Statistics and Probability Letters

Mathematical Institute, University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

A note on martingale inequalities for fluid models

Zbigniew Palmowski and Tomasz Rolski

Report no 71 **June 13, 2002**

A note on martingale inequalities for fluid models

Zbigniew Palmowski
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

Tomasz Rolski [*]
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

June 13, 2002

Abstract

We discuss how to prove exponential upper bounds for simple fluid models driven by a finite state CTMC. In particular we consider the fluid model of Anick, Mitra and Sondhi, in which the fluid is generated by N independent 0-1 Markovian sources. We also give a result on a generalized eigenvector for fluid models driven by reversible CTMC's.

Keywords: queueing fluid model, buffer content, exponential bound, generalized eigenvalue problem.

AMS 1991 Subject Classification: Primary 60K25, Secondary 68M20, 90B22.

---

[*] Work supported by KBN under grant 1026/803/95/08

1 Introduction

We say that a random variable X admits an exponential bound if for some $C, \gamma > 0$

$$\mathbb{P}(X > x) \le Ce^{-\gamma x}, \qquad x \ge 0 \ . \tag{1.1}$$

Of course in (1.1) we look for best possible constants C, γ. In the theory of queues Kingman (1964) and Ross (1974) proved exponential bounds for the steady state waiting time in the GI/GI/1 queue and in the earlier paper a method of proving was proposed, from which it comes the name of martingales inequalities. In fluid models a counterpart of the steady state waiting time is the steady state buffer content, which in many cases has a representation

$$X =_{\mathrm{d}} \sup_{t \ge 0} \int_0^t u_{Z(s)}\, ds \ , \tag{1.2}$$

where $\{Z(t),\ t \ge 0\}$ is a finite state, continuous time Markov chain (CTMC) with right-continous trajectories and $\boldsymbol{u} = (u_0, \ldots, u_m)$ is a given vector. We denote the generator matrix of Z by $\boldsymbol{Q}$ and without loss of generality we suppose the state space is $E = \{0, 1, \ldots, m\}$. Then random variable in (1.2) is finite almost surely if

$$\sum_{i=0}^{m} \pi_i u_i < 0 \ , \tag{1.3}$$

where $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_n)$ is the steady state probability vector of the CTMC (i.e. $\boldsymbol{\pi Q} = \boldsymbol{0}$).

The following model is already classical in queueing fluid theory and plays a similar role as M/M/1 in standard queueing theory. Anick *et al* (1982) considered N independent sources, each transmitting a *fluid* according to a 0-1 CTMC. The mean time a source is *on* is λ^{-1} and is *off* is μ^{-1} and we denote $\alpha = \mu + \lambda$. The buffer is emptied with constant rate $c > 0$ and if a source is *on* it is sending fluid with rate r. Here $m = N$,

$$\boldsymbol{Q} = \begin{pmatrix} -N\mu & N\mu & 0 & 0 & 0 & 0 \\ \lambda & -\lambda-(N-1)\mu & (N-1)\mu & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & (N-1)\lambda & -(N-1)\lambda-\mu & \mu \\ & & & & N\lambda & -N\lambda \end{pmatrix} \tag{1.4}$$

and $u_j = rj - c$. Thus the stationary number of *on* units is $\pi_j = \binom{N}{j} f^j (1-f)^{N-j}$, where $f = \mu/\alpha$. The stability condition (1.3) is fulfilled when the drift

$$d = Nfr - c < 0 \ . \tag{1.5}$$

We call the above fluid model AMS model. The key idea for obtaining martingale bounds is the following result, for which we refer to Ethier & Kurtz (1986,page 175).Let $\boldsymbol{h} = (h_0, \ldots, h_m)^{\mathrm{T}}$. If $h_j > 0$ ($j = 0, \ldots, m$) then

$$\frac{h_{Z(t)}}{h_i} \exp\left(- \int_0^t \frac{(\boldsymbol{Qh})_{Z(s)}}{h_{Z(s)}}\, ds \right), \qquad t \ge 0 \tag{1.6}$$

is a martingale with respect to $\mathbb{P}^i$, where as usual $\mathbb{P}^i$ denote the underlying probability measure such that $Z(0) = i$.

In the case of the AMS model, choosing $h_j = e^{-\theta j}$ we get

$$\frac{(\boldsymbol{Qh})_j}{h_j} = j(\lambda e^{\theta} + \mu)(1 - e^{-\theta}) - N\mu(1 - e^{-\theta}).$$

Let $a = (\lambda e^{\theta} + \mu)(1 - e^{-\theta})$ and $b = N\mu(1 - e^{-\theta})$. Then

$$-\int_0^t (aZ(s) - b)\, ds = -\frac{a}{r}\int_0^t (rZ(s) - c)\, ds + t(b - \frac{c}{r}a) \ .$$

Choosing

$$\theta = \log(\frac{\mu}{\lambda}\frac{(Nr - c)}{c}) \ ,$$

so $b = \frac{c}{r}a$, we get that

$$M(t) = \exp\left(-\theta(Z(t) - i)\right) \exp\left(\gamma \int_0^t (rZ(s) - c)\, ds\right) \ , \tag{1.7}$$

is a martingale with respective to $\mathbb{P}^i$, where

$$\gamma = \frac{N}{c}\frac{\alpha c - \mu N r}{Nr - c} = \frac{\alpha(1 - \rho)}{r - \frac{c}{N}} \ ,$$

where $\rho = Nfr/c < 1$. For the above choice of θ, which in view of the stability assumption (1.5) is negative, we have $h_j = \kappa^j$, where

$$\kappa = \frac{\lambda c}{\mu(Nr - c)} > 1 \ . \tag{1.8}$$

Further details are in Section 2. For a general CTMC we have to find h_j $(j = 0, \ldots, m)$ to reduce (1.6) to

$$M(t) = \frac{h_{Z(t)}}{h_i} \exp\left(\gamma \int_0^t u_{Z(s)}\, ds\right), \qquad t \geq 0 \ .$$

This requires more delicate considerations worked out in Section 5.

Some related fluid models have been recently studied by Kulkarni & Rolski (1994), Buffet & Duffield (1994), Duffield (1994).

2 AMS model

In this section we consider the AMS model. Denote X as the steady-state fluid content and $\psi = c/r$.

Theorem 2.1

$$\mathbb{P}(X > x) \leq Ce^{-\gamma x}, \qquad x \geq 0 \tag{2.1}$$

where

$$C = \left(\frac{f}{1-f}\right)^{\psi} \left(\frac{N - \psi}{\psi}\right)^{\psi} (1 - f)^N \left(1 - \frac{\psi}{N}\right)^{-N} , \tag{2.2}$$

$$\gamma = \frac{N}{c}\frac{\alpha\psi - \mu N}{N - \psi} \ . \tag{2.3}$$

Proof. First observe that the following statements hold true.

For $x > 0$ and $Z(0) = i$

$$\begin{aligned}
\left[\sup_{t\geq 0}\int_0^t (rZ(s)-c)\,ds > x\right] &\equiv \left[\sup_{t:Z(t)\geq \frac{C}{r}}\int_0^t (rZ(s)-c)\,ds > x\right] \\
\equiv \left[\sup_{t:Z(t)\geq \frac{C}{r}} \exp\left(\gamma\int_0^t (rZ(s)-c)\,ds\right) > e^{\gamma x}\right] &\equiv \left[\sup_{t:Z(t)\geq \frac{C}{r}} e^{\theta(Z(t)-i)}M(t) > e^{\gamma x}\right] \\
&\subset \left[\sup_{t\geq 0} M(t) > e^{-\theta(\frac{C}{r}-i)}e^{\gamma x}\right] ,
\end{aligned}$$

where M is from (1.7). In the first equivalence we used the following fact. It suffices to inspect only all points of increase of continous function $\{\int_0^t (rZ(s)-c)\,ds\}$. However this set is enclosed in the set $\{t:\ Z(t)\geq \frac{c}{r}\}$, because realizations of $\{rZ(t)-c,\ t\geq 0\}$ are right-continous. Hence

$$\mathbb{P}^i(X > x) = \mathbb{P}^i\left(\sup_{t\geq 0}\int_0^t (rZ(s)-c)\,ds > x\right) \leq \mathbb{P}^i\left(\sup_{t\geq 0} M(t) > e^{-\theta(\frac{C}{r}-i)}e^{\gamma x}\right) .$$

By Doob-Kolmogorov inequality we have

$$\mathbb{P}^i\left(\sup_{t\geq 0} M(t) > e^{-\theta(\frac{C}{r}-i)}e^{\gamma x}\right) \leq e^{-\gamma x}e^{\theta(\frac{C}{r}-i)} ,$$

which yields

$$\mathbb{P}(X > x) \leq e^{-\gamma x}\sum_{i=0}^{N} \pi_i e^{\theta(\frac{C}{r}-i)} .$$

□

3 Ornstein-Uhlenbeck limiting case.

In this section, following [11] we consider a class of AMS models parametrized by the number of sources N with λ and μ fixed, input rate r_N and output rate c_N. We suppose that for some $c, r > 0$

$$r_N = N^{-1/2}\frac{r}{\sqrt{f(1-f)}} , \qquad c_N = c + N^{1/2}r\sqrt{\frac{f}{1-f}} . \tag{3.1}$$

Thus

$$\psi_N = \frac{c_N}{r_N} = \beta N^{1/2} + fN ,$$

where $\psi = c/r$ and $\beta = \psi\sqrt{f(1-f)}$.

Denote by C_N the constant C in (2.2) for the Nth model and γ_N the constant γ in (2.3) respectively. Kulkarni & Rolski (1994) proved that under condition (3.1)

$$X_N \xrightarrow{\mathcal{D}} X$$

where X_N, X are steady state buffer contents in Nth AMS model and in the fluid model driven by the Ornstein-Uhlenbeck process with drift parameter $-\alpha z$ and variance parameter 2α respectively. They also got exponential bound:

$$\mathbb{P}(X > x) \leq \exp\left(-\phi\psi x\right) \exp\left(-\frac{\psi^2}{2}\right) ,$$

where $\phi = \alpha/r$. In the following proposition we demonstrate that from Theorem 2.1 straightforward computations lead us to exactly the same limit.

Proposition 3.1 *Under condition (3.1), as* $N \to \infty$

$$C_N \to e^{-\frac{\psi^2}{2}} \tag{3.2}$$

and

$$\gamma_N \to \phi\psi \ . \tag{3.3}$$

Proof. From Theorem 2.1 we obtain

$$\gamma_N = \frac{N}{c + N^{1/2} r \sqrt{\frac{f}{1-f}}} \frac{\alpha c \sqrt{f(1-f)}}{(1-f) r N^{1/2} - c\sqrt{f(1-f)}}$$

and

$$C_N = \left(1 - \frac{\beta}{1-f} N^{-1/2}\right)^{\beta N^{1/2}} \left(1 + \frac{\beta}{f} N^{-1/2}\right)^{-\beta N^{1/2}} \frac{\left(1 + \frac{\beta}{f} N^{-1/2}\right)^{-fN}}{\left(1 - \frac{\beta}{1-f} N^{-1/2}\right)^{(1-f)N}} \ . \tag{3.4}$$

Standard calculations shows that $\gamma_N \to \phi\psi$ for $N \to \infty$. Now the limits in the first and second factors of (3.4) are obvious. For the third factor by de L'Hospital theorem

$$\begin{aligned} \lim_{N\to\infty} \frac{\left(1 + \frac{\beta}{f} N^{-1/2}\right)^{-fN}}{\left(1 - \frac{\beta}{1-f} N^{-1/2}\right)^{(1-f)N}} &= \exp\left(\lim_{y\to 0} y^{-1} \log \frac{\left((1 + \frac{\beta}{f} y^{1/2}\right)^{-f}}{\left((1 + \frac{\beta}{1-f} y^{1/2}\right)^{1-f}}\right) \\ &= \exp\left(\frac{1}{2}\beta^2 (\frac{1}{f} + \frac{1}{1-f})\right) \ . \end{aligned}$$

Since $\beta^2(\frac{1}{f} + \frac{1}{1-f}) = \psi^2$ we obtain (3.2), which completes the proof.

□

Remark 3.1 The limit in (3.2) was observed in Kulkarni & Rolski (1994).

In Anick *et al* (1982) there is given an asymptotic relationship

$$\mathbb{P}(X > x) \sim A e^{-\gamma x}, \qquad x \to \infty. \tag{3.5}$$

However the computation of the constant A is difficult, especially for big N. In next section we discuss (3.5), looking at the formula from another point of view than in Anick *et al* (1982). Note however that (3.5) is not good enough if we want to estimate $\mathbb{P}\left(X > x\right)$ for x and N large.

4 Exponential change of measure

In this section we show an alternative proof of (2.1) using the exponential change of measure. Following Asmussen (1994) or Asmussen & Rubinstein (1995) consider another AMS model with N, r, c as before but now take $\tilde{\mu} = \kappa\mu$ and $\tilde{\lambda} = \kappa^{-1}\lambda$, where κ was defined in (1.8). To formulate the problem precisely we take now $\Omega = D[0,\infty)$ the space of right continuous function with left hand limits and $\mathcal{F}$ the σ-field of Borel subsets as defined in Chapter 3 of Ethier & Kurtz (1986). The stochastic process is the canonical process $Z(\omega, t) = \omega(t)$, for $\omega \in \Omega$. Let $\mathbb{P}$ be the probability measure for which the process $\{Z(t),\ t \geq 0\}$ is a finite state Markov process with the generator $\boldsymbol{Q}$ as defined in (1.4). Consider also another probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ for which $\{Z(t)\}$ is a finite state Markov process with the generator like $\boldsymbol{Q}$ with λ and μ changed to $\tilde{\lambda} = \kappa^{-1}\lambda$ and $\tilde{\mu} = \kappa\mu$ respectively. Let $\mathbb{P}_t^i$ and $\tilde{\mathbb{P}}_t^i$ denote the restrictions of $\mathbb{P}$ and $\tilde{\mathbb{P}}$ to the partial histories to time t respectively. Then from Proposition 4.6 in Asmussen (1994) we have

$$\frac{d\tilde{\mathbb{P}}_t^i}{d\mathbb{P}_t^i} = \frac{h_{Z(t)}}{h_i} e^{\gamma \int_0^t (rZ(s) - c)\, ds} \qquad (= M_t) \ . \tag{4.1}$$

The process $\{Z(t)\}$ under $\tilde{\mathbb{P}}$ has the unique stationary distribution

$$\tilde{\pi}_j = \binom{N}{j} \tilde{f}^j (1 - \tilde{f})^{N-j} \ ,$$

where $\tilde{f} = \tilde{\mu}/(\tilde{\mu} + \tilde{\lambda})$ and therefore its trend is now $\tilde{d} = r\sum_{j=0}^N j\tilde{\pi}_j - c = rN\tilde{f} - c > 0$. Hence

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (rZ(s) - c)\, ds = \tilde{d}$$

with $\tilde{\mathbb{P}}$-probability 1, which yields that the stopping time

$$\tau(x) = \inf\{t \geq 0 : \int_0^t (rZ(s) - c)\, ds > x\}$$

is $\tilde{\mathbb{P}}^i$-a.s. finite. Note that $\mathbb{P}(X > x) = \mathbb{P}(\tau(x) < \infty)$. Let $F_t = \{\omega : \tau(x) \leq t\}$. Following Asmussen (1994), using the fact that stopping time $\tau(x)$ is $\tilde{\mathbb{P}}^i$-a.s. finite and the strong Markov property we can write

$$\begin{aligned}
\mathbb{P}^i(\tau(x) < \infty) &= \lim_{t\to\infty} \mathbb{P}_t^i(F_t) \\
= \lim_{t\to\infty} \tilde{\mathbb{E}}_t^i \left[\tilde{\mathbb{E}}_t^i \left(M_t^{-1} \mathbb{I}\{F_t\} | \mathcal{F}_{\tau(x)}\right)\right] &= \lim_{t\to\infty} \tilde{\mathbb{E}}_t^i \left[M_{\tau(x)}^{-1} \mathbb{I}\{F_t\} \tilde{\mathbb{E}}_t^i \left(M_{\tau(x)} M_t^{-1} | \mathcal{F}_{\tau(x)}\right)\right] \\
= \lim_{t\to\infty} \tilde{\mathbb{E}}_t^i \left(M_{\tau(x)}^{-1} \mathbb{I}\{F_t\}\right) &= \tilde{\mathbb{E}}^i \left(M_{\tau(x)}^{-1}\right) \ .
\end{aligned}$$

Hence

$$\mathbb{P}(\tau(x) < \infty) = C(x) e^{-\gamma x} \ ,$$

where

$$C(x) = \sum_{i=0}^n \pi_i \kappa^i \tilde{\mathbb{E}}^i \kappa^{-Z(\tau(x))} \ ,$$

from which we easily obtain (2.1) because $Z(\tau(x)) \geq c/r$. It is not difficult to see that $\{Z(\tau(x)), x \geq 0\}$ is again an irreducible (and so ergodic) CTMC, from which we derive (3.5). The problems how to compute the generator of the Markov process $\{Z(\tau(x)),\ x \geq 0\}$ was recently discussed in Asmussen (1995) and Rogers (1994).

Remark 4.1 Result (4.1) is a special case of the Girsanov transformation, however referred Asmussen's proof is elementary. Notice that (4.1) follows from a more general result, which has been applied by Fukushima & Stroock (1986) and Ethier & Kurtz (1993). Adapted to our case of CTMC's, given a strictly positive function h, consider the perturbation of generator $\mathcal{A}$ of a Markov process in the following sense:

$$\mathcal{A}^h f = \frac{\mathcal{A}(fh) - f\mathcal{A}h}{h} = \mathcal{A}f + \frac{< h, f >_{\mathcal{A}}}{h} ,$$

where $< f, g >_{\mathcal{A}} = \mathcal{A}(fg) - f\mathcal{A}g - g\mathcal{A}f$. Then the following result holds. Let $\mathbb{P}$ be the distribution for a CTMC with generator $\mathcal{A}$ and suppose that $h > 0$. Then

$$M(t) = \frac{h(Z(t))}{h(Z(0))} \exp\left(- \int_0^t \frac{(\mathcal{A}h)(Z(s))}{h(Z(s))}\, ds \right)$$

is a mean-one $\mathcal{F}_t$-martingale under probability measure $\mathbb{P}$ on the space of trajectories. Defining $\tilde{\mathbb{P}}$ on this trajectories space by $d\tilde{\mathbb{P}}_t / d\mathbb{P}_t = M(t)$ for all $t \geq 0$. We can do that by Daniel-Kolmogorov theorem because $\{\tilde{\mathbb{P}}_t,\ t \geq 0\}$ is a consitent family of distributions. Then $\tilde{\mathbb{P}}$ is the distribution of the unique Markov process with generator $\mathcal{A}^h$.

Remark 4.2 In the large deviation theory the change of measure argument, utilizing martingale (1.7) is known under the name of twisting; see e.g. Schwartz and Weiss (1995), page 75. However large deviation techniques gives only asymptotical results of type (3.5), with the same exponent γ.

5 Markovian fluid models

Suppose now the fluid model is governed by a finite state CTMC with generator $\boldsymbol{Q}$ and stationary distribiution $\boldsymbol{\pi}$. Let $\boldsymbol{\Delta} = \mathrm{diag}(\boldsymbol{u})$ denote a diagonal matrix with $u_0, \ldots, u_m$ on the diagonal. We have to find $\lambda < 0$ and a vector $\boldsymbol{h} = (h_0, \ldots, h_m)$ such that $h_j > 0$ $(j = 1, \ldots, m)$ and

$$\frac{(\boldsymbol{Q}\boldsymbol{h})_j}{h_j} = \lambda u_j,$$

or in matrix notations we have to solve a generalized eigenvalue problem

$$\boldsymbol{Q}\boldsymbol{h} = \lambda \boldsymbol{\Delta} \boldsymbol{h} \ . \tag{5.1}$$

In this case each λ is called a generalized eigenvalue and $\boldsymbol{h}$ a generalized eigenvector. Sonneveld (1988), Mitra (1986), London *et al* (1982) and Kulkarni (1992) discuss when generalized eigenvalues for (5.1) are real and how they are located. However we need some information on generalized eigenvectors in the form of the following lemma, which proof is hidden in Asmussen (1995) and Rogers (1994). We denote by $\boldsymbol{e}'$ the column vector consisting of 1's.

Lemma 5.1 *Suppose that* $\boldsymbol{Q}$ *is irreducible with negative trend* $d = \boldsymbol{\pi}\boldsymbol{\Delta}\boldsymbol{e}' < 0$. *Then a generalized eigenvalue with the biggest negative real part is real and the generalized eigenvector corresponding to it has all coordinates positive.*

Proof. Asmussen (1995) considers CTMC $\{Z(\tau(x)), x \geq 0\}$, which under probability measure $\mathbb{P}$ is transient and let $\boldsymbol{U}$ be its generator. Theorem 5.2 in Asmussen (1995) says that for $\Re s > 0$ we have

$$(\boldsymbol{Q} + s\boldsymbol{\Delta}) \begin{pmatrix} \boldsymbol{h}^{(+)} \\ \boldsymbol{h}^{(-)} \end{pmatrix} = \boldsymbol{0}$$

if and only if $-s$ is an eigenvalue of $\boldsymbol{U}$ such that

$$\boldsymbol{U}\boldsymbol{h}^{(+)} = -s\boldsymbol{h}^{(+)} \ . \tag{5.2}$$

Besides in this case $\boldsymbol{h}^{(-)} = \boldsymbol{\alpha}^{(-)}\boldsymbol{h}^{(+)}$, where $\boldsymbol{\alpha}^{(-)}_{ij} = \mathbb{P}^i(Z(\tau(0)) = j)$. Since $\boldsymbol{U}$ is transient, its Perron-Frobenius eigenvalue is negative and the corresponding eigenvector $\boldsymbol{h}^{(+)}$ has positive entries. From the irreducibility vector $\boldsymbol{h}^{(-)}$ has also positive entries.

□

Let λ and $\boldsymbol{h}$ be as in Lemma 5.1. If $Z(0) = i$ then

$$M_i(t) = \frac{h_{Z(t)}}{h_i} \exp\left(-\lambda \int_0^t u_{Z(s)}\, ds\right)$$

is a martingale.

Proposition 5.1

$$\mathbb{P}(X > x) \le Ce^{-\gamma x} \ , \tag{5.3}$$

where

$$C = \frac{\sum_i \pi_i h_i}{\min_i h_i} \qquad \text{and} \qquad \gamma = -\lambda \ . \tag{5.4}$$

Proof. Let $\mathbb{P}^i$ denotes the underlying probability measure if the process $\{Z(t)\}$ starts from i. We first bound above $\mathbb{P}^i(X > x)$. Similarly as in Section 2, using that components $h_i > 0$, we get

$$\left[\sup_{t\ge 0} \int_0^t u_{Z(s)}\, ds > x\right] \quad \Rightarrow \quad \left[\sup_{t\ge 0} M_i(t) > \min_j h_j h_i^{-1} e^{\gamma x}\right] \ .$$

Hence by Doob-Kolmogorov inequality

$$\mathbb{P}^i(\sup_{t\ge 0} M_i(t) > \min_j h_j h_i^{-1} e^{\gamma x}) \le \max_j \frac{h_i}{h_j} e^{-\gamma x} \ .$$

We now get (5.3) from $\mathbb{P} = \sum_i \pi_i \mathbb{P}^i$.

□

Remark 5.1 The constant C in (5.4) is not optimal.

Acknowledgement

We thank Søren Asmussen for providing the manuscript of his paper with Rubinstein before publication and remarks on Lemma 5.1. We thank also the referee for pointing us references Fukushima & Stroock (1986) and Ethier & Kurtz (1993).

References

[1] Anick, D., Mitra, D. & Sondhi, M.M. (1982) Stochastic theory of a data-handling system with multiple sources. *The Bell System Technical Journal* . **61**, 1871–1894.

[2] Asmussen, S. (1994) Busy period analysis, transient behaviour and rare events for fluid flow models. *J. Appl. Math. Stoch. Anal.* **7**, 269–299.

[3] Asmussen, S. (1995) Stationary distributions for fluid models with or without Brownian noise. *Stochastic Models* **11**, 21–49.

[4] Asmussen, S. & Rubinstein, R.Y. (1995) Steady state rare events simulation in queueing models and its complexity properties. *Advances in Queueing.* J. Dhashalow, ed. CRC Press, Boca Raton, Florida, 429–462.

[5] Buffet, E. & Duffield, N.G. (1994) Exponential upper bounds via martingales for multiplexers with Markovian arrivals. *J. Appl. Probab.* **31**, 1049–1060.

[6] Duffield, N.G. (1994) Exponential bounds for queues with Markovian arrivals. *Queueing Systems.* **17**, 413–430.

[7] Ethier, H.J. & Kurtz, T.G. (1986) *Markov Processes, Characterization and Convergence.* J. Wiley & Sons, New York.

[8] Ethier, H.J. & Kurtz, T.G. (1993) Fleming-Viot processes in population genetics. *SIAM J. Control and Optimization.* **31**, 345–386.

[9] Fukushima, M. & Stroock, D. (1986) Reversibility of solutions to martingale problems. *Adv. Math. Suppl. Stud.* **9**, 107–123.

[10] Kingman, J.F.C. (1964) A martingale inequality in the theory of queues. *Proc. Camb. Philos. Soc.* **60**, 359–361.

[11] Kulkarni, V. & Rolski, T. (1994) Fluid model driven by an Ornstein-Uhlenbeck process. *Prob. Eng. Inf. Sci.* **8**, 403–417.

[12] Kulkarni, V. (1992) A fluid flow model of a buffer in random environment. Technical Report No. UNC/OR TR-91/7.

[13] London, R.R., McKean, H.P., Rogers, L.C.G., & Williams, D. (1982) A martingale approach to some Wiener-Hopf problems II. *Séminaire de probabilités XVI*, Lect. Notes Math 920, 68–90. Springer, Berlin, 1982.

[14] Mitra, D. (1988) Stochastic theory of a fluid model of producer and consumers coupled by a buffer. *Advances in Appl. Prob.* **20**, 646–676.

[15] Rogers, C.L.G. (1994) Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Ann. Appl. Prob.* **4**, 390–413.

[16] Ross, S. (1974) Bounds on the delay distribution in GI/G/1 queues. *J. Appl. Probab.* **11**, 417–421.

[17] Sonneveld, P. (1988) Some properties of the generalized eigenvalue problem $Mx = \lambda(\Gamma - cI)x$, where M is the infinitesimal generator of a Markov process and Γ is a real diagonal matrix. Report, Faculty of Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands, 1988.

[18] Schwartz, A. & Weiss, A. (1995) *Large Deviations For Performance Analysis.* Chapman & Hall, London.