Quantile hedging for equity-linked contracts

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Abstract

We consider an equity-linked contract whose payoff depends on the lifetime of policy holder and the stock price. We provide best strategy for an insurance company assuming the limited capital for the hedging.

Keywords: quantile hedging, equity-linked contract

JEL subject classification: Primary G10; Secondary G12

1 Introduction

Equity-linked insurance contracts have been studied since the middle of the 1970s (see Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Delbaen (1986)). Among later authors on this topic are Bacinello and Ortu (1993), Aase and Persson (1994), Ekern and Persson (1996), Boyle and Hardy (1997), Bacinello (2001), Moeller (2001), Melnikov (2004a,b), Melnikov and Romanyuk (2008). The payoff of equity-linked policies depends on two factors: the value of stock price (hence the term equity-linked), and some insurance-type event in the life of the owner of the contract (death, retirement, survival to a certain date, etc.). As such, the payoff contains both financial and insurance risk elements, which have to be priced so that the resulting premium is fair to both the seller and the buyer of the contract. This is not completely possible, because the mortality risk makes the market incomplete. There are very few methods providing an appropriate risk-management in connection of such a contracts which exploit some imperfect hedging forms.

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In most of these papers the insurance risk is simply not priced explicitly disregarding mortality risk increasing the premium rate. It is called "unsystematic risk," for which the insurer does not receive any compensation. The usual argument is that mortality risk can be diversified by selling the large number of equity-linked contracts. However in reality there are only a finite number of contracts. Hodges and Neuberger (1989) used utility-based indifference pricing approach where the premium for the contract is calculated in such a way as to make the hedger indifferent between including and not including a specified number of contracts in their portfolio (see also Young and Zriphopoulou (2002, 2005)). Moeller (1989, 2001) proposed another approach exploiting the mean-variance hedging theory of Föllmer-Sondermann-Schweizer. In this paper, following Sekine (2000) and Föllmer and Leukert (1999), we use hedging methodology, whose goal is to:

- maximizes the probability of success of hedge under a given initial capital or
- maximizes the the expected success ratio.

There are a number of papers adapting the quantile hedging approach to the insurance setting. Krutchenko and Melnikov (2001), Melnikov (2004a,b), Kirch and Melnikov (2005), Melnikov and Skornyakova (2005), Melnikov and Romanyuk (2008) consider the contracts pure endowment with flexible guarantees, that is one or more risky assets modeled by correlated Wiener processes or by jump-diffusion processes and the payoff $D\mathbf{1}_{\{\tau(x)>T\}}$, where $\tau(x)$ is the remaining lifetime of a person of current age x, D is independent financial payoff (usually the largest of the values of the risky assets at the maturity T). Thus the contract is exercised if the insured is still alive at the maturity time. Note that this payoff could be represented as D_E, where $D_2 = D$, $D_1 = 0$ and $E = \mathbf{1}_{\{\tau(x)>T\}}+1$ is independent of the financial market random variable. In this paper we consider more complex payoff function D_E, where D_i are financial contracts and independent random variable $E \in \{1, 2, \ldots, n\}$ describes the state of insured. The results are illustrated by a numerical actuarial analysis.

The paper is organized as follows. The Section 2 introduces a model of financial market and the structure of an insurance product we consider. We also state and give solution of both problems of hedging. In Section 3 we present numerical example. Finally, in Section 4 we give the proofs of the main results.

2 Main Results

Consider Brownian motion $W = (W_t)_{t\in[0,T]}$ on a probability space $(\Omega_1, \mathcal{F}, \mathbb{P}_1)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, where $\mathcal{F}_t \subset \mathcal{F}$, denote a natural filtration of W. Here T denotes a fixed time horizon describing the expiry date of the equity-linked contract. Consider a process of discounted price $X = (X_t)_{t\in[0,T]}$ given by equation:

$$\mathrm{d}X_t = X_t m \mathrm{d}t + X_t \sigma \mathrm{d}W_t.$$

As is typical in such a setting, we assume that it is possible to trade in the financial market without any frictions (all assets are perfectly divisible, there are no transaction costs, there are no restrictions on borrowing, etc.), and that the market admits no arbitrage opportunities.

Consider an insurance product, which pays an amount D_E where E is an integer valued random variable independent of W (hence also X) defined on a probability space $(\Omega_2, \mathcal{E}, \mathbb{P}_2)$. For example E will be related with the lifetime of the policy holder. We will consider probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F} \times \mathcal{E}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}_1 \times \mathbb{P}_2)$, where $\mathcal{G}_t = \mathcal{F}_t$ for $t < T$ and $\mathcal{G}_T = \mathcal{F}_T \times \mathcal{E}$, with the convention $\mathbb{P}(A_E) = \mathbb{P}((\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \omega_1 \in A_{E(\omega_2)})$ for $A_i \in \mathcal{F}$. For $i = 1, 2, \ldots, n$ we will assume:

$$\mathbb{P}(E = i) = p_i > 0,$$

where $\sum_{i=1}^n p_i = 1$. We will assume that D_i are:

- $$\mathcal{F}_T\text{-measurable random variables,} \tag{2.1}$$

- $$0 \leq D_1 \leq D_2 \leq \ldots \leq D_n \qquad \mathbb{P}\text{ −a.s.} \tag{2.2}$$

- $$\mathbb{E}^{\mathbb{P}} D_n < \infty, \tag{2.3}$$

  where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation with respect to some measure $\mathbb{Q}$.

For convenience we will take $D_0 = 0$. Denote the set of all equivalent martingale measures on $(\Omega, \mathcal{G})$ by $\mathcal{P}$. It means that $\mathcal{P}$ is the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{G})$ under which X follows a $\{\mathcal{G}_t\}$-martingale. Define a measure $\mathbb{R} \in \mathcal{P}$ by:

$$\left.\frac{\mathrm{d}\mathbb{R}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = M_t, \tag{2.4}$$

where

$$M_t = \exp\left\{-\frac{m}{\sigma}W_t - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 t\right\}. \tag{2.5}$$

The measure $\mathbb{R}$ is unique martingale measure on $(\Omega_1, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}_1)$.

We will consider the contingent claim H being $\mathcal{G}_T$-measurable, nonnegative random variable and the replicating investment strategies, which are expressed in terms of the integrals with respect to X. That is, we will deal with the self-financing admissible trading strategies (V_0, ξ) on $[0,T]$ for which the value process

$$V_t = V_0 + \int_0^t \xi_u \,\mathrm{d}X_u, \qquad t \in [0,T],$$

is well defined and that generate non-negative wealth:

$$V_t \geq 0, \qquad \mathbb{P} - \text{a.s.}$$

for all $t \in [0,T]$.

Next we state two main problems which will be solved in this paper:

Problem 1

Fix an initial capital $\tilde{V}_0$. Among all admissible strategies such that $V_0 \leq \tilde{V}_0$ find one that maximizes $\mathbb{P}(V_T \geq D_E)$.

The insurance company which wants to hedge with the strategy defined in the **Problem 1** wants to minimize the probability of loss, but it does not control the size of the potential loss. To take into account also the situation when the strategy may not give the successful hedge we will also consider the following **Problem 2**.

Problem 2

Fix an initial capital $\tilde{V}_0$. Among all admissible strategies satisfying $V_0 \leq \tilde{V}_0$ find one that maximizes *expected success ratio*:

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{V_T \geq D_E)} + \mathbf{1}_{\{V_T < D_E\}}\frac{V_T}{D_E}\right]. \tag{2.6}$$

Solution of the Problem 1. Denote $C_i = (D_i - D_{i-1})M_T$ for $i \geq 1$. Define sets:

$$\begin{aligned}
\tilde{A}_{n+1} &:= \emptyset \qquad (2.7)\\
\tilde{A}_n &:= \{p_n - kC_n > 0\}\\
\tilde{A}_{n-1} &:= \{p_{n-1} - kC_{n-1} + \mathbf{1}_{\tilde{A}_n}(p_n - kC_n) > 0\}\\
\tilde{A}_{n-2} &:= \{p_{n-2} - kC_{n-2} + \mathbf{1}_{\tilde{A}_{n-1}}(p_{n-1} - kC_{n-1} + \mathbf{1}_{\tilde{A}_n}(p_n - kC_n)) > 0\}\\
\dots &\qquad \dots
\end{aligned}$$

and

$$A_i^* := \bigcap_{j \le i} \tilde{A}_j. \tag{2.8}$$

Assume that constant k is chosen in that way that equality:

$$\mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} D_i \mathbf{1}_{A_i^*/A_{i+1}^*}\right] = \tilde{V}_0 \tag{2.9}$$

is satisfied.

Theorem 2.1. *Define a contingent claim:*

$$H = \sum_{i=1}^{n} D_i \mathbf{1}_{A_i^*/A_{i+1}^*}, \tag{2.10}$$

where sets A_i^* are defined in (2.8)-(2.9). Consider the strategy $(\tilde{V}_0, \tilde{\xi})$ being the perfect hedge for H:

$$H = \mathbb{E}^{\mathbb{R}} H + \int_0^T \xi_s \mathrm{d}X_s = \tilde{V}_0 + \int_0^T \xi_s \mathrm{d}X_s.$$

Then $(\tilde{V}_0, \tilde{\xi})$ is the super replication strategy of $\mathbf{1}_{A_E^*} D_E$ and it is the solution of the Problem 1. The maximal probability is equal to $\mathbb{P}(A_E^*) = \sum_{i=1}^{n} p_i \mathbb{P}(A_i^*)$.

Solution of the Problem 2.

For every positive real number k denote

$$\begin{aligned} \phi_n^* &:= \sum_{i=1}^{n} \frac{D_i - D_{i-1}}{D_n} \mathbf{1}_{\{\sum_{j=i}^{n} \frac{p_j}{D_j M_T} \le k\}} \\ \phi_i^* &:= \min\left(1, \frac{\phi_n^* D_n}{D_i} \mathbf{1}_{\{D_i > 0\}}) + \mathbf{1}_{\{D_i = 0\}}\right) \quad \text{for } i < n \end{aligned} \tag{2.11}$$

where M_t is defined in (2.5) and k satisfies

$$\mathbb{E}^{R} \phi_n^* D_n = \mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} (D_i - D_{i-1}) \mathbf{1}_{\{\sum_{j=i}^{n} \frac{p_j}{D_j M_T} \le k\}}\right] = \tilde{V}_0. \tag{2.12}$$

Theorem 2.2. *Define a contingent claim:*

$$H = \max_{i=1,2,\ldots,n} \phi_i^* D_i,$$

where ϕ_i^* are defined in (2.11)-(2.12). If strategy $(\tilde{V}_0, \tilde{\xi})$ gives a perfect hedge for H:

$$H = \mathbb{E}^{\mathbb{R}} H + \int_0^T \xi_s \mathrm{d}X_s = \tilde{V}_0 + \int_0^T \xi_s \mathrm{d}X_s,$$

then $(\tilde{V}_0, \tilde{\xi})$ is the super replication strategy of $\phi_E^* D_E$ and it is the solution of the Problem 2. The maximal expected success ratio is equal to $\mathbb{E}^{\mathbb{P}} \phi_E^* = \sum_{i=1}^n p_i \mathbb{E}^{\mathbb{P}} \phi_i^*$.

3 Numerical examples

3.1 Maximizing of probability of successful hedge

Consider an insurance product paying a person an age x an amount F ($\mathcal{F}_T$-measurable random variable), if he/she survives T years. Using actuarial notation $D_1 = 0$, $D_2 = F$, $p_1 = {}_T q_x$ and $p_2 = {}_T p_x$. We will hedge this product using capital $\tilde{V}_0$.

We will use the Theorem 2.1 with the following inputs:

$$C_1 = 0$$

and

$$C_2 = F \exp\left\{ -\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 t \right\}.$$

Moreover,

$$\tilde{A}_3 = \emptyset,$$

$$\tilde{A}_2 = \left\{ {}_T p_x - kF \exp\left\{ -\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 t \right\} > 0 \right\},$$

and

$$\tilde{A}_1 = \left\{ {}_T q_x + 1_{\tilde{A}_2} \left({}_T p_x - kF \exp\left\{ -\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 t \right\} \right) > 0 \right\}.$$

Thus

$$A_3^* = \tilde{A}_3, \qquad A_2^* = \tilde{A}_2, \qquad A_1^* = \tilde{A}_1.$$

Constant k is defined by (2.9), i.e.

$$E^{\mathbb{P}_1}\left[F\exp\left\{-\frac{m}{\sigma}W_T-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2T\right\}1_{\left\{{}_Tp_x>kF\exp\left\{-\frac{m}{\sigma}W_T-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2T\right\}\right\}}\right]=\tilde{V}_0.$$

We will describe an algorithm which produce the probability of hedging in this model:

1. Simulate W_T^i $(i=1,2,\ldots,N)$ according to the normal distribution $N(0,T)$ and calculate $Y^i=F^i\exp\left\{-\frac{m}{\sigma}W_T^i-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2T\right\}$.

2. Denote by $Y^{(l)}$ the number on the lth position of the sorted $\{Y^i\}$ in the descending order. Find j satisfying: $\tilde{V}_0\in\left[\frac{1}{N}\sum_{i=1}^{j}Y^{(i)},\frac{1}{N}\sum_{i=1}^{j+1}Y^{(i)}\right)$. We estimate k using the estimator

$$\hat{k}=\frac{{}_Tp_x}{Y^{(j)}}.$$

3. We estimate the probabilities $\mathbb{P}(A_1^*)$ and $\mathbb{P}(A_2^*)$ by:

$$r_1:=\frac{\sum_{i=1}^{N}1_{\left\{{}_Tq_x+1_{\left\{{}_Tp_x>\hat{k}F\exp\left\{-\frac{m}{\sigma}W_T^i-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2T\right\}\right\}}\left({}_Tp_x-\hat{k}F\exp\left\{-\frac{m}{\sigma}W_T^i-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2t\right\}\right)\right\}}}{N}$$

   and

$$r_2:=\frac{\sum_{i=1}^{N}1_{\left\{{}_Tp_x>\hat{k}F\exp\left\{-\frac{m}{\sigma}W_T^i-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2T\right\}\right\}}}{N},$$

   respectively.

4. Finally, the probability of the optimal hedging can be estimated by: ${}_Tp_xr_1+{}_Tq_xr_2$.

For calculations we took: $\sigma=0.15,m=0.05,N=100000,T=5,S_0=100,F=(S_T-100)^+,\tilde{V}_0=\alpha E^{\mathbb{R}}F$. See Figure 1 for the results.

3.2 Maximizing of success ratio

Let us consider a market index with the following dynamics:

$$\mathrm{d}X_t=0.05X_t\mathrm{d}t+0.15X_t\mathrm{d}W_t,\quad X_0=100.$$

We consider an insurer selling an insurance product paying $\max(100,X_1)$ at the end of the year if the person is alive. We will assume that an insurer

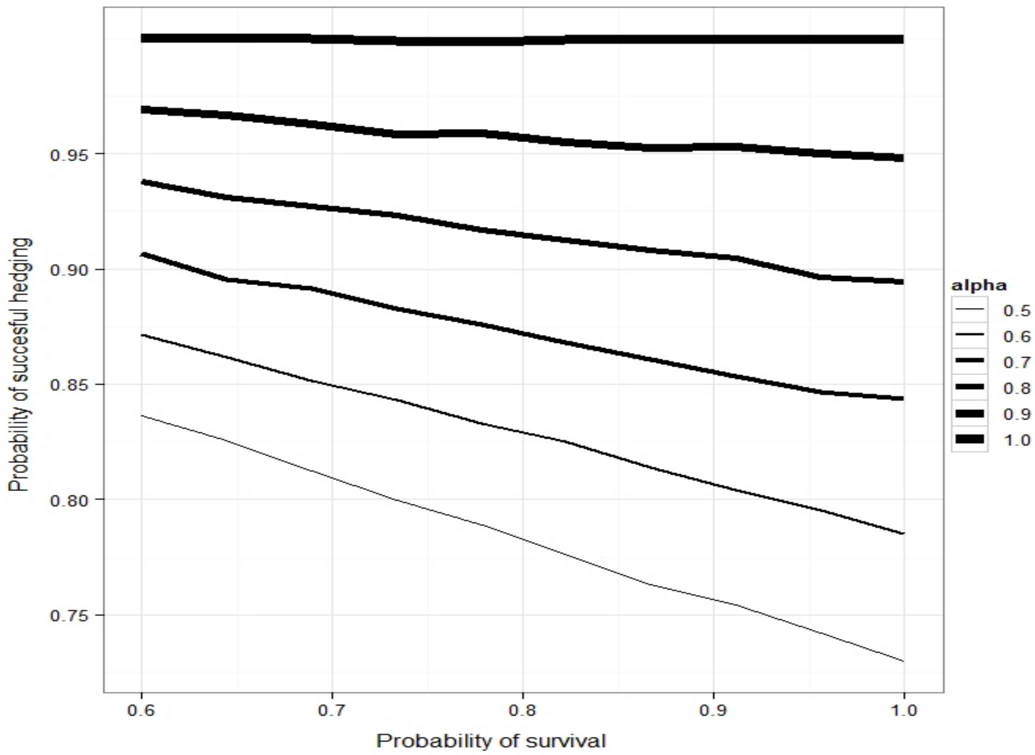

Figure 1: Maximal probability of hedging with respect to the probability of surviving one year. Each line represent different α (starting capital).

sells this product for fifty persons. Thus at the time 1 the insurer pay $D_i = (i-1)\max(100, X_1)$ with probability $p_{i-1} = \binom{n}{i-1} p^{i-1}(1-p)^{n-i+1}, i = 1, 2, \ldots, n = 51$, where p denotes the probability of surviving one year.

The Figure 2 shows the minimal capital (wealth) necessary to achieve claimed expected success ratio. Each line represents different probability of surviving one year.

The algorithm producing each of this curves is based on the Theorem 2.2 and shortly could be described as follows:

1. Take a sample of $(\omega_m)_{m=1,\ldots,N}$'s from Ω_1, which corresponds to the choice of N independent standard normal r.v.'s.

2. Create a data matrix Ξ of the size $(nN) \times 5$, where each row of this

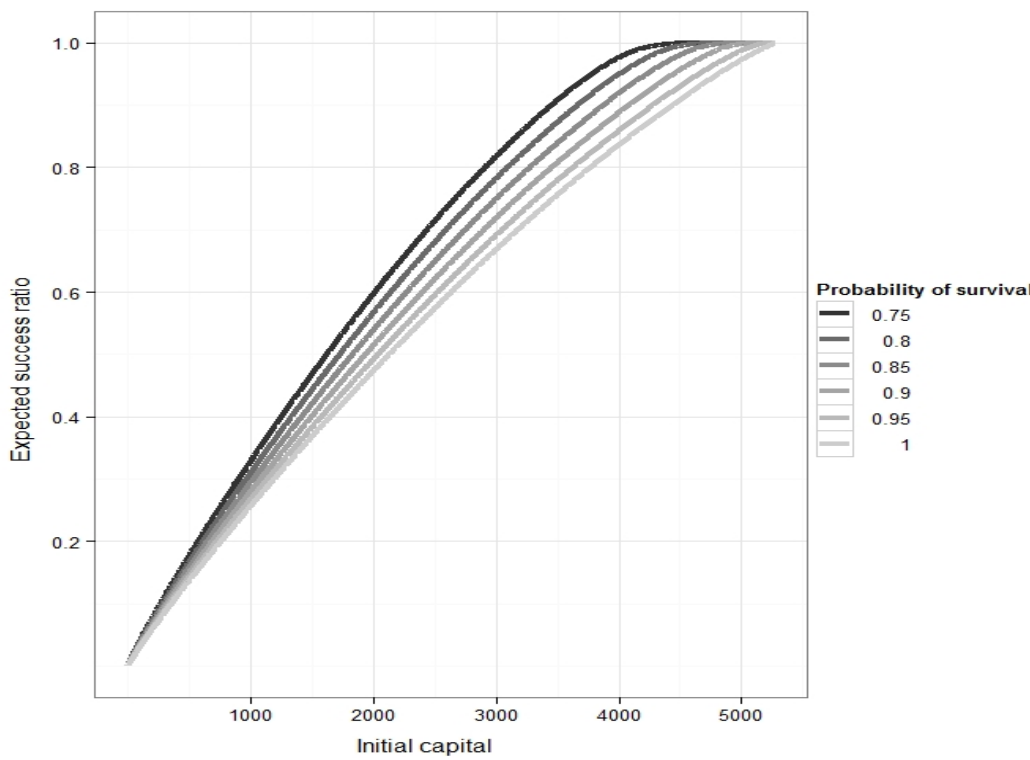

Figure 2: Maximal expected success ratio with respect to the cost of hedging strategy for a portfolio of 50 insurance products. Each line represent different mortality.

matrix is a vector

$$\Big(\omega_m,\ i,\ M_T(\omega_m)(D_i(\omega_m)-D_{i-1}(\omega_m)),\\ \sum_{j=i}^{n}\frac{p_j}{D_j(\omega_m)M_T(\omega_m)},\ \frac{D_i(\omega_m)-D_{i-1}(\omega_m)}{D_n(\omega_m)}\Big)$$

for $m = 1, 2 \ldots, N$ and $i = 1, 2 \ldots, n$.

3. Sort rows of this matrix in ascending order with respect to the forth column.

4. Consider a submatrix consisting of some subset of first rows of Ξ. Calculate the sum of numbers appearing in the third column of this submatrix and divide it by N. We denote this sum by x. In next step

for each m we calculate the sum of the numbers from the fifth column appearing in the rows having ω_m in the first column. We denote this sum by $\hat{\phi}_n(m)$. If there are no rows in this submatrix with ω_m in the first column, then $\hat{\phi}_n(m) = 0$. Calculate

$$y := \frac{\sum_{m=1}^{N} \sum_{i=1}^{n} p_i \min\left(1, \frac{\hat{\phi}_n(m) D_n(\omega_m)}{D_i(\omega_m)} \mathbf{1}_{\{D_i(\omega_m)>0\})} + \mathbf{1}_{\{D_i(\omega)=0\}}\right)}{N}.$$

The estimate of minimal capital necessary to obtain the expected success ratio y equals x.

4 Proofs of the main results

In the proofs we will use the following preliminary lemma.

Lemma 4.1. *For $\alpha = (\alpha_1, \ldots, \alpha_n) \in R^n$ and sequence $(\phi_i)_{i=1,2,\ldots n}$ of $\mathcal{F}_T$-measurable functions with values in $[0,1]$, we define a random variable:*

$$L = \min_{i=1,2,\ldots,n} \{i : \phi_i D_i = \max_{j=1,2,\ldots,n} \phi_j D_j\}.$$

Consider the measure $\mathbb{R}^\alpha$ given by:

$$\frac{\mathrm{d}\mathbb{R}^\alpha}{\mathrm{d}\mathbb{R}} = \mathbf{1}_{\{E \neq L\}} \frac{1 - p_L \alpha_L}{1 - p_L} + \alpha_L \mathbf{1}_{\{E=L\}}. \tag{4.1}$$

Then $\mathbb{R}^\alpha \in \mathcal{P}$. Moreover,

$$\lim_{\substack{\alpha_i \to 1/p_i \\ i=1,2,\ldots,n}} \mathbb{E}^{\mathbb{R}^\alpha} [\phi_E D_E] = \mathbb{E}^{\mathbb{R}} \left[\max_{i=1,2,\ldots,n} (\phi_i D_i) \right].$$

Proof. We first show that $\mathbb{E}^{\mathbb{R}^\alpha}[X_s|\mathcal{G}_t] = X_t$. Indeed,

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}^\alpha} [X_s|\mathcal{G}_t] &= \mathbb{E}^{\mathbb{R}} \left[X_s [\mathbf{1}_{\{E \neq L\}} \frac{1 - p_L \alpha_L}{1 - p_L} + \alpha_L \mathbf{1}_{\{E=L\}}] \middle| \mathcal{G}_t \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[M_t X_s \mathbb{E}^{\mathbb{P}_2} \left[\mathbf{1}_{\{E \neq L\}} \frac{1 - p_L \alpha_L}{1 - p_L} + \alpha_L \mathbf{1}_{\{E=L\}} \right] \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[M_t X_s \left[(1 - p_L) \frac{1 - p_L \alpha_L}{1 - p_L} + \alpha_L p_L \right] \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{R}} [X_s|\mathcal{F}_t] \\
&= X_t.
\end{aligned}$$

Furthermore, by the dominated convergence theorem, condition (2.3) and fact that $\phi_i \leq 1$ $(i = 1, \ldots, n)$ we have:

$$\begin{aligned}
\lim_{\substack{\alpha_i \to 1/p_i \\ i=1,2,\ldots,n}} \mathbb{E}^{\mathbb{R}^\alpha}[\phi_E D_E] &= \lim_{\substack{\alpha_i \to 1/p_i \\ i=1,2,\ldots,n}} \mathbb{E}^{\mathbb{R}} \left[\phi_E D_E \left[\mathbf{1}_{\{E \neq L\}} \frac{1 - p_L \alpha_L}{1 - p_L} + \alpha_L \mathbf{1}_{\{E=L\}} \right] \right] \\
&= \mathbb{E}^{\mathbb{R}} \left[\phi_E D_E \frac{\mathbf{1}_{\{E=L\}}}{p_L} \right] \\
&= \mathbb{E}^{\mathbb{P}_1} \left[M_T \frac{\phi_L D_L}{p_L} \mathbb{P}_2(E = L) \right] \\
&= \mathbb{E}^{\mathbb{R}} [\phi_L D_L] \\
&= \mathbb{E}^{\mathbb{R}} \left[\max_{i=1,2,\ldots,n} (\phi_i D_i) \right].
\end{aligned}$$

This completes the proof. □

4.1 Proof of the Theorem 2.1

To solve the **Problem 1** we introduce two other equivalent problems.
Problem 1'
Find a sequence of the decreasing sets $(A_i)_{i=1,2,\ldots,n}$ (with $A_{n+1} = \emptyset$) belonging to $\mathcal{F}_T$ that maximizes $\mathbb{P}(A_E)$ subject to the condition

$$\mathbb{E}^{\mathbb{R}} \left[\sum_{i=1}^{n} \mathbf{1}_{A_i / A_{i+1}} D_i \right] \leq \tilde{V}_0. \tag{4.2}$$

Problem 1"
Find a sequence of sets $(A_i)_{i=1,2,\ldots n}$ belonging to $\mathcal{F}_T$ that maximizes $\mathbb{P}(A_E)$ subject to the condition $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{A_E} D_E] \leq \tilde{V}_0$ for all $\mathbb{Q} \in \mathcal{P}$.

Proposition 1. *The sequence $(A_i^*)_{i=1,2\ldots,n}$ given in (2.8)-(2.9) is the solution of the* **Problem 1'**.

Proof. For any decreasing $A_i \in \mathcal{F}_T$ $(i = 1, 2 \ldots, n)$ satisfying (4.2) we have:

$$\begin{aligned}
(\mathbf{1}_{\tilde{A}_n} - \mathbf{1}_{A_n})(p_n - kC_n) &\geq 0 \\
(\mathbf{1}_{\tilde{A}_{n-1}} - \mathbf{1}_{A_{n-1}})(p_{n-1} - kC_{n-1} + \mathbf{1}_{\tilde{A}_n}(p_n - kC_n)) &\geq 0 \\
&\ldots
\end{aligned}$$

where sets $\tilde{A}_i$ are defined in (2.7). Thus,

$$\begin{aligned}
&\mathbb{P}(A_E) - k\mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} \mathbf{1}_{A_i/A_{i+1}} D_i\right] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{A_1}p_1 + \ldots + \mathbf{1}_{A_n}p_n] \\
&\quad -k\mathbb{E}^{\mathbb{R}}[\mathbf{1}_{A_1}(D_1 + \mathbf{1}_{A_2}(D_2 - D_1 + \mathbf{1}_{A_3}(\ldots + +\mathbf{1}_{A_n}(D_n - D_{n-1})))] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{A_1}p_1 + \ldots + \mathbf{1}_{A_n}p_n] \\
&\quad -k\mathbb{E}^{\mathbb{P}}[\mathbf{1}_{A_1}(C_1 + \mathbf{1}_{A_2}(C_2 + \mathbf{1}_{A_3}(C_3 + \mathbf{1}_{A_4}(\ldots + \mathbf{1}_{A_{n-1}}(C_{n-1} + \mathbf{1}_{A_n}C_n)\ldots))] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{A_1}(p_1 - kC_1 + \mathbf{1}_{A_2}(p_2 - kC_2 + \mathbf{1}_{A_3}(\ldots + \mathbf{1}_{A_n}(p_n - kC_n)))))] \\
&\leq \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\tilde{A}_1}(p_1 - kC_1 + \mathbf{1}_{\tilde{A}_2}(p_2 - kC_2 + \mathbf{1}_{\tilde{A}_3}(\ldots + \mathbf{1}_{\tilde{A}_n}(p_n - kC_n)))))] \\
&= \mathbb{P}(A_E^*) - k\mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} \mathbf{1}_{A_i^*/A_{i+1}^*} D_i\right].
\end{aligned}$$

Finally,

$$\mathbb{P}(A_E) - \mathbb{P}(A_E^*) \leq k\mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} \mathbf{1}_{A_i/A_{i+1}} D_i\right] - k\mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} \mathbf{1}_{A_i^*/A_{i+1}^*} D_i\right] \leq 0,$$

which completes the proof. □

Proposition 2. *The sequence* $(A_i^*)_{i=1,2\ldots,n}$ *given in (2.8)-(2.9) is the solution of the* **Problem 1”**.

Proof. For any equivalent martingale measure $\mathbb{Q}$ we have:

$$\mathbb{E}^{\mathbb{Q}}\mathbf{1}_{A_E} D_E \leq \mathbb{E}^{\mathbb{Q}}[\max_{i=1,2,\ldots,n} \mathbf{1}_{A_i} D_i] = \mathbb{E}^{\mathbb{R}}[\max_{i=1,2,\ldots,n} \mathbf{1}_{A_i} D_i], \tag{4.3}$$

where the last equality follows from the fact that $\max_{i=1,2,\ldots,n} \mathbf{1}_{A_i} D_i$ is a $\mathcal{F}_T$-measurable random variable and measure R is the unique martingale measure on $\mathcal{F}_T$. From Lemma 4.1 it follows that the upper bound (4.3) is achieved:

$$\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}\mathbf{1}_{A_E} D_E = \mathbb{E}^{\mathbb{R}}\left[\max_{i=1,2,\ldots,n} \mathbf{1}_{A_i} D_i\right].$$

Note that:

$$\mathbb{E}^{\mathbb{R}}\left[\max_{i=1,2,\ldots,n} \mathbf{1}_{A_i} D_i\right] = \mathbb{E}^{\mathbb{R}}\left[\sum_{i=1}^{n} \mathbf{1}_{\overline{A}_i/\overline{A}_{i+1}} D_i\right]$$

and that

$$\mathbb{P}(A_E) \leq \mathbb{P}\left(\overline{A}_i\right),$$

where $\overline{A}_i = \bigcup_{j \geq i} A_j$. Since sets $\overline{A}_i$ are decreasing and

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}} \mathbf{1}_{A_E} D_E \leq \tilde{V}_0 \qquad \text{iff} \qquad \mathbb{E}^{\mathbb{R}} \left[\sum_{i=1}^{n} \mathbf{1}_{\overline{A}_i / \overline{A}_{i+1}} D_i \right] \leq \tilde{V}_0,$$

it follows from the Proposition 1 that $P(\overline{A}_i)$ is maximal when $\overline{A}_i = A_i^*$. Hence always $P(A_i) \leq P(A_i^*)$ and the upper bound is achieved for $A_i = A_i^*$. This completes the proof. □

Proposition 3. *For sets $(A_i^*)_{i=1,2\dots,n}$ given in (2.8)-(2.9) the super replicating strategy of the claim $\mathbf{1}_{A_E^*} D_E$ is the solution of the* **Problem 1***.*

Proof. Take any strategy (V_0, ξ) with $V_0 \leq \tilde{V}_0$. We will show that the sequence $A_i = \{E = i\} \cap \{V_T \geq D_i\}$ is in the domain of the **Problem 1**, that is for all $\mathbb{Q} \in \mathcal{P}$,

$$\begin{aligned} \tilde{V}_0 &\geq V_0 \\ &\geq \mathbb{E}^{\mathbb{Q}} V_T \\ &\geq \mathbb{E}^{\mathbb{Q}} \left[D_E \mathbf{1}_{\{V_T \geq D_E\}} \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[D_E \mathbf{1}_{A_E} \right]. \end{aligned}$$

The maximum considered in the **Problem 1** is less or equal to the maximum analyzed in the **Problem 1"**: $P[V_T \geq D_E] = \mathbb{P}[A_E] \leq \mathbb{P}(A_E^*)$. Considering the super replicating strategy of the claim $\mathbf{1}_{A_E^*} D_E$ we have $V_0 = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}} \mathbf{1}_{A_E^*} D_E \leq \mathbb{E}^{\mathbb{R}} \left[\sum_{i=1}^{n} \mathbf{1}_{A_i^* / A_{i+1}^*} D_i \right] \leq \tilde{V}_0$ by the Proposition 1 and

$$\mathbb{P}(V_T \geq D_E) \geq \mathbb{P}(\mathbf{1}_{A_E^*} D_i \geq D_E) = \mathbb{P}(D_E = 0) + \mathbb{P}(A_E^*, D_E > 0) \geq \mathbb{P}(A_E^*).$$

Hence the maximum is indeed achieved for the super replicating strategy of the claim $\mathbf{1}_{A_E^*} D_E$. □

Proof of the Theorem 2.1. The proof follows from the Proposition 3 and the following equalities:

$$\begin{aligned} \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{P}} E^{\mathbb{Q}} [\mathbf{1}_{A_E^*} D_E | \mathcal{F}_t] &= E^{\mathbb{R}} [\max_{i=1,2,\dots,n} \mathbf{1}_{A_i^*} D_i | \mathcal{F}_t] \\ &= E^{\mathbb{R}} \left[\sum_{i=1}^{n} D_i \mathbf{1}_{A_i^* / A_{i+1}^*} | \mathcal{F}_t \right] \\ &= E^{\mathbb{R}} [H | \mathcal{F}_t]. \end{aligned}$$

□

4.2 Proof of the Theorem 2.2

To solve the **Problem 2** similarly like in the previous section we introduce two other equivalent problems.
Problem 2'
Find a sequence of $\mathcal{F}$-measurable random variables $(\phi_i)_{i=1,2,...n}$ such that $0 \le \phi_i \le 1$ and that maximizes $\mathbb{E}^{\mathbb{P}}(\phi_E)$ subject to the condition

$$\mathbb{E}^{\mathbb{R}}[\max_{i=1,2,\ldots,n} \phi_i D_i] \le \tilde{V}_0. \tag{4.4}$$

Problem 2"
Find a sequence of $\mathcal{F}$-measurable random variables $(\phi_i)_{i=1,2,...n}$ such that $0 \le \phi_i \le 1$ and that maximizes $\mathbb{E}^{\mathbb{P}}(\phi_E)$ subject to condition $\mathbb{E}^{\mathbb{Q}}[\phi_E D_E] \le \tilde{V}_0$ for all $\mathbb{Q} \in \mathcal{P}$.

Proposition 4. *The solution of the* **Problem 2'** *is given by* $(\phi_i^*)_{i=1,2,\ldots,n}$ *defined in (2.11)-(2.12).*

Proof. Let $\Upsilon := \max_{i=1,2,\ldots,n}(\tilde{\phi}_i D_i)$, where the sequence $\tilde{\phi}_i$ is a solution of the **Problem 2'**. We will express a solution of the **Problem 2'** ϕ_i^* (possibly different than $\tilde{\phi}_i$) in terms of Υ. Note that for given Υ under assumption $D_i\phi_i^* \le \Upsilon$, the expression $\mathbb{E}^{\mathbb{P}}[\phi_i^*]$, hence also $\mathbb{E}^{\mathbb{P}}[\phi_E^*] = \sum_{i=1}^n p_i \mathbb{E}^{\mathbb{P}}[\phi_i^*]$, is maximized for

$$\phi_i^* = \min\left(1, \frac{\Upsilon}{D_i}\mathbf{1}_{\{D_i>0\}} + \mathbf{1}_{\{D_i=0\}}\right).$$

Moreover, $\tilde{\phi}_i D_i \le \tilde{\phi}_i D_n \le D_n$ $(i = 1, 2, \ldots, n)$, hence $\Upsilon \le D_n$. Thus

$$\phi_n^* = \frac{\Upsilon}{D_n}\mathbf{1}_{\{D_n>0\}} + \mathbf{1}_{\{D_n=0\}}$$

and $\phi_n^* D_n = \Upsilon$. Finally, $\max_{i=1,2,\ldots,n} D_i\phi_i^* = \Upsilon$ and one of the solutions of the **Problem 2'** has the following form:

$$\phi_i^* = \min\left(1, \frac{\phi_n^* D_n}{D_i}\mathbf{1}_{\{D_i>0\}} + \mathbf{1}_{\{D_i=0\}}\right)$$

for $i < n$. We will find now ϕ_n^* maximizing

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n p_i \min\left(1, \frac{\phi_n^* D_n}{D_i}\mathbf{1}_{\{D_i>0\}} + \mathbf{1}_{\{D_i=0\}}\right)\right] \tag{4.5}$$

subject to the condition

$$\mathbb{E}^{\mathbb{R}}[\phi_n^* D_n] \le \tilde{V}_0. \tag{4.6}$$

Define a random variable $F(s)$:

$$F(s) := M_T^{-1} \sum_{i=1}^{n} p_i \min\left(\frac{1}{D_n}, \frac{s}{D_i}\right), \qquad s \in [0,1].$$

Note that F is a $\mathbb{P}$-a.s. concave function of s and thus for $s \in [0,1]$:

$$F(s) \leq F(\phi_n^*) + k(s - \phi_n^*), \tag{4.7}$$

where ϕ_n^* defined in (2.11)-(2.12). Consider now any random variable $\phi_n : \Omega \to [0,1]$ satisfying (4.6) maximizing (4.5). Note that $\mathbb{E}^{\mathbb{R}}[\phi_n D_n] = \mathbb{E}^{\mathbb{R}}[\phi_n^* D_n] = \tilde{V}_0$. Thus from (4.7),

$$\mathbb{E}^{\mathbb{R}}[F(\phi_n)D_n] \leq \mathbb{E}^{\mathbb{R}}[F(\phi_n^*)D_n] + k\left(\mathbb{E}^{\mathbb{R}}[\phi_n D_n] - \mathbb{E}^{\mathbb{R}}[\phi_n^* D_n]\right) = \mathbb{E}^{\mathbb{R}}[F(\phi_n^*)D_n].$$

and hence ϕ_n^* maximizes (4.5). This completes the proof. □

Proposition 5. *The random variables $(\phi_i^*)_{i=1,2,\ldots n}$ defined in (2.11)-(2.12) are solution of the* **Problem 2"**.

Proof. For any equivalent martingale measure $\mathbb{Q}$ we have:

$$\mathbb{E}^{\mathbb{Q}} \phi_E D_E \leq \mathbb{E}^{\mathbb{Q}}\left[\max_{i=1,2,\ldots,n} \phi_i D_i\right] = \mathbb{E}^{\mathbb{R}}\left[\max_{i=1,2,\ldots,n} \phi_i D_i\right].$$

From the Lemma 4.1 it follows that

$$\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}} \phi_E D_E = \mathbb{E}^{\mathbb{R}}[\max_{i=1,2,\ldots,n} \phi_i D_i],$$

which together with Proposition 4 completes the proof. □

Proposition 6. *The random variables $\phi_i^*, i = 1, 2\ldots, n$ given in (2.11)-(2.12) solve* **Problem 2**.

Proof. Take any admissible strategy (V_0, ξ) from the domain of the **Problem 2**, i.e. such that $V_0 \leq \tilde{V}_0$. The sequence $\phi_i = \mathbf{1}_{\{V_T \geq D_i)} + \mathbf{1}_{\{V_T < D_i\}} \frac{V_T}{D_i}$ is in domain of the **Problem 2"**, since for all $\mathbb{Q} \in \mathcal{P}$:

$$\begin{aligned} \tilde{V}_0 &\geq V_0 \\ &\geq \mathbb{E}^{\mathbb{Q}} V_T \\ &\geq \mathbb{E}^{\mathbb{Q}}\left[D_E \mathbf{1}_{\{V_T \geq D_E\}} + V_T \mathbf{1}_{\{V_T < D_E\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[D_E \phi_E\right] \geq \mathbb{E}^{\mathbb{Q}}\left[D_E \phi_E\right]. \end{aligned}$$

Hence from Proposition 5

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{V_T \geq D_E)} + \mathbf{1}_{\{V_T < D_E\}}\frac{V_T}{D_E}\right] \leq \mathbb{E}^{\mathbb{P}}[\phi_E^*].$$

The super replicating strategy (V_0^*, ξ^*) of $\phi_E^* D_E$ costs also not more than $\tilde{V}_0$, since $\mathbb{E}^{\mathbb{Q}}[\phi_E^* D_E] \leq \tilde{V}_0$ for all $\mathbb{Q} \in \mathcal{P}$. Taking $\tilde{\phi}_i = \mathbf{1}_{\{V_T^* \geq D_i)} + \mathbf{1}_{\{V_T^* < D_i\}}\frac{V_T^*}{D_i} \leq 1$ and using fact that $\phi_E^* \leq 1$, we have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[\phi_E^*] &\leq \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{V_T^* \geq D_E)} + \mathbf{1}_{\{V_T^* < D_E\}}\phi_E^*\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{V_T^* \geq D_E)} + \mathbf{1}_{\{V_T^* < D_E\}}\frac{D_E \phi_E^*}{D_E}\right] \\
&\leq \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{V_T^* \geq D_E)} + \mathbf{1}_{\{V_T^* < D_E\}}\frac{V_T^*}{D_E}\right] \\
&= \mathbb{E}^{\mathbb{P}}[\tilde{\phi}_E].
\end{aligned}$$

This completes the proof. □

Proof of the Theorem 2.2. The proof follows from the Proposition 6 and the following equalities:

$$\begin{aligned}
\operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}[\phi_E^* D_E | \mathcal{F}_t] &= \mathbb{E}^{\mathbb{R}}[\max_{i=1,2,\ldots,n} \phi_i^* D_i | \mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{R}}[H | \mathcal{F}_t].
\end{aligned}$$

□

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