = 2, 3, \ldots \)), we have the formula

\[
\hat{\varphi}_a(t) = \frac{\hat{\mu}(\hat{a}t)}{\mu((n-1)t)}
\]

whence the relation

\[
\lim_{n \to \infty} \max_{1 \leq k \leq n} \left| \frac{\varphi_k(\frac{1}{n}) - 1}{\frac{1}{n}} \right| = 0
\]

(1)
follows. Let \( \{X_n\} \) be a sequence of independent random variables with probability distributions \( v_1, v_2, \ldots \). By (1) we have the convergence
\[
\lim_{n \to \infty} \max_{1 \leq k \leq n} P(|X_k| > cn) = 0
\]
for every positive number \( c \). Thus \( \{X_n\} \) generates a triangular array. Given a positive number \( c \), we put \( X_{kn} = X_{(cn + k)} \) \((k = 1, 2, \ldots, n; n = 1, 2, \ldots)\). Let \( \lambda_n \) be the probability distribution of the sum \((1/n) \sum_{k=1}^{n} X_{kn}\). Then
\[
\lambda_n(t) = \frac{\mu((1 + c)t)}{\mu(ct)},
\]
which yields the relation
\[
\lambda_n(t) \to \mu(1 + ct) / \mu(ct).
\]
Now taking into account the equation \( \mu((1 + c)t) = \mu(ct) \mu_c((1 + c)t) \), we infer that the probability measure \((1 + c)t \mu_{c,(1 + c)t})\) is a limit distribution of the array \( \{X_{kn}\} \). This probability measure, being an element of \( L_{m-1} \), is, by the induction assumption, a limit distribution of an array generated by a sequence from \( S_{m-1} \). Thus \( \{X_{kn}\} \) is equivalent to an array generated by a sequence from \( S_{m-1} \) and, consequently, \( \{X_n\} \) \( \in S_m \). Since \( \mu \) is the probability distribution of the normed sums \( (1/n) \sum_{k=1}^{n} X_k \) \((n = 1, 2, \ldots)\), we have \( \mu \in L_m \), which completes the proof.

As a direct consequence of Proposition 1 we get the following

**Corollary.** A probability measure \( \mu \) belongs to \( L_m \) if and only if every number \( a \in (0, 1) \) there exists a probability measure \( \mu_a \) \( \in L \cdot \) such that \( \mu = a \mu + \mu_a \).

Our next aim is to give a representation of the characteristic functions of probability measures from \( L_m \). The following representation formula was established in [3]: a measure \( \mu \) is self-decomposable, i.e., belongs to \( L_0 \), if and only if its characteristic function is of the form
\[
\hat{\mu}(t) = \exp\left(itg + \int_{-\infty}^{\infty} \left( \frac{1}{v} \exp(iv) - 1 \right) dv - it \arctan u \right) \frac{Q(du)}{\log(1 + u^2)},
\]
where \( g \) is a real constant, \( Q \) is a finite Borel measure on the real line, and the integrand is defined as its limiting value \(-1/2\) when \( v = 0 \). Moreover, the measure \( \mu \) determines the couple \( g, Q \) uniquely. We shall call \( Q \) the spectral measure for \( \mu \). It is evident that the convolution of measures corresponds to the sum of spectral measures of factors. Further, it is easy to check that if \( Q \) is the spectral measure for \( \mu \), then
\[
Q_a = \int_{-\infty}^{\infty} \frac{\log(1 + u^2)}{\log(1 + (u/a)^2)} Q(du)
\]
is the spectral measure for \( a \).

By \( L_m^+ \) \((m = 0, 1, \ldots, \infty)\) we shall denote the subset of \( L_m \) consisting of probability measures with vanishing constant \( q \) in (2) and whose spectral measure is concentrated on the open right half-line \((0, \infty)\). If \( a \in L_m^+ \), then, of course, \((-1)a \in L_m^+ \) and its spectral measure is concentrated on the half-line \((-\infty, 0)\). Let us introduce the operation \( \mu \mapsto \mu^+ \), which associates with every measure \( \mu \) from \( L_m \) having the spectral measure \( Q \) a probability measure \( \mu^+ \) from \( L_m^+ \) with the spectral measure \( Q^+ \) \((E = Q(E) \cap (0, \infty))\). It is easy to verify the equation \((a\mu)^+ = a\mu^+ \ast \delta_a \) for \( a > 0 \) where
\[
b = \int_{-\infty}^{\infty} \left( \arctan u - \arctan au \right) \frac{Q(du)}{\log(1 + u^2)}.
\]

Further, it is clear that \((\mu \ast \lambda)^+ = \mu^+ \ast \lambda^+ \). Thus the relation \( \mu = a\mu \ast \mu_a \) yields the following one \( \mu = a\mu^+ \ast (\mu_a^+) \ast \delta_a \). Hence, by a simple induction, we get the relation \( \mu^+ \in L_m \) whenever \( \mu \in L_m \). Suppose that, \( \mu \in L_m \) and \( \hat{\mu} \) is given by (2). Denoting by \( \mu^0 \) the Gaussian probability measure with the characteristic function
\[
\hat{\mu}^0(t) = \exp\left(itq + \frac{1}{2} Q((0))^2\right),
\]
we have the equation
\[
\mu = \mu^0 \ast \mu^+ \ast ((-1))(-1)(-1)\mu^+.
\]

It is evident that Gaussian measures belong to \( L_\infty \). Consequently, formula (4) reduces the investigation of \( L_m \) to that of \( L_m^+ \). Moreover, we have the following criterion: \( \lambda \in L_m^+ \) if and only if for every \( a \in (0, 1) \), \( \lambda = a\lambda \ast \gamma_a \ast \delta_c \) where \( c \) is a constant and \( \lambda_a \) \( \in L_m \) \((m = 1, 2, \ldots, \infty)\).

With every probability measure \( \lambda \) from \( L_0^+ \) we associate a function \( F_\lambda \) defined on the real line by means of the formula
\[
F_\lambda(x) = \int_{-\infty}^{\infty} \frac{Q(du)}{\log(1 + u^2)},
\]
where \( Q \) denotes the spectral measure for \( \lambda \). It is clear that this correspondence is one-to-one and
\[
Q(E) = -\int_E \log(1 + x^2) dF_\lambda(\log x)
\]
for any Borel subset \( E \) of \((0, \infty)\). Moreover
\[
F_\lambda = F_\lambda + F_\nu
\]
and, by (3), for \( a > 0 \)
\[
F_{\lambda a}(x) = F_\lambda(x - \log a).
\]
Proposition 2. Let \( m = 0, 1, \ldots \). A function \( F \) is associated with a probability measure \( \lambda \) from \( L_{m+}^{+} \), i.e., \( F = F_{\lambda} \), if and only if
\[
F(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} H(dy)}{\log^m(1 + y^{2/(m+1)})},
\]
where \( H \) is a finite Borel measure on \((0, \infty)\).

**Proof.** We shall prove Proposition 2 by induction with respect to \( m \). For \( m = 0 \) it is a direct consequence of formula (5). Suppose now that \( m > 0 \) and that for indices less than \( m \) the statement is true. Let \( \lambda \) be a probability measure from \( L_{m+}^{+} \). Since it belongs to \( L_{m-1}^{+} \), we have, by the induction assumption,
\[
F_{\lambda}(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} H_{\lambda}(dy)}{\log^m(1 + y^{2/m})},
\]
where \( H_{\lambda} \) is a finite Borel measure on \((0, \infty)\). Moreover, for any \( a \in (0, 1) \) we have the equation \( \lambda = a \lambda \ast \delta_\mu \ast \delta_a \) where \( \lambda \) is \( L_{m-1}^{+} \). Consequently, by the induction assumption,
\[
F_{\lambda}(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} G_{\lambda}(dy)}{\log^m(1 + y^{2/m})},
\]
where \( G_{\lambda} \) is a finite Borel measure on \((0, \infty)\). On the other hand, by (7), (8), and (10),
\[
F_{\lambda}(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1}}{\log^m(1 + y^{2/m})} \left( H_{\lambda}(dy) - \frac{\log^m(1 + y^{2/m})}{\log^m(1 + (y/a)^{2/m})} H_{\lambda}(a^{-1} dy) \right).
\]
Hence we get the equation
\[
G_{\lambda}(E) = H_{\lambda}(E) - \int_E \frac{\log^m(1 + y^{2/m})}{\log^m(1 + (y/a)^{2/m})} H_{\lambda}(a^{-1} dy)
\]
for any Borel subset \( E \) of \((0, \infty)\). Consequently,
\[
\int_E \frac{H_{\lambda}(dy)}{\log^m(1 + y^{2/m})} - \int_E \frac{H_{\lambda}(a^{-1} dy)}{\log^m(1 + (y/a)^{2/m})} \geq 0.
\]
\[
\int_E \frac{H_{\lambda}(dy)}{\log^m(1 + y^{2/m})} - \int_E \frac{H_{\lambda}(a^{-1} dy)}{\log^m(1 + (y/a)^{2/m})} \geq 0.
\]
Put
\[
g(x) = \int_{e^x}^{\infty} \frac{H_{\lambda}(dy)}{\log^m(1 + y^{2/m})}.
\]
By (11) for any \( a \in (0, 1) \) and \( u < v \) we have the inequality
\[
g(u) - g(v) - g(u - \log a) + g(v - \log a) \geq 0,
\]
which for \( u = v + \log a \) yields
\[
g(v) \leq \frac{1}{2}(g(v - \log a) + g(v + \log a).
\]
Thus the function \( g \) is convex and, consequently, can be represented in the form
\[
g(x) = \int_e^{\infty} h(u) du
\]
where \( h \) is a nonnegative monotone nonincreasing function. Further, by (12) we have the formula
\[
H_{\lambda}(E) = \int_E \log^m(1 + y^{2/m}) h(\log y) \frac{dy}{y}.
\]
Consequently,
\[
\int_0^{\infty} \log^m(1 + y^{2/m}) h(\log y) \frac{dy}{y} < \infty.
\]
Moreover, by (10),
\[
F_{\lambda}(x) = \int_{e^x}^{\infty} \frac{(\log y - x)^{m-1} h(\log y) \frac{dy}{y}}{\log^m(1 + y^{2/m})}
\]
By (13) the limit inferior of the function \( h(\log y) \log^{m+1}(1 + y^{2/(m+1)}) \) at 0 and \( \infty \) is equal to 0. Consequently, integration of (14) by parts gives
\[
F_{\lambda}(x) = -\frac{1}{m} \int_{e^x}^{\infty} (\log y - x)^m dh(\log y).
\]
Moreover, by (13),
\[
-\int_0^{\infty} \log^{m+1}(1 + y^{2/(m+1)}) dh(\log y)
\]
\[
= 2 \int_0^{\infty} \frac{y^{2/(m+1)}}{1 + y^{2/(m+1)}} \log^m(1 + y^{2/(m+1)}) h(\log y) \frac{dy}{y} < \infty.
\]
Thus the measure \( H \) defined by means of the formula
\[
H(E) = -\frac{1}{m} \int_E \log^{m+1}(1 + y^{2/(m+1)}) dh(\log y)
\]
is finite on \((0, \infty)\). Setting it into (15), we get desired representation (9).

Suppose now that the function \( F \) is given by formula (9). We note that \( F \) can be written in the form
\[
F(x) = \int_{e^x}^{\infty} \frac{H_{\lambda}(dy)}{\log^m(1 + y^{2/m})}
\]
where the measure
\[
G(E) = m \int_E \log^m(1 + y^{2/m}) \int_y^{\infty} \log^{m-1}(1 + x^{2/(m+1)}) H(dx) \frac{dy}{y}
\]
is also finite on \((0, \infty)\). Consequently, by the induction assumption, \( F = F_{\lambda} \) for a measure \( \lambda \) from \( L_{m-1}^{+} \). In order to prove that \( \lambda \in L_{m+}^{+} \), consider for any

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We note that the function \( \phi \) determines the measure \( N \) in (17). Indeed, by a simple calculation (17) can be transformed into formula (2), whence, by the uniqueness of the spectral measure, Eq. (16) follows. Now it is obvious that the spectral measure, and consequently the function \( \phi \), determines the measure \( N \).

Integration by parts shows that

\[
k_m(t, y) = \left( \frac{it}{(m+1)!} \right) \int_0^y e^{ix} \left( \log \frac{y}{x} \right)^{m+1} \, dx - it \arctan y
\]

\[
\times \log^{-m-1}(1 + y^{2/(m+1)}) + it \tau(y)
\]

where \( \tau(y) \) is a bounded continuous function on the right half-line tending to 0 as \( y \to 0 \). Moreover, it is easy to check that

\[
\lim_{y \to 0} k_m(t, y) = \frac{t^2}{2^{m+2}}.
\]

We turn now to probability measures from \( L_m \) (\( m = 1, 2, \ldots \)). Given \( \mu \in L_m \), we denote by \( N_+ \) and \( N_- \) the Borel measures corresponding by Proposition 4 to \( \mu^+ \) and \( \mu^- \), respectively. Further, by \( d\mu \) we denote the variance of the Gaussian component \( \mu_0 \) of \( \mu \). Put for any Borel subset \( E \) of the real line

\[
M(E) = N_+(E \cap (0, \infty)) + N_-((-E) \cap (0, \infty)) + 2^{m+1} d\delta_0(E)
\]

where \( -E = \{ -x : x \in E \} \). From decomposition formula (4) and Proposition 4 we get a characterization of \( L_m \) in terms of the measures \( M \). Using relations (18) and (19), we finally obtain the following theorem.

**Theorem 1.** Let \( m = 1, 2, \ldots \). A function \( \phi \) is the characteristic function of a probability measure from \( L_m \) if and only if

\[
\phi(t) = \exp \left[ it + \frac{1}{(m+1)!} \int_0^\phi \exp(itx) \left( \log \frac{y}{x} \right)^{m+1} \, dx - it \arctan x \right]
\]

\[
\times \left( \frac{1}{m+1} \log^{m+1}(1 + y^{2/(m+1)}) \right)
\]

where \( c \) is a constant, \( M \) is a finite Borel measure on the real line, and the integrand is defined as its limiting value if \( 2^{m+2} \) when \( y = 0 \). Moreover, the function \( \phi \) determines \( c \) and \( M \) uniquely.

We proceed now to a characterization of the set \( L_m \). We begin with that of \( L_{\infty}^+ \).
Proposition 5. A function $F$ is associated with a probability measure $\lambda$ from $L_\infty^+$, i.e., $F = F_\lambda$, if and only if

$$F(x) = \int_0^2 \sin[(\pi/2)y]e^{-xy}N(dy),$$

(20)

where $N$ is a finite Borel measure on $(0, 2)$.

Proof. Suppose that $\lambda \in L_\infty^+$. Then, by Proposition 2, for every $m$ we have the formula

$$F_m(x) = \int_0^\infty (\log y - x)^m H_m(dy)$$

(21)

for some finite measures $H_m$ on $(0, \infty)$. Hence it follows that $F_\lambda$ is infinitely differentiable and

$$F_m^{(m)}(x) = (-1)^m m! \int_0^\infty H_m(dy)$$

(22)

Thus the function $F_\lambda$ is completely monotonic on the real line, and consequently, by Bernstein's theorem [4, p. 155], has a representation

$$F_\lambda(x) = \int_0^\infty e^{-xy}R(dy)$$

(23)

where $R$ is a finite Borel measure on the right half-line. By (6) for the spectral measure $Q$ corresponding to $\lambda$ we have the equation

$$Q((0, \infty)) = \int_0^\infty \left( \int_{-\infty}^\infty \log(1 + x^2)e^{-xy} dx \right)R(dy).$$

Since the integrand is equal to $\pi/[(\sin(\pi/2)y)]$ in the interval $(0, 2)$ and is infinite outside it, we infer that the measure $R$ is concentrated on $(0, 2)$ and

$$\int_0^2 \frac{R(dy)}{\sin(\pi/2)y}$$

is finite. Setting the measure

$$N(E) = \int_E R(dy)$$

into (23), we get representation (20). The necessity of the condition is thus proved.

Suppose now that the function $F$ is given by formula (20). Note first that

$$\int_0^\infty \log^{m+1}(1 + x^{2/(m+1)}) \frac{dx}{x^{1+y}} \leq \frac{b}{y^{m+2}} + \frac{c}{2 - y}$$

for $y \in (0, 2)$, where $b$ and $c$ are constants. Hence it follows that the measure $H_m$ defined on $(0, \infty)$ by the means of the formula

$$H_m(E) = \frac{1}{m!} \int_0^m \log^{m+1}(1 + x^{2/(m+1)}) dx$$

(24)

is finite. Consequently, by Proposition 2, the function

$$G_m(x) = \int_0^\infty \log^{m+1}(1 + y^{2/(m+1)})$$

(25)

is associated with a probability measure, say, $\lambda_m$, from $L_m^+$. By a simple calculation we get the formula

$$G_m^{(m)}(x) = (-1)^m \int_0^m \log^{m+1}(1 + y^{2/(m+1)})$$

(26)

which, by (20), yields the equation $G_m^{(m)} = F^{(m)}$. Since both functions $F$ and $G_m$ approach 0 at infinity, the last equation implies $G_m = F$. Thus $F = F_m$ ($m = 1, 2, \ldots$). Since the correspondence between measures $\lambda$ and functions $F_\lambda$ is one-to-one, we have the equations $\lambda_1 = \lambda_2 = \cdots$, which show that $F$ is associated with a measure belonging to all sets $L_m^+$ and, consequently, to $L_\infty^+$. The sufficiency of the condition is thus proved.

Proposition 5 and formula (6) yield the following characterization of $L_\infty^+$ in terms of the spectral measures.

Proposition 6. A measure $Q$ defined on $(0, \infty)$ is the spectral measure for a probability distribution from $L_\infty^+$ if and only if it is of the form

$$Q(E) = \int_E \log(1 + x^2) dx$$

(27)

where $N$ is a finite Borel measure on $(0, 2)$.

Setting this expression into (2) we get the following result.

Proposition 7. A function $\varphi$ is the characteristic function of a probability measure from $L_\infty^+$ if and only if

$$\varphi(t) = \exp \int_0^2 k_\varphi(t, y)N(dy)$$

(28)

where $N$ is an arbitrary finite Borel measure on $(0, 2)$ and the kernel $k_\varphi$ is defined by the formula

$$k_\varphi(t, y) = y \sin \frac{\pi}{2} \int_0^\infty \left( \int_0^\infty \frac{e^u - 1}{u} \frac{du}{\arctan x} \right) \frac{dx}{x^{1+y}}.$$
In the same way as in Proposition 4 we conclude that the function \( \varphi \) determines the measure \( N \) uniquely. Moreover, integrating by parts, we have

\[
k_\varphi(t, y) = \sin \frac{\pi}{2} \int_0^\infty \left( e^{ix} - 1 - \frac{ix}{1 + x^2} \right) \frac{dx}{x^{1 + \gamma}}.
\]

This integral occurs in the investigation of stable laws [2, p. 329]. For \( y \in (0, 1) \cup (1, 2) \) we have the formula

\[
k_\varphi(t, y) = -it \tan \frac{\pi}{2} y - \frac{\Gamma(2 - y) \sin(\pi/2) y}{y(1 - y)} \frac{1}{|t|} \left( \cos \frac{\pi}{2} y - i \frac{t}{|t|} \sin \frac{\pi}{2} y \right),
\]

which by the continuity of the kernel yields

\[
k_\varphi(t, 1) = it(1 - C) - it \log |t| - \frac{\pi}{2} |t|
\]

where \( C \) is Euler’s constant. Further, by a simple calculation we get the formula

\[
k_\varphi(t, y) = - \frac{\Gamma(2 - y) \sin(\pi/2) y}{y(1 - y)} \left[ |t|^\gamma \left( \cos \frac{\pi}{2} y - i \frac{t}{|t|} \sin \frac{\pi}{2} y \right) + ity \right] + itr(y)
\]

(23)

where \( r \) is a bounded continuous function on \((0, 2)\) tending to \(-\pi/2\) as \( y \to 2 \). For \( y = 1 \) we take the limiting value of the kernel. It is also easy to verify the formula

\[
\lim_{y \to 2} k_\varphi(t, y) = -\frac{\pi t^2}{4}.
\]

(24)

We note that the function \( y^{-1} \Gamma(2 - y) \sin(\pi/2) y \) is bounded and has a positive greatest lower bound in the interval \((0, 2)\). Consequently, we may replace the measure \( N \) in representation formula (22) by the measure \( I \) as defined as follows.

\[
I(E) = \int_E y^{-1} \Gamma(2 - y) \sin \frac{\pi}{2} y N(dy).
\]

Taking into account (23), we get an equivalent form of representation (22)

\[
\varphi(t) = \exp \left( it - \int_0^\infty \left[ |t|^\gamma \left( \cos \frac{\pi}{2} y - i \frac{t}{|t|} \sin \frac{\pi}{2} y \right) + ity \right] I(dy) \right) 1 - y
\]

(25)

where \( I \) is an arbitrary finite Borel measure on \((0, 2)\) and \( b \) a suitably chosen constant.

Consider a probability measure \( \mu \) from \( L_\infty \). Let \( I_+ \) and \( I_- \) be the measures corresponding in representation (25) to \( \mu^+ \) and \((-1)\mu)^+\), respectively. Suppose that the characteristic function of the Gaussian component \( \mu^0 \) of \( \mu \) is given by the formula

\[
\hat{\mu}^0(t) = \exp(it - d^2 t^2).
\]

For any subset \( E \) of \((-2, 0) \cup (0, 2)\) we put

\[
M(E) = I_+(E \cap (0, 2)) + I_-((-E) \cap (0, 2)) + d^2 \delta_4(E).
\]

From decomposition formula (4) and representation (25) we get a characterization of \( L_\infty \) in terms of the measures \( M \). Namely, we have the following theorem.

**Theorem 2.** A function \( \varphi \) is the characteristic function of a probability measure from \( L_\infty \) if and only if

\[
\varphi(t) = \exp \left( i c t - \int_0^\infty \left[ |t|^\gamma \left( \cos \frac{\pi}{2} y - i \frac{t}{|t|} \sin \frac{\pi}{2} y \right) + ity \right] I(dy) \right) 1 - |y|
\]

where \( c \) is a real constant, \( M \) is a finite Borel measure on \((-2, 0) \cup (0, 2)\), and the integrand is defined as its limiting values \((\pi/2) |t| + it \log |t| - it \) and \((\pi/2) |t| - it \log |t| + it \) when \( y = 1 \) and \( y = -1 \), respectively.

**Corollary 1.** The set \( L_\infty \) is the smallest set containing all stable probability measures and closed under convolutions and passages to the limit.

**Corollary 2.** Each sequence of independent random variables with stable probability distributions (not necessarily with the same exponent) generating a convergent triangular array is slowly varying.

**REFERENCES**