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Heavy traffic Gaussian asymptotics of on-off fluid model

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On-off fluid models in heavy traffic environment

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Abstract

We consider fluid models with infinite buffer size. Let $\{Z_N(t)\}$ be the net input rate to the buffer where $\{Z_N(t)\}$ is a superposition of N homogeneous alternating *on-off* flows. Under *heavy traffic environment* $\{Z_N(t)\}$ converges in distribution to a centred Gaussian process with covariance function of a single flow. The aim of this paper is to prove the convergence of the stationary buffer content process $\{X_N^*(t)\}$ in the Nth model to the buffer content process $\{X^*(t)\}$ in the limiting Gaussian model.

Keywords: fluid model, alternating on-off process, Gaussian process, buffer content, large deviation, heavy traffic environment, long range dependence.

AMS Classification: Primary: 60K25; Secondary: 68M20, 90B22, 60G15.

1 Introduction

Fluid model $\{X(t), t \geq 0\}$ *driven by* stationary and ergodic process $\{Z(t), t \in \mathbb{R}\}$ is a stochastic process whose dynamics is described by

$$\frac{dX(t)}{dt} = \begin{cases} Z(t) - c & X(t) > 0 \\ (Z(t) - c)_+ & X(t) = 0 \;, \end{cases} \tag{1.1}$$

where $c > 0$ is a constant. We consider only models with infinite buffer capacity. Let

$$\Psi(u) = \mathbb{P}(\sup_{t \leq 0} \int_t^0 (Z(s) - c) ds > u)$$

be the probability that under steady state conditions the buffer content exceeds a given level u. The particularly interesting case is when $\{Z(t), t \in \mathbb{R}\}$ is a stationary and ergodic centred Gaussian process. Such fluid models were studied among others by Dębicki & Rolski [9] and Norros [20]. The Gaussian driving process appears as the limiting case of the fluid model that is driven by the superposition of *on-off* alternating renewal processes in *heavy traffic environment* (see Section 2).

In this paper we consider fluid models $\{X_N(t), t \geq 0\}$ that are driven by $\{r_N Z_N(t), t \in \mathbb{R}\}$ - an appropriate scaled superposition of N stationary, ergodic alternating renewal *on-off* flows($N = 1, 2, ...$). Motivation for our work were the following papers: Heath *et al* [14], Leland *et al* [18], Jelenković & Lazar [15], Jelenković *et al* [16], Willinger *et al* [24], Norros [20], Crovella & Bestavros [8], in which in a numerical way were found that fluid models which are driven by appropriate Gaussian processes generate good approximation for modelling traffic on data networks such as Ethernet LANs, ATM network (voice and video) or for traditional voice traffic. That is $\{X_N(t), t \geq 0\}$ can be well approximated by process $\{X(t), t \geq 0\}$ which is driven by a Gaussian process.

Kulkarni & Rolski [17] proved that under *heavy traffic environment* parametrization, in the case of multiplexing Anick *et al* [2] model, the steady state buffer content X_N^* in Nth model converges in distribution to the steady state buffer content in limiting Gaussian model X^*:

$$X_N^* \xrightarrow{\mathcal{D}} X^* \;.$$

In Corollary 3.2 we generalise this result. Moreover, in Propsition 3.1 we give some mild conditions under which the stationary buffer content process $\{X_N^*(t), t \geq 0\}$ in the Nth model converges to the stationary buffer content process $\{X^*(t), t \geq 0\}$ in limiting Gaussian model. We also compare the fluid model with Fractional Brownian Motion input with the limiting model obtained by multiplexing *on-off* sources with heavy-tailed activity period under *heavy traffic environment* parametrization. Such a model plays a great role in the analysis of data traffic in high speed communication networks.

2 Fluid models

Let $\{X(t), t \geq 0\}$ be a fluid model driven by process $\{Z(t),\ t \in \mathbb{R}\}$ as described in the introduction. Following Theorem 13 in Borovkov [6] (see also Kulkarni & Rolski [17]), we can prove the existence of stationary buffer content process $\{X^*(t), t \geq 0\}$ for fluid model (1.1).

Proposition 2.1 *If* $\{Z(t),\ t \in \mathbb{R}\}$ *is a stationary (in strict sense), ergodic process and* $\mathbb{E}Z(t) < c$, *then*

$$\{X^*(t), t \geq 0\} =_d \{\sup_{s\leq t} \int_s^t (Z(v) - c)\, dv, t \geq 0\}\ ,$$

where

$$\lim_{s\to\infty} \mathbb{P}(X(s) > x) = \mathbb{P}(X^*(t) > x) \quad \text{for all } t \geq 0\ .$$

Let $\{Z_N(t),\ t \in \mathbb{R}\}$ be the superposition of N independent homogeneous alternating *on-off* flows. That is, $Z_N(t) = \sum_{n=1}^N \xi_n(t)$, where by *on-off* flow $\{\xi_n(t),\ t \in \mathbb{R}\}$ we mean $0-1$ stationary process in which consecutive 0-periods (*off* times) alternate with 1-periods (*on* times). The *on* and *off* times are i.i.d. variables having the same distribution for all sources.

By $T_{\rm off}$ and $T_{\rm on}$ we denote the generic *off* and *on* times respectively. We assume that $T_{\rm off}$ and $T_{\rm on}$ are i.i.d. with distribution functions $F_{\rm on}$ and $F_{\rm off}$ respectively and that $\mathbb{E}(T_{\rm on} + T_{\rm off}) < \infty$ (hence there exists a stationary version of the process $\{\xi_n(t),\ t \in \mathbb{R}\}$). Let the covariance function of $\{\xi_n(t),\ t \in \mathbb{R}\}$ be $R(t)$.

By $\{X_N(t),\ t \geq 0\}$ we mean the fluid model driven by $\{r_N Z_N(t),\ t \in \mathbb{R}\}$, whose dynamics is described by

$$\frac{dX_N(t)}{dt} = \begin{cases} r_N Z_N(t) - c_N & X_N(t) > 0 \\ (r_N Z_N(t) - c_N)_+ & X_N(t) = 0\ , \end{cases} \tag{2.2}$$

where r_N, c_N are respectively chosen constants. Note that if we assume the stability condition

$$N f r_N < c_N\ , \tag{2.3}$$

where

$$f = \frac{\mathbb{E}T_{\rm on}}{\mathbb{E}(T_{\rm on} + T_{\rm off})}\ ,$$

then by Proposition 2.1 there exists a stationary buffer content process $\{X_N^*(t), t \geq 0\}$ fulfilling (2.2), and

$$\{X_N^*(t), t \geq 0\} =_d \{\sup_{s\leq t} \int_s^t (r_N Z_N(v) - c_N)\, dv,\ t \geq 0\}\ .$$

Remark 2.1 It is clear that we must have $N r_N > c_N$. Otherwise the buffer is empty in steady state. Notice that f is the steady state probability that the flow is in the state *on*.

3 Main theorem

Consider the following asymptotic parametrization, called *the heavy traffic environment*:

$$r_N \sqrt{N} = r \tag{3.4}$$

$$c_N - N f r_N = c \tag{3.5}$$

for some $r, c > 0$. Let

$$Z_N^*(t) = \frac{Z_N(t) - fN}{\sqrt{N}}\ .$$

and $\{Z^*(t), t \in \mathbb{R}\}$ be the stationary centred Gaussian process with covariance function $R(t)$.

In the following theorem we find sufficient conditions for the weak convergence

$$\{X_N^*(t), t \geq 0\} \xrightarrow{\mathcal{D}} \{X^*(t), t \geq 0\}$$

in $C[0,\infty)$, where $\{X^*(t), t \geq 0\}$ is the stationary buffer content process

$$\{X^*(t), t \geq 0\} =_d \{\sup_{s \leq t} \int_s^t (rZ^*(v) - c)\, dv,\ t \geq 0\}$$

driven by $\{Z(t) = rZ^*(t), t \in \mathbb{R}\}$.

Theorem 3.1 *If*

$$\{Z_N^*(t), t \in \mathbb{R}\} \xrightarrow{\mathcal{D}} \{Z^*(t), t \in \mathbb{R}\} \tag{3.6}$$

for $N \to \infty$ in such a way that (3.4) and (3.5) hold, then

$$\{X_N^*(t), t \geq 0\} \xrightarrow{\mathcal{D}} \{X^*(t), t \geq 0\}. \tag{3.7}$$

We give the proof of Theorem 3.1 in Section 4.

Remark 3.1 Note that since $\xi_n(t)$ are stochastically continuous, then from Hahn [13] $\{Z^*(t), t \in \mathbb{R}\}$ has continuous sample paths a.s.

Theorem 3.1 says that to obtain the convergence $\{X_N^*(t), t \geq 0\} \xrightarrow{\mathcal{D}} \{X^*(t), t \geq 0\}$ it is enough to study the convergence of $\{Z_N^*(t), t \in \mathbb{R}\} \xrightarrow{\mathcal{D}} \{Z^*(t), t \in \mathbb{R}\}$. In the following we recall the result which deals with this convergence. Let

$$F_{\text{off}}^{(0)}(t) = \int_0^t \frac{\bar{F}_{\text{off}}(s)}{\mathbb{E}T_{\text{off}}}\, ds\ , \qquad F_{\text{on}}^{(0)}(t) = \int_0^t \frac{\bar{F}_{\text{on}}(s)}{\mathbb{E}T_{\text{on}}}\, ds$$

and

$$\tilde{F}(t) = \max\{F_{\text{off}}(t), F_{\text{off}}^{(0)}(t), F_{\text{on}}(t), F_{\text{on}}^{(0)}(t)\}\ .$$

Theorem 3 of Szczotka [23] adapted to our case says following.

Lemma 3.1 *Suppose that*

$$\lim_{t \to 0} \tilde{F}(t) t^{-\alpha} < \infty \tag{3.8}$$

for $\alpha > 1/2$ and functions

$$H_0(t) = (1-f) F_{\text{off}}^{(0)} * F_{\text{on}} * H(t) + f F_{\text{on}}^{(0)} * H(t)$$

and

$$H_1(t) = (1-f) F_{\text{off}}^{(0)} * H(t) + f F_{\text{on}}^{(0)} * F_{\text{off}} * H(t)$$

are continuous, where $H = \sum_{n=1}^{\infty} F_{\text{on}}^{*n} * F_{\text{off}}^{*n}$. Then $\{\xi(t),\ t \in \mathbb{R}\}$ satisfy the CLT in the $D(\mathbb{R}, d)$, where d is a Skorokhod metric, that is

$$\{Z_N^*(t), t \in \mathbb{R}\} \xrightarrow{\mathcal{D}} \{Z^*(t), t \in \mathbb{R}\}\ . \tag{3.9}$$

Corollary 3.1 *If F_{off} and F_{on} are absolutely continuous with densities $f_{\text{off}}(t)$ and $f_{\text{on}}(t)$ such that $f_{\text{on}}(0+) < \infty$ and $f_{\text{off}}(0+) < \infty$, then convergence (3.9) holds.*

Proof. Under above assumptions H_0 and H_1 are obviously continuous and $\max\{F_{\text{off}}(t), F_{\text{on}}(t)\}t^{-1} < \infty$. Furthermore, $F_{\text{on}}^{(0)}(t) \leq t/\mathbb{E}T_{\text{on}}$ and $F_{\text{off}}^{(0)}(t) \leq t/\mathbb{E}T_{\text{off}}$ and therefore the assumptions of Lemma 3.1 are fulfilled for $\alpha = 1$.

□

Remark 3.2 The result of Szczotka excludes F_{off} and F_{on} with densities that are not bounded in the neighbourhood of 0. We belive that this restriction can be weakened.

From Theorem 3.1 and Corollary 3.1 we have the following proposition.

Proposition 3.1 *If on and off time distributions are absolutely continuous with densities such that $f_{\text{on}}(0+) < \infty$ and $f_{\text{off}}(0+) < \infty$, then for $N \to \infty$ in such a way that (3.4) and (3.5) hold we have*

$$\{X_N^*(t), t \geq 0\} \xrightarrow{\mathcal{D}} \{X^*(t), t \geq 0\} . \tag{3.10}$$

One of the important characteristics considered in fluid model theory is the steady state buffer content. Let

$$X_N^* =_d X_N^*(0), \qquad X^* =_d X^*(0)$$

be the steady state buffer content in the Nth model and in the limiting Gaussian model respectively. Under conditions (3.4) and (3.5) Kulkarni & Rolski [17] proved the convergence $X_N^* \xrightarrow{\mathcal{D}} X^*$ for Markovian *on-off* process $\xi(t)$. In Corollary 3.2 we extend this result.

Corollary 3.2 *If on and off time distributions are absolutely continuous with densities such that $f_{\text{on}}(0+) < \infty$ and $f_{\text{off}}(0+) < \infty$, then for $N \to \infty$ in such a way that (3.4) and (3.5) hold we have*

$$X_N^* \xrightarrow{\mathcal{D}} X^* . \tag{3.11}$$

In the rest of this section we give application of Theorem 3.1 and Corollary 3.2 to fluid models considered in the literature. We assume that the T_{on} and T_{off} times are absolutely continuous distributed with densities bounded in the neighbourhood of 0.

Example 3.1 Palmowski & Rolski [21] considered fluid models that are driven by the superposition of N alternating *on-off* flows with T_{on} and T_{off} times such that $\mathbb{E}(T_{\text{on}} + T_{\text{off}})^3 < \infty$. They proved that under *heavy traffic environment* parametrization the tail of distribution of steady state buffer content in the Nth model has the same asymptotic (in logarithmic sense) like in the limiting fluid model driven by a Gaussian process (which was studied by Dębicki & Rolski [9] or Dębicki, Michna & Rolski [10]). However, they did not prove that $X_N^* \xrightarrow{\mathcal{D}} X^*$. Note that from Corollary 3.2 it is obvious that this convergence holds. Moreover, from Proposition 3.1 we have the weak convergence of the stationary buffer content process in the Nth model to the stationary buffer content process in limiting Gaussian model:

$$\{X_N^*(t), t \geq 0\} \xrightarrow{\mathcal{D}} \{X^*(t), t \geq 0\} .$$

Example 3.2 Recent measurement of data traffic in computer networks has pointed out that data traffic is best approximated by models possessing *long range dependence* structure (see for example Leland [18]). We consider a special model that generates this type of structure. Let $\xi(t)$ be a stationary alternating *on-off* process with $T_{\rm on}$ and $T_{\rm off}$ times such that:

$$\bar{F}_{\rm on}(t) \sim At^{2H-3} , \quad t \to \infty \tag{3.12}$$

$$\bar{F}_{\rm off}(t) = o(\bar{F}_{\rm on}(t)) , \quad t \to \infty , \tag{3.13}$$

where $1/2 < H < 1$ and $A > 0$ is a given constant. A fluid model which is driven by $\xi(t)$ process was analysed by Willinger *et al* [24] and Heath *et al* [14]. Note that $1 < 3-2H < 2$ and hence $\mathbb{E}(T_{\rm on}+T_{\rm off}) < \infty$. Thus there exists the stationary version of the input process.

We study the asymptotic of the tail of the stationary distribution of buffer content $\Psi(u) = \mathbb{P}(\sup_{t\leq 0}(\int_0^t Z(s)ds - ct) > u)$ for limiting case of multiplexing this model in *heavy traffic environment*, where $Z(t)$ is stationary centred Gaussian process with covariance function $R(t)$. Without loss of generality we can assume that in (3.4) and (3.5) $r = 1$ and $A = 1$. From Heath *et al* [14] the covariance function $R(t)$ of $\xi(t)$ process has the following asymptotic

$$R(t) \sim \frac{\mu_{\rm off}^2}{(2-2H)(\mu_{\rm on}+\mu_{\rm off})^3} t^{2H-2} , \quad t \to \infty . \tag{3.14}$$

Note that $\int_0^\infty R(t)dt = \infty$ so this model posses *long range dependence* structure. Moreover, we have $\mathbb{E}T_{\rm on}^2 = \infty$. From Corollary 3.2 we have that $X_N^* \xrightarrow{\mathcal{D}} X^*$. It is remarkable that from Dębicki [11]

$$\lim_{u\to\infty} \frac{\log(\Psi(u))}{u^{2-2H}} = -\alpha\delta ,$$

where

$$\alpha = \frac{1}{2}\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H}$$

$$\delta = \frac{(2-2H)(2H-1)H(\mu_{\rm on}+\mu_{\rm off})^3}{\mu_{\rm off}^2} .$$

This is related to Norros [20] who considered a fluid model with the Fractional Brownian Motion with Hurst parameter $1/2 < H < 1$ (**FBM**) input. It seems to be interesting that (in the logarithmic sense) the asymptotic of the $\Psi(u)$ in the Norros's model is the same as the asymptotic of the model presented in this example. But in contrast to **FBM** case in the model analysed here the input process has differentiable sample paths almost surely.

4 Proof of theorem

Before the proof of Theorem 3.1 we give some lemmas that are of technical interest.

Lemma 4.1 *If*

$$\mathbb{E} \sup_{0\leq s\leq 1} |Z(s)| < \infty ,$$

then for all $\eta, \epsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\mathbb{P}(\sup_{0\leq s\leq\delta} |Z(s) - c| \geq \frac{\epsilon}{\delta}) \leq \eta\delta . \tag{4.15}$$

Proof. Assume that (4.15) does not hold. Then there exist $\epsilon, \eta > 0$ such that for all $\delta \in (0,1)$

$$\mathbb{P}(\sup_{0\le s\le \delta} |Z(s)| \ge \rho) \ge \eta\delta \ , \tag{4.16}$$

where $\rho = \frac{\epsilon}{\delta}$. We have

$$\begin{aligned} \mathbb{E} \sup_{0\le s\le 1} |Z(s)| &\ge \int_\epsilon^\infty \mathbb{P}(\sup_{0\le s\le 1} |Z(s)| \ge \rho)\, d\rho \\ &\ge \int_\epsilon^\infty \mathbb{P}(\sup_{0\le s\le \delta} |Z(s)| \ge \rho)\, d\rho \\ &\ge \int_\epsilon^\infty \frac{\epsilon\eta}{\rho}\, d\rho = \infty \ , \end{aligned} \tag{4.17}$$

where (4.17) follows from (4.16). So we obtained the contradiction to the assumption of lemma.

□

Let

$$\{Y_N(t) = \int_0^t (r_N Z_N(s) - c_N)\, ds,\ t \in \mathbb{R}\}$$

and

$$\{Y(t) = \int_0^t (r Z^*(s) - c)\, ds,\ t \in \mathbb{R}\} \ ,$$

where $\{Z_N(t),\ t \in \mathbb{R}\}$ is the superposition of alternating *on-off* flows and $\{Z^*(t),\ t \in \mathbb{R}\}$ is a centred stationary Gaussian process with the covariance function $R(t)$ of the single flow. Note that

$$\begin{aligned} Y_N(t) &= \int_0^t (r_N Z_N(s) - c_N)\, ds \\ &= \int_0^t (r_N \sqrt{N} \frac{\sum_{n=1}^N \xi_n(s)}{\sqrt{N}} - c_N)\, ds \\ &= \int_0^t (r_N \sqrt{N} Z_N^*(s) - (c_N - N f r_N))\, ds \\ &= \int_0^t (r Z_N^*(s) - c)\, ds \\ &= \int_0^t (r Z_N^*(s) - c)_+\, ds - \int_0^t (r Z_N^*(s) - c)_-\, ds \ , \end{aligned}$$

where $(f(t))_+ = \max\{0, f(t)\}$ and $(f(t))_- = -\min\{0, f(t)\}$. Similarly,

$$\begin{aligned} Y(t) &= \int_0^t (r Z^*(s) - c)\, ds \\ &= \int_0^t (r Z^*(s) - c)_+\, ds - \int_0^t (r Z^*(s) - c)_-\, ds \ . \end{aligned}$$

Thus

$$\begin{aligned} Y_N(t) &= Y_N^+(t) - Y_N^-(t) \\ Y(t) &= Y^+(t) - Y^-(t) \ , \end{aligned}$$

where

$$Y_N^{+/-}(t) = \int_0^t (rZ_N^*(s) - c)_{+/-}\, ds, \qquad Y^{+/-}(t) = \int_0^t (rZ^*(s) - c)_{+/-}\, ds$$

are nondecreasing processes. Moreover, let

$$\begin{aligned}\beta_{k,N}^{+/-} &= Y_N^{+/-}(k) - Y_N^{+/-}(k-1)\\ \beta_k^{+/-} &= Y^{+/-}(k) - Y^{+/-}(k-1)\ .\end{aligned}$$

The following lemma follows from Theorem 4 from Borovkov [7], p. 216.

Lemma 4.2 *If*
(I) the process $\{(Y_N^+(t), Y_N^-(t)), t \in \mathbb{R}\}$ *has jointly stationary increments;*
(II) the sequence $\{\beta_k = \beta_k^+ - \beta_k^-\}$ *is ergodic and* $\mathbb{E}\beta_1 < 0$*;*
(III) the finite-dimensional distributions of $Y_N^{+/-}(t)$ *weakly converge to the finite-dimensional distributions of* $Y^{+/-}(t)$ *for* $N \to \infty$*;*
(IV)

$$\mathbb{E}\beta_{1,N}^+ \to \mathbb{E}\beta_1^+\ ,$$

then for $N \to \infty$ *the finite-dimensional distributions of* $\{X_N^*(t), t \geq 0\}$ *weakly converge to the finite-dimensional distributions of* $\{X^*(t), t \geq 0\}$.

Proof of Theorem 3.1. First we show that finite dimensional distributions of the processes $\{X_N^*(t), t \geq 0\}$ converge to finite dimensional distributions of $\{X^*(t), t \geq 0\}$. To do it we check conditions (I)-(IV) of Lemma 4.2. By stationarity of the processes $\{Z_N^*(t),\ t \in \mathbb{R}\}$ and $\{Z^*(t),\ t \in \mathbb{R}\}$ we have that $\{(Y_N^+(t), Y_N^-(t)), t \in \mathbb{R}\}$ $(N = 1, 2, ...)$ have stationary nonnegative increments. From assumption of Theorem 3.1 we have

$$\{rZ_N^*(t) - c,\ t \in \mathbb{R}\} \stackrel{\mathcal{D}}{\to} \{rZ^*(t) - c,\ t \in \mathbb{R}\}\ . \tag{4.18}$$

Because $(f(t))_+ = \max\{0, f(t)\}$ and $(f(t))_- = -\min\{0, f(t)\}$ are continuous functionals and for given $t_1, t_2, \ldots, t_k$ mapping

$$\{X(t), t \geq 0\} \longrightarrow \left(\int_0^{t_1} X(s)\, ds, \ldots, \int_0^{t_k} X(s)\, ds\right)$$

is continuous on $D(\mathbb{R}, d)$, then from (4.18) we get that finite dimensional distributions of $\{Y_N^{+/-}(t),\ t \in \mathbb{R}\}$ weakly converge to finite dimensional distributions of $\{Y^{+/-}(t),\ t \in \mathbb{R}\}$. Hence conditions (I) and (III) are fulfilled.

To prove condition (II) note that from Fubini theorem

$$\begin{aligned}\mathbb{E}\beta_1 &= \mathbb{E}\int_0^1 (rZ^*(s) - c)\, ds\\ &= \mathbb{E}(rZ^*(0) - c)\\ &= -c < 0\ .\end{aligned}$$

Similarly,

$$\mathbb{E}\beta_{1,N}^{+/-} = \mathbb{E}(rZ_N^*(0) - c)_{+/-} \tag{4.19}$$

$$\mathbb{E}\beta_1^{+/-} = \mathbb{E}(rZ^*(0) - c)_{+/-}\ . \tag{4.20}$$

Note that $\{\beta_k = \int_k^{k+1} (rZ^*(s) - c)\, ds\}_{k=1,2,\ldots}$ is Gaussian with $\mathbb{E}\beta_k = -c$. Thus to prove the ergodicity of $\{\beta_k\}_{k=1,2,\ldots}$ it suffices to show that $r(k) = \mathbf{Cov}(\beta_k, \beta_0) \to 0$ for $k \to \infty$ (see Shirayev [22], p. 385). This follows from the chain of equalities

$$\begin{aligned} r(k) &= \left[\mathbb{E} \int_0^1 (rZ^*(s) - c)\, ds \int_k^{k+1} (rZ^*(s) - c)\, ds\right] - c^2 \\ &= \int_0^1 \int_k^{k+1} (\mathbb{E}(rZ^*(t) - c)(rZ^*(s) - c))\, dt\, ds - c^2 \\ &= r^2 \int_0^1 \int_k^{k+1} \mathbb{E} Z^*(s) Z^*(t)\, dt\, ds \\ &= r^2 \int_0^1 \int_k^{k+1} R(t-s)\, dt\, ds \to 0 \end{aligned}$$

and the fact that for ergodic alternating processes $R(t) \to 0$ for $t \to \infty$. Hence condition (II) is fulfilled.

Finally, to prove condition (IV) by (4.19) and (4.20), it suffices to show that following convergence holds

$$\mathbb{E}(rZ_N^*(0) - c)_+ \longrightarrow \mathbb{E}(rZ^*(0) - c)_+ \tag{4.21}$$

for $N \to \infty$. From (4.18) and the fact that functional $(.)_+$ is continuous we get

$$(rZ_N^*(0) - c)_+ \xrightarrow{\mathcal{D}} (rZ^*(0) - c)_+ \; . \tag{4.22}$$

To obtain (4.21) we prove that second moments of $(rZ_N^*(0) - c)_+$ are bounded:

$$\begin{aligned} \mathbb{E}((rZ_N^*(0) - c)_+)^2 &\leq \mathbb{E}(rZ_N^*(0) - c)^2 \\ &= r^2 f(1-f) + c^2 < \infty \; . \end{aligned}$$

This completes the proof of the convergence of finite dimensional distributions of the processes $\{X_N^*(t), t \geq 0\}$ to finite dimensional distributions of $\{X^*(t), t \geq 0\}$. Besides it, to show convergence (3.7) we need to verify that sequence $\{X_N^*(t), t \geq 0\}_{N=1,2,\ldots}$ is tight. From Theorem 8.3 of Billingsley [3] and Lindvall [19] it suffices to check that
(i) for each $\eta > 0$ there exists a such that $\mathbb{P}(|X_N^*(0)| > a) \leq \eta$ for all $N \geq 1$;
(ii) for all $\epsilon, \eta > 0$ there exists $0 < \delta < 1$ and n_0 such that

$$\mathbb{P}(\sup_{t \leq s \leq t+\delta} |X_N^*(s) - X_N^*(t)| \geq \epsilon) \leq \eta\delta$$

for all $N \geq n_0$ and $t > 0$.
Condition (i) follows from stability condition (2.3) and the fact that $X_N^*(0) \xrightarrow{\mathcal{D}} X^*(0)$.

To prove (ii) notice that from Proposition 1 of Mazumdar & Rainer [5]

$$X_N^*(s) - X_N^*(t) = \int_t^s 1_{[X_N^*(u) > 0]} (r_N Z_N(u) - c_N) du \; , \tag{4.23}$$

which immediately gives

$$\sup_{t \leq s \leq t+\delta} |X_N^*(s) - X_N^*(t)| \leq_d \delta \sup_{0 \leq s \leq \delta} |r_N Z_N(s) - c_N| \tag{4.24}$$

and

$$\mathbb{P}(\sup_{t\le s\le t+\delta}|X_N^*(s)-X_N^*(t)|\ge\epsilon)\le\mathbb{P}(\sup_{0\le s\le\delta}|r_NZ_N(s)-c_N|\ge\frac{\epsilon}{\delta})\ . \tag{4.25}$$

By assumption of Theorem 3.1

$$\{Z_N^*(t),t\in\mathbb{R}\}\stackrel{\mathcal{D}}{\to}\{Z^*(t),t\in\mathbb{R}\}$$

for $N\to\infty$ and consequently

$$\mathbb{P}(\sup_{0\le s\le\delta}|r_NZ_N(s)-c_N|\ge\frac{\epsilon}{\delta})\to\mathbb{P}(\sup_{0\le s\le\delta}|Z(s)-c|\ge\frac{\epsilon}{\delta}) \tag{4.26}$$

for $N\to\infty$.

Since $Z^*(t)$ has continuous sample paths a.s. (see Remark 3.1), thus $\mathbb{E}(\sup_{0\le s\le 1}|Z(s)-c|)<\infty$. From Lemma 4.1 for all $\epsilon,\eta>0$ there exists $0<\delta<1$ such that

$$\mathbb{P}(\sup_{0\le s\le\delta}|Z(s)-c|\ge\frac{\epsilon}{\delta})\le\eta\delta\ , \tag{4.27}$$

which combined with (4.25) and (4.26) gives condition (ii). This completes the proof. □

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