TAIL ASYMPTOTICS FOR THE SUPREMUM OF A REGENERATIVE PROCESS

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Abstract. We give precise asymptotic estimates of the tail behavior of the distribution of the supremum of a process with regenerative increments. Our results cover four qualitatively different regimes, involving both light tails and heavy tails, and are illustrated with examples arising in queueing theory and insurance risk.

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1. Introduction

Regenerative processes are a versatile tool in stochastic modeling, as they are general enough to cover many applications, and at the same time provide a natural and tractable extension of the random walk. In particular, the computation of overflow probabilities in (fluid) queues and ruin probabilities in insurance can often be reduced to the study of the maximum of a process, of which the increments are regenerative. Specifically, let $S(t), t \geq 0$, be a càdlàg process a.s. drifting to $-\infty$ such that $S(0) = 0$. Suppose that there exists a renewal process with renewal epochs $0 \leq T_0 < T_1 < ...$ such that

$$(S(t))_{0\leq t<T_0}, \qquad (S(T_0+t)-S(T_0))_{0\leq t<T_1-T_0}, \qquad \ldots$$

are independent, and the distribution of $(S(T_k+t)-S(T_k))_{0\leq t<T_{k+1}-T_k}$ is identical for all $k \geq 0$. We call $T_n, n \geq 0$, the regeneration or renewal epochs for $S(t), t \geq 0$. If $T_0 = 0$, we say that $(S(t))$ is zero-delayed. Define

$$M = \sup_{t\geq 0} S(t), \qquad M_{n+1} = \sup_{T_n \leq t < T_{n+1}} S(t) - S(T_n), \quad \text{for } n \geq 0,$$

and denote $X_{n+1} = S(T_{n+1}) - S(T_n), S_n = S(T_n), n \geq 0$. This is strongly related to the setting considered in Asmussen *et al.* [4] and many other papers. Typically, the distribution of M is too complicated to compute exactly. Therefore, one is often concerned with the tail behavior of M, i.e. the behavior of $\mathbb{P}\{M > x\}$ as x grows large. This forms the motivation of the present paper.

In this paper, we focus on the zero-delayed case. Under this assumption, we have the identity

$$M = \sup_{n\geq 1}[S_{n-1} + M_n]. \tag{1}$$

The sequence $M_n, n \geq 1$, is i.i.d. but depends on the random walk $S_n, n \geq 1$, since M_n and X_n are dependent. Note that the sequence of pairs $(M_n, X_n), n \geq 1$, is i.i.d. Thus, the regenerative setting can be viewed a special case of the more general framework of the *perturbed random walk*, which is considered in a recent paper by Araman & Glynn [1]. The authors investigate the tail behavior of M in a variety of cases.

The main goal of this work is to analyze the tail behavior of M in the perturbed random walk setting (1), under conditions that are general enough to be applicable to regenerative processes. For random walk maxima, it is well known (see e.g. Bertoin & Doney [5], Embrechts & Veraverbeke [12] and Korshunov [21]) that the description of the tail behavior can be classified by three main regimes: (i) the Cramér case, (ii) the intermediate case and (iii) the heavy-tailed case. Our main results cover these cases for perturbed random walks. In addition, we identify a fourth main case, in which the perturbations are dominating the tail behavior of M. Specifically, our results are as follows:

- The first scenario we consider is when the Cramér condition holds for X_1; i.e. we assume that there exists a strictly positive solution κ to the equation $\mathbb{E}\{e^{\kappa X_1}\} = 1$. In addition, we assume that the tail of M_1 is not too heavy (in a sense we make precise later on). These assumptions allow us to apply the implicit renewal theory developed by Goldie [15] to obtain the tail behavior of M. The results in [15] have mostly been applied to autoregressive processes but have much wider applicability. Special cases of our result have been derived before by Araman & Glynn [1] and by Schmidli [25]. We note that the main result in [1], which covers the case where the perturbations (M_n) form a stationary sequence independent of the random walk (S_n), is not covered in the present paper.
- A qualitatively different case occurs when M_1 is light-tailed, but heavier than the tail of $\sup_n S_n$. Again, we exploit theory developed for autoregressive processes. In particular, we utilize stochastic ordering arguments proposed in Grey [17] to extend and unify both Theorem 3 in [1] and Example 2 in [17].
- We apply again stochastic ordering arguments to analyze the intermediate case, which also occurs in the case of a standard random walk. We derive the tail behavior of M under the assumption that the right tail of X_1 is in the class $\mathcal{S}(\alpha)$, and that the right tail of M_1 is not heavier than the right tail of X_1.
- A fourth regime we consider is the case where $M_1^* = \max\{M_1, X_1\}$ has a heavy tail, in a sense we make precise later on. This case has been investigated before in [1], under the additional assumption that X_1 is light-tailed. We also extend a result of Asmussen *et al.* [4], who assume that the tails of M_1^* and X_1 asymptotically coincide (note that our M_1^* is identical to their M_1). Other related papers are Foss & Zachary [13] and Foss *et al.* [14], who consider a class of modulated random walks and Lévy processes with heavy-tailed increments.

We illustrate our results by considering some specific models that arise in telecommunications and insurance. We first consider a basic on-off fluid model, where $S(t), t \geq 0$, either increases with rate $r - c$ ($r > c$), or decreases with rate c. After that, we investigate an insurance risk model. More precisely, we consider a Cox-type process which has been introduced and motivated in Schmock [26]. We apply our general result to derive an exponential estimate for the ruin probability, and obtain bounds on the pre-factor.

The rest of this paper is organized as follows. Section 2 contains the main results of this paper. The four different cases are investigated in Subsections 2.1–2.4. Section 3 is devoted to the on-off model. The insurance risk model is investigated in Section 4. Section 5 concludes.

2. General results

In this section we present the main results of this work. We focus on four cases: first, we consider the situation where M_1^* is heavy-tailed in Section 2.1. After that, we assume that both X_1 and M_1 are light-tailed. Three further distinctions arise here, which are treated in Sections 2.2, 2.3 and 2.4.

We always assume that the joint distribution of (X_1, M_1) satisfies $\mathbb{E}\{X_1\} \in (-\infty, 0)$, $\mathbb{E}\{M_1\} < \infty$, and $\mathbb{P}\{M_1 = -\infty\} = 0$. The last assumption is not restrictive: for regenerative processes we have the representation $M_1 = \sup_{t \in [0,T_1)} S(t) \geq 0$. Throughout this section, we use various standard results for the classes $\mathcal{L}$ of long-tailed distributions, the class $\mathcal{S}$ of subexponential distributions, as well as the class $\mathcal{S}(\gamma), \gamma \geq 0$. A standard reference on such distributions is the textbook Embrechts *et al.* [11]. For two functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if $f(x) = g(x)(1 + \mathrm{o}(1))$ as $x \to \infty$.

2.1. **The heavy-tailed case.** Our first result concerns the case where $M_n^* = \max\{X_n, M_n\}$ is heavy-tailed.

Theorem 1. *If* $\mathbb{E}\{M_1^*\} < \infty$ *and* $\min\{1, \int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u\}$ *is subexponential, then*

$$\mathbb{P}\{M > x\} \sim \frac{1}{\mu} \int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u, \tag{2}$$

as $x \to \infty$*, with* $\mu = -\mathbb{E}\{X_1\}$.

This is an extension of Theorem 3.3 of Asmussen *et al.* [3], where it was assumed that $\mathbb{P}\{M_1^* > x\} \sim \mathbb{P}\{X_1 > x\}$. We give examples in Section 3 to show that this condition is sometimes too restrictive. Theorem 1 is also related to Theorem 4 of Araman & Glynn [1]. There it is assumed that X_1 is light-tailed, and that the marginal distribution of M_1 has a hazard rate converging to 0.

Proof. The proof consists in deriving lower and upper bounds, which asymptotically coincide.

We start with the lower bound, for which we adapt a standard (cf. [4, 14, 13, 28]) technique to our setting. The idea is to identify a way in which the event $M > x$ occurs. Informally speaking, we choose an event on which $S_{n-1} - M_n, n \geq 1$, behaves in a typical way up to some time k for which $M_{k+1}^* = \max\{M_{k+1}, X_{k+1}\}$ is large. By also including the event that M_{k+2} is not too small, we ensure that $M > x$.

Let $0 < \delta < \mu$ be given and define for $n \geq 1$, the event $E_n = E_n(\delta, K)$ as

$$E_n = \{S_k \in (-k(\mu + \delta) - K, -k(\mu - \delta) + K), k \leq n\}.$$

In addition, consider the event $F_n = F_n(\delta, K)$ which is defined by

$$F_n = \{M_k < \delta k + K, k \leq n\}.$$

Also define $G(x) = \mathbb{P}\{M_n < x\} = \mathbb{P}\{M_1 < x\}$ and let K be such that $\bar{G}(K) = 1 - G(K) < 1/2$. Since $\log(1 - x) \geq -2x$ if $x \in (0, 1/2)$, we see that

$$\log \mathbb{P}\{F_n\} = \sum_{k=1}^{n} \log(1 - \bar{G}(\delta k + K)) \geq -2 \sum_{k=1}^{\infty} \bar{G}(\delta k + K) = -2\mathbb{E}\{\lfloor (M_1 - K)^+/\delta \rfloor\},$$

with $y^+ = \max\{y, 0\}$. Since $\mathbb{E}\{M_1\} < \infty$, the last expression converges to 0 if $K \to \infty$ for any $\delta > 0$. Combining this fact with the weak law of large numbers for $S_n, n \geq 1$, we arrive at the following conclusion: for every $\epsilon > 0$ there exists a K such that $\mathbb{P}\{E_n \cap F_n\} \geq 1 - \epsilon$.

For $n \geq 1$, we define the event

$$G_n = F_n \cap E_n \cap \{M_{n+1}^* > x + n(\delta + \mu) + 2K\} \cap \{M_{n+2} > -K\} \tag{3}$$

Observe that the events $G_n, n \geq 1$, are disjoint and that G_n implies $M > x$ for every $n \geq 1$. Consequently,

$$\begin{aligned}
\mathbb{P}\{M > x\} &\geq \mathbb{P}\{\cup_{n=1}^\infty G_n\} = \sum_{n=1}^{\infty} \mathbb{P}\{G_n\} \\
&\geq (1 - \epsilon) \sum_{n=1}^{\infty} \mathbb{P}\{M_{n+1}^* > x + n(\delta + \mu) + 2K\}\mathbb{P}\{M_{n+2} > -K\} \\
&\sim \frac{1 - \epsilon}{\delta + \mu} \mathbb{P}\{M_1 > -K\} \int_{x+K}^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u \\
&\sim \frac{1 - \epsilon}{\delta + \mu} \mathbb{P}\{M_1 > -K\} \int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u,
\end{aligned}$$

where in the last two steps, we have used the fact that M_1^* is long-tailed. This implies

$$\liminf_{x \to \infty} \frac{\mathbb{P}\{M > x\}}{\int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u} \geq \frac{1 - \epsilon}{\delta + \mu} \mathbb{P}\{M_1 > -K\}.$$

The proof of the lower bound follows by letting $K \to \infty$ and $\delta, \epsilon \downarrow 0$.

To obtain an asymptotic upper bound, let $y > 0$ be given and construct the random walk $S_n^y, n \geq 0$: set $S_0^y = 0$. For $k \geq 1$, set $X_k^y = X_k$ if $M_k^* \leq y$ and $X_k^y = M_k^*$ if $M_k^* > y$. Set finally $S_n^y = X_1^y + \ldots + X_n^y, n \geq 1$. Informally, the increments of the random walk $S_n^y, n \geq 0$, are the same as those of $S_n, n \geq 0$, except when a large value of M_n^* occurs.

Obviously, we have that $S_n \leq S_n^y$ for any $y > 0$ and $n \geq 1$. Moreover, we have the following crucial bound:

$$\sup_{n\geq 1}[S_{n-1} + M_n] \leq \sup_{n\geq 0} S_n^y + y. \tag{4}$$

For $x > y$, we have $\mathbb{P}\{X_k^y > x\} = \mathbb{P}\{M_k^* > x\}$, which implies that the integrated tail of X_k^y is subexponential. Thus, we can apply Veraverbeke's theorem (see e.g. [27] or [28]), yielding

$$\mathbb{P}\{\sup_{n\geq 1} S_n^y > x\} \sim \frac{1}{-\mathbb{E}\{X_1^y\}} \int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u. \tag{5}$$

Putting everything together, we conclude that

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{M > x\}}{\int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u} \leq \limsup_{x\to\infty} \frac{\mathbb{P}\{\sup_{n\geq 1} S_{T_n}^y > x - y\}}{\int_x^\infty \mathbb{P}\{M_1^* > u\}\mathrm{d}u} \leq \frac{1}{-\mathbb{E}\{X_1^y\}}.$$

By dominated convergence, we have that $-\mathbb{E}\{X_1^y\} \to \mu$ as $y \to \infty$. □

2.2. **Dominating perturbations.** We now suppose that M_1 is light-tailed. More precisely, we assume that

$$\mathbb{P}\{M_1 > x\} = m_1(x)\mathrm{e}^{-\nu x}, \tag{6}$$

with $m_1(\cdot)$ such that $m_1(x + y) \sim m_1(x)$ for any fixed y as $x \to \infty$. This is equivalent to the requirement that the right tail of the distribution of $\exp\{M_1\}$ is regularly varying with index $-\nu$.

To formulate our result, we need to make two more assumptions. The first additional assumption we invoke is that

$$\mathbb{E}\{\mathrm{e}^{\nu X_1}\} < 1. \tag{7}$$

Our final assumption is of a more technical nature: let $\tilde{X}_1$ be an independent copy of X_1 which is also independent of M_1, and suppose that

$$\mathbb{P}\{\tilde{X}_1 + M_1 > x\} \sim \mathbb{E}\{\mathrm{e}^{\nu X_1}\}\mathbb{P}\{M_1 > x\}. \tag{8}$$

This assumption is a consequence of (6), (7) and some minor additional regularity condition. In particular, one of the following three conditions suffices: (i) $\mathbb{E}\{\mathrm{e}^{\eta X_1}\} < \infty$ for some $\eta > \nu$ (Breiman's theorem), (ii) $\lim_{x\to\infty} m_1(x) > 0$, or (iii) $m_1(x)$ is decreasing and in $\mathcal{S}^*$; see Denisov & Zwart [10] for details. Moreover, assumption (8) implies that $\mathbb{P}\{X_1 > x\} = \mathrm{o}(\mathbb{P}\{M_1 > x\})$.

The following result extends and unifies Example 2 of Grey [17] (where condition (i) is assumed), and Theorem 4 of [1] (where it is assumed that $\lim_{x\to\infty} m_1(x) > 0$). To prove our result, we adapt the arguments in [17] to our setting.

Theorem 2. *If (6)–(8) hold, then*

$$\mathbb{P}\{M > x\} \sim \frac{1}{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}}\mathbb{P}\{M_1 > x\}.$$

In the remainder of this subsection, we assume that (6)–(8) are in force. The proof of Theorem 2 involves the following lemma.

Lemma 1. *Suppose that $Y \geq 0$ is independent of (M_1, X_1) and that $\mathbb{P}\{Y > x\} \sim c_Y\mathbb{P}\{M_1 > x\}$ for a constant $c_Y \in (0, \infty)$. Then $\mathbb{P}\{\max\{M_1, X_1 + Y\} > x\} \sim (1 + c_Y\mathbb{E}\{\mathrm{e}^{\nu X_1}\})\mathbb{P}\{M_1 > x\}$.*

Proof. An asymptotic upper bound simply follows from the assumptions, the bound $\mathbb{P}\{\max\{M_1, X_1+Y\} > x\} \leq \mathbb{P}\{M_1 > x\} + \mathbb{P}\{X_1 + Y > x\}$, and the fact that (8) is closed under tail equivalence. To prove that this upper bound is tight, it suffices to show that $\mathbb{P}\{\min\{M_1, X_1 + Y\} > x\} = \mathrm{o}(\mathbb{P}\{M_1 > x\})$. Note that there exists a function $h(x) \to \infty$ such that $x - h(x) \to \infty$ and $\mathbb{P}\{X_1 > h(x)\} = \mathrm{o}(\mathbb{P}\{M_1 > x\})$. Now, distinguish between the two cases $X_1 > h(x)$ and $X_1 \leq h(x)$ to obtain

$$\mathbb{P}\{M_1 > x, X_1 + Y > x\} \leq \mathbb{P}\{X_1 \geq h(x)\} + \mathbb{P}\{M_1 > x\}\mathbb{P}\{Y > x - h(x)\} = \mathrm{o}(\mathbb{P}\{M_1 > x\}),$$

which implies the statement. □

Proof of Theorem 2. As in [17], we use the fact that $M \stackrel{d}{=} \max\{M_1, X_1 + \tilde{M}\}$, with $\tilde{M}$ an independent copy of M which is independent of X_1 and M_1. In addition, we use the following stochastic comparison argument. If $Y_1^{\downarrow}$ and $Y_1^{\uparrow}$ are two random variables independent of (X_1, M_1), then $Y_1^{\downarrow} \geq_{st} M$ if and only if $Y_1^{\downarrow} \geq_{st} \max\{M_1, X_1 + Y_1^{\downarrow}\}$. Similarly, $Y_1^{\uparrow} \leq_{st} M$ if and only if $Y_1^{\uparrow} \leq_{st} \max\{M_1, X_1 + Y_1^{\uparrow}\}$.

To prove an asymptotic lower bound, define a sequence of random variables $Y_n^{\uparrow}, n \geq 1$, where $Y_1^{\uparrow} = M_1$, and $Y_{n+1}^{\uparrow} = \max\{M_{n+1}, X_{n+1} + Y_n^{\uparrow}\}$. Using the above stochastic comparison argument n times, we obtain that $\mathbb{P}\{M > x\} \geq \mathbb{P}\{Y_n^{\uparrow} > x\}$ for every n. By a repeated application of Lemma 1, we have

$$\mathbb{P}\{Y_n^{\uparrow} > x\} \sim \frac{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}^n}{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}}\mathbb{P}\{M_1 > x\}.$$

Consequently,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{M > x\}}{\mathbb{P}\{M_1 > x\}} \geq \frac{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}^n}{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}},$$

which implies the desired asymptotic lower bound by letting $n \to \infty$.

To prove an asymptotic upper bound, we use again an idea from Grey [17]. Take $C > 1/(1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\})$, and let Y be a random variable independent of (M_1, X_1) such that $\mathbb{P}\{Y > x\} \sim C\mathbb{P}\{M_1 > x\}$. By Lemma 1, there exists $x_0 < \infty$ such that $\mathbb{P}\{Y > x\} \geq \mathbb{P}\{\max\{M_1, X_1+Y\} > x\}$ for $x \geq x_0$.

Define $Y_1^{\downarrow}$ as an independent random variable such that $\mathbb{P}\{Y_1^{\downarrow} > x\} = \mathbb{P}\{Y > x\}/\mathbb{P}\{Y > x_0\}$. As in Lemma 3 of Grey [17], it is easy to verify that $Y_1^{\downarrow} \geq_{st} \max\{M_1, X_1 + Y_1^{\downarrow}\}$. Now, define $Y_{n+1}^{\downarrow} = \max\{M_{n+1}, X_{n+1} + Y_n^{\downarrow}\}$ for $n \geq 1$. The remainder of the proof is similar to the proof of the lower bound. Applying the comparison argument, we see that $\mathbb{P}\{M > x\} \leq \mathbb{P}\{Y_n^{\downarrow} > x\}$. Set $C_0 = C/\mathbb{P}\{Y > x_0\}$. By Lemma 2 and induction, we have that

$$\mathbb{P}\{Y_n^{\downarrow} > x\} \sim \left(\frac{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}^{n-1}}{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}} + C_0\mathbb{E}\{\mathrm{e}^{\nu X_1}\}^{n-1}\right)\mathbb{P}\{M_1 > x\}.$$

Combining these results, we obtain that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{\mathbb{P}\{M > x\}}{\mathbb{P}\{M_1 > x\}} \leq \frac{1}{1 - \mathbb{E}\{\mathrm{e}^{\nu X_1}\}},$$

which completes the proof of the upper bound. □

2.3. **The intermediate case.** We now consider the case where X_1 satisfies

(9) $\quad \mathbb{P}\{X_1 > x + y\} \sim \mathrm{e}^{-\alpha y}\mathbb{P}\{X_1 > x\}$

for every fixed y, as $x \to \infty$, and

(10) $\quad \mathbb{P}\{X_1 + X_2 > x\} \sim 2\mathbb{E}\{\mathrm{e}^{\alpha X_1}\}\mathbb{P}\{X_1 > x\}.$

This is equivalent to the condition that $X_1^+ \in \mathcal{S}(\alpha)$. The case $\alpha = 0$ is treated in Section 2.1 so we assume $\alpha > 0$. Another assumption we make is that

(11) $\quad \mathbb{E}\{\mathrm{e}^{\alpha X_1}\} < 1,$

which implies that Cramér condition is not satisfied. Finally, we specify the tail behavior of M_1. The case where M_1 has a heavier tail than X_1 is already covered in the previous section. Motivated by this, we assume that

$$\lim_{x\to\infty} \frac{\mathbb{P}\{M_1 > x\}}{\mathbb{P}\{X_1 > x\}} < \infty \tag{12}$$

(we allow the limit to equal 0). Furthermore, we assume that there exists a bounded function f such that

$$\lim_{x\to\infty} \frac{\mathbb{P}\{M_1 > x; X_1 \leq x - a\}}{\mathbb{P}\{X_1 > x\}} = f(a), \tag{13}$$

for all real values of a. This covers the case where M_1 and X_1 are independent (in which f is constant), and the random walk case (in which case $M_1 = 0$ so that $f(a) = 0$). An example in the regenerative setting can be found in Section 3.

We are now ready to state and prove our third main result. The method we use is similar to the one employed in the previous section, and also provides a new proof of the random walk case. As a preliminary result, we need the following lemma, which plays the same role as Lemma 1 in the previous subsection.

Lemma 2. *Suppose that (9)–(13) are satisfied, and that* Y *is independent of* (M_1, X_1) *with* $\mathbb{P}\{Y > x\} \sim C_Y \mathbb{P}\{X_1 > x\}$. *Then*

$$\mathbb{P}\{\max\{M_1, X_1 + Y\} > x\} \sim (\mathbb{E}\{f(Y)\} + \mathbb{E}\{e^{\alpha Y}\} + C_Y \mathbb{E}\{e^{\alpha X_1}\})\mathbb{P}\{X_1 > x\}.$$

Proof. Write

$$\mathbb{P}\{\max\{M_1, X_1 + Y\} > x\} = \mathbb{P}\{X_1 + Y > x\} + \mathbb{P}\{M_1 > x; X_1 + Y \leq x\}.$$

For the first term, we observe that $(\mathbb{E}\{e^{\alpha Y}\} + C_Y \mathbb{E}\{e^{\alpha X_1}\})\mathbb{P}\{X_1 > x\}$, using a result of Cline [8]. To estimate the second term, note that (12) allows us to apply the bounded convergence theorem, which yields that

$$\lim_{x\to\infty} \frac{\mathbb{P}\{M_1 > x, X_1 + Y \leq x\}}{\mathbb{P}\{X_1 > x\}} = \lim_{x\to\infty} \frac{\int \mathbb{P}\{M_1 > x, X_1 \leq x - y\} \mathrm{d}\mathbb{P}\{Y \leq y\}}{\mathbb{P}\{X_1 > x\}} = \mathbb{E}\{f(Y)\},$$

which completes the proof of the lemma. □

Theorem 3. *Suppose that (9)–(13) are satisfied. Then*

$$\mathbb{P}\{M > x\} \sim \frac{\mathbb{E}\{e^{\alpha M}\} + \mathbb{E}\{f(M)\}}{1 - \mathbb{E}\{e^{\alpha X_1}\}} \mathbb{P}\{X_1 > x\}.$$

Proof. As in the proof of Theorem 2, we make use of stochastic ordering arguments. The sequence $(Y_n^{\uparrow})$ is defined as before: $Y_1^{\uparrow} = M_1$, and $Y_{n+1}^{\uparrow} = \max\{M_{n+1}, X_{n+1} + Y_n^{\uparrow}\}$. Using the stochastic comparison argument n times, we obtain that $\mathbb{P}\{M > x\} \geq \mathbb{P}\{Y_n^{\uparrow} > x\}$ which holds for every n. By a repeated application of Lemma 2, we have

$$\mathbb{P}\{Y_n^{\uparrow} > x\} \sim C_{Y_n^{\uparrow}} \mathbb{P}\{X_1 > x\}, \qquad n \geq 2$$

with $C_{Y_1^{\uparrow}}$ being the limit in (12), and

$$C_{Y_{n+1}^{\uparrow}} = \mathbb{E}\{f(Y_n^{\uparrow})\} + \mathbb{E}\{e^{\alpha Y_n^{\uparrow}}\} + C_{Y_n^{\uparrow}} \mathbb{E}\{e^{\alpha X_1}\}, \qquad n \geq 1.$$

Consequently,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{M > x\}}{\mathbb{P}\{X_1 > x\}} \geq C_{Y_n^{\uparrow}}, \qquad n \geq 1.$$

Since $Y_n^{\uparrow} \leq_{st} Y_{n+1}^{\uparrow} \leq_{st} M$ and $Y_n^{\uparrow} \xrightarrow{d} M$ we obtain by bounded and monotone convergence that $C_{Y_n^{\uparrow}} \to C_M$. C_M satisfies

$$C_M = \mathbb{E}\{f(M)\} + \mathbb{E}\{e^{\alpha M}\} + C_M \mathbb{E}\{e^{\alpha X_1}\}.$$

This completes the proof of the asymptotic lower bound.

To obtain an upper bound, we need to construct a random variable $Y_1^\downarrow$ with the property that $Y_1^\downarrow \geq_{st} \max\{M_1, X_1+Y_1^\downarrow\}$. Let X be an independent copy of X_1 and set $Y = (X+T)I(X \geq y)$, with y and T large constants. Note that $\mathbb{P}\{Y > x\} \sim \mathrm{e}^{\alpha T}\mathbb{P}\{X_1 > x\}$ and $\mathbb{E}\{\mathrm{e}^{\alpha Y}\} = \mathrm{e}^{\alpha T}\mathbb{E}\{\mathrm{e}^{\alpha X_1} I(X_1 \geq y)\}$. By Lemma 2, we have that

$$\mathbb{P}\{\max\{M_1, X_1 + Y\} > x\} \sim (\mathbb{E}\{f(Y)\} + \mathbb{E}\{\mathrm{e}^{\alpha Y}\} + \mathrm{e}^{\alpha T}\mathbb{E}\{\mathrm{e}^{\alpha X_1}\})\mathbb{P}\{X_1 > x\}.$$

Using the fact that $\mathbb{E}\{f(Y)\}$ is bounded and $\mathbb{E}\{\mathrm{e}^{\alpha X_1}\} < 1$, we can choose T and y such that the pre-factor $(\mathbb{E}\{f(Y)\} + \mathbb{E}\{\mathrm{e}^{\alpha Y}\} + \mathrm{e}^{\alpha T}\mathbb{E}\{\mathrm{e}^{\alpha X_1}\})$ is smaller than $\mathrm{e}^{\alpha T}$. Consequently, there exists a value x_0 such that $\mathbb{P}\{Y > x\} \geq \mathbb{P}\{\max\{M_1, X_1 + Y\} > x\}$ for $x \geq x_0$.

Now, define $Y_1^\downarrow$ as a random variable independent of (M_1, X_1) such that $\mathbb{P}\{Y_1^\downarrow > x\} = \mathbb{P}\{Y > x\}/\mathbb{P}\{Y > x_0\} = \mathbb{P}\{Y > x \mid Y > x_0\}$. As in Lemma 3 of Grey [17], we see that, for $x \geq x_0$,

$$\begin{aligned}
\mathbb{P}\{\max\{M_1, X_1 + Y_1^\downarrow\} > x\} &= \mathbb{P}\{\max\{M_1, X_1 + Y\} > x \mid Y > x_0\} \\
&\leq \frac{\mathbb{P}\{\max\{M_1, X_1 + Y\} > x\}}{\mathbb{P}\{Y > x_0\}} \\
&\leq \frac{\mathbb{P}\{Y > x\}}{\mathbb{P}\{Y > x_0\}} = \mathbb{P}\{Y_1^\downarrow > x\}.
\end{aligned}$$

The inequality is trivial for $x < x_0$, so we see that $Y_1^\downarrow \geq_{st} \max\{X_1, M_1 + Y_1^\downarrow\}$. The proof is now completed by defining $Y_n^\downarrow$ for $n \geq 2$ as in the proof of Lemma 1. Since $M \leq_{st} Y_n^\downarrow$ for any n we obtain, applying Lemma 2,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{M > x\}}{\mathbb{P}\{X_1 > x\}} \leq C_{Y_n^\downarrow},$$

with $C_{Y_n^\uparrow}$ defined in a similar way as we have done for the proof of the lower bound. Using the same arguments as in the proof of the lower bound, we have that $C_{Y_n^\downarrow} \to C_M$, which completes the proof. □

2.4. **The Cramér case.** In this subsection, we review the extension of the classical Cramér case from random walks to perturbed random walks and regenerative processes. This problem has also been considered in Schmidli [25] and Araman & Glynn [1]. The result presented here is an extension of these two works, and it follows from Theorem 5.2 in Goldie [15]. Goldie actually considers the equation $R \stackrel{d}{=} \max\{A\tilde{R}, B\}$, which can easily be reduced to our equation for M by taking logarithms.

Theorem 4. *(Goldie [15], Theorem 5.2). Assume that there exists a solution $\kappa > 0$ to the equation*

$$\mathbb{E}\{\mathrm{e}^{\kappa X_1}\} = 1 \text{ such that } m = \mathbb{E}\{X_1 \mathrm{e}^{\kappa X_1}\} < \infty.$$

Assume furthermore that X_1 is non-lattice and that $\mathbb{E}\{\mathrm{e}^{\kappa M_1}\} < \infty$. Then

$$\mathbb{P}\{M > x\} \sim K\mathrm{e}^{-\kappa x}$$

with $K = \frac{1}{\kappa m}\mathbb{E}\{\mathrm{e}^{\kappa M_1} - \mathrm{e}^{\kappa(\tilde{M}+X_1)}; M_1 > \tilde{M} + X_1\}$.

It is easy to see that K is bounded from above by $\bar{K} = \mathbb{E}\{\mathrm{e}^{\kappa M_1}\}/(\kappa m)$. If M_1 is nonnegative, K is bounded from below by the pre-factor C_W in the Cramér-Lundberg expansion $\mathbb{P}\{W > x\} \sim C_W \mathrm{e}^{-\kappa x}$, with $W = \sup_{n\geq 0} S_n$, cf. Asmussen [3].

3. A fluid model

To illustrate the general theory developed in the previous section, we now investigate a simple example. Let $J(t), t \geq 0$, be an alternating renewal (0-1) process with generic on-period T_{on} and generic off-period T_{off}, i.e. T_{on} is the period where $J(s) = 1$ and T_{off} is the period where $J(s) = 0$. Let $Y(t) = r\int_0^t J(s)\mathrm{d}s, t \geq 0$, be the associated integrated on-off process. The constant $r > 0$ is called the *on rate*. Assume that $J(t)$ is such that an on-period starts at time 0. Let the sequence $(T_{on,i}, T_{off,i}), i \geq 1$ representing on-times and off-times be i.i.d. with $(T_{on,1}, T_{off,1}) \stackrel{d}{=} (T_{on}, T_{off})$.

We allow T_{on} and T_{off} to be dependent and assume that $T_{on} + T_{off}$ has finite mean. Assume further that $\mathbb{E}\{J(t)\} \to \rho \in (0, c)$ for some constant $c > 0$ which is called the *drain rate*. Under these conditions, the process $S(t) = Y(t) - ct, t \geq 0$, is converging a.s. to $-\infty$. The renewal epochs for the process $S(t), t \geq 0$, are given by $T_i = \sum_{k=1}^{i}(T_{on,k} + T_{off,k}), i \geq 0$. In this setting, the distribution of M can be viewed as the Palm-stationary distribution of the amount of fluid in a buffer fed by an on-off source. This is a simple and well-known model (see e.g. Heath *et al.* [18] and Kella & Whitt [20]), and as such it provides simple applications. of the theory developed in the previous section. In the setting of that section, we have $X_1 = (r - c)T_{on} - cT_{off}$, $M_1 = (r - c)^+ T_{on}$ and hence $M_1^* = M_1$.

3.1. **An application of Theorems 1, 2 and 4.** Assume that $r > c$, and that T_{on} and T_{off} are dependent in the following way: let E_0, E_1 and E_2 be independent random variables with finite means, and suppose that $T_{on} = E_0 + E_1$ and $T_{off} = E_0 + E_2$. In this case, we have that $M_1 = (r - c)E_0 + (r - c)E_1$, $X_1 = (r - c)E_1 + (r - 2c)E_0 - cE_2$. Moreover, $\mathbb{E}\{X_1\}$ is assumed to be strictly negative. We now focus on two different scenarios.

- Assume that $\mathbb{P}\{E_0 > x\}$ is long-tailed, that $\int_x^\infty \mathbb{P}\{E_0 > u\}\mathrm{d}u$ is subexponential, and that E_1 has a finite moment generating function in a neighborhood of the origin. Then $\mathbb{P}\{M_1 > x\} \sim \mathbb{P}\{(r - c)E_0 > x\}$, implying that the conditions of Theorem 1 are satisfied. The property $\mathbb{P}\{M_1 > x\} \sim \mathbb{P}\{X_1 > x\}$ is clearly not satisfied. If E_0 has a lognormal or heavy-tailed Weibull distribution, one can actually show that $\mathbb{P}\{X_1 > x\} \sim \mathrm{o}(\mathbb{P}\{M_1 > x\})$; we omit the details.
- Assume that $\mathbb{P}\{E_0 > x\} \sim c_0 \mathrm{e}^{-\nu_0 x}$, and that $\mathbb{E}\{\mathrm{e}^{\nu_0 E_1}\} < \infty$. Then
$$\mathbb{P}\{M_1 > x\} \sim c_0 \mathbb{E}\{\mathrm{e}^{\nu_0 E_1}\}\mathrm{e}^{-\frac{\nu_0}{r-c}x} =: \bar{c}_0 \mathrm{e}^{-\bar{\nu}_0 x}.$$
Furthermore, we have $\mathbb{E}\{\mathrm{e}^{\bar{\nu}_0 X_1}\} < \infty$. If this quantity is strictly larger than 1, we are in the Cramér case covered by Theorem 4. If the quantity is strictly less than 1, the tail behavior of $\mathbb{P}\{M > x\}$ follows from Theorem 2.

3.2. **An application of Theorem 3.** Assume that $r > c$, that T_{on} and T_{off} are independent, and that T_{on} is in $\mathcal{S}(\alpha), \alpha > 0$. Then M_1 is in $\mathcal{S}(\alpha)$ as well, and $\mathbb{P}\{X_1 > x\} \sim \mathbb{E}\{\mathrm{e}^{-\alpha c T_{off}}\}\mathbb{P}\{M_1 > x\}$. Since the class $\mathcal{S}(\alpha)$ is closed under tail equivalence, we see that $X_1^+ \in \mathcal{S}(\alpha)$, which implies (9) and (10). Assume further that (11) holds. To apply Theorem 3, it remains to verify condition (13). For this, write

$$\begin{aligned}
\mathbb{P}\{M_1 > x; X_1 \leq x - a\} &= \mathbb{P}\{M_1 > x; M_1 - cT_{off} \leq x - a\} \\
&= \int_0^\infty \mathbb{P}\{x < M_1 \leq x + u\}\mathrm{d}\mathbb{P}\{cT_{off} - a \leq u\} \\
&\sim \mathbb{P}\{M_1 > x\}\int_0^\infty (1 - \mathrm{e}^{-\alpha u})\mathrm{d}\mathbb{P}\{cT_{off} - a \leq u\} \\
&= \mathbb{P}\{M_1 > x\}(\mathbb{P}\{cT_{off} \geq a\} - \mathbb{E}\{\mathrm{e}^{-\alpha(cT_{off} - a)}I_{(cT_{off} \geq a)}\}),
\end{aligned}$$

where we applied the property (9) for M_1 in the third step. The tail behavior of M now follows from Theorem 3, with $f(a) = \mathbb{E}\{(1 - \mathrm{e}^{-\alpha(cT_{off} - a)})I_{(cT_{off} \geq a)}\}\left(\mathbb{E}\{\mathrm{e}^{-\alpha c T_{off}}\}\right)^{-1}$.

3.3. **A fluid model with noise.** In this subsection, we provide another application of Theorem 1. Set $r = c = 1$, and $S(t) = Y(t) + W(t) - t$, with $W(t), t \geq 0$, an independent standard Wiener process. The process $S(t), t \geq 0$, can be interpreted as the net input process of an on-off fluid model perturbed by Brownian motion, with the additional feature that the on rate equals the drain rate (both are equal to 1).

The sequence $T_n, n \geq 1$, representing the starting points of on-periods is again a renewal sequence for the process $S(t), t \geq 0$. It is clear that

$$(M_1, X_1) \stackrel{d}{=} \left(\sup_{0 < t \leq T_{on} + T_{off}} [W(t) - (t - T_{on})^+], W(T_{on} + T_{off}) - T_{off}\right).$$

Assume that $\mathbb{P}\{T_{on} > x\}$ is regularly varying of index $-\nu, \nu > 1$. We first state some preliminary results. Combining Theorem 2.1 and Proposition 2.1(i) of Dębicki *et al.* [9] we obtain:

$$(14)\qquad \mathbb{P}\{\sup_{0<t<T_{on}} W(t) > x\} \sim \frac{1}{\sqrt{\pi}} 2^{\nu+1}\Gamma(\nu+\frac{1}{2})\mathbb{P}\{T_{on} > x^2\}.$$

Using a famous result of P. Lévy we have that:

$$\mathbb{P}\{\sup_{0<t<T_{on}} W(t) > x\} = \mathbb{P}\{|W(T_{on})| > x\}.$$

By using the symmetry around 0 of the standard normal distribution, we conclude that

$$(15)\qquad \mathbb{P}\{W(T_{on}) > x\} = \frac{1}{2}\mathbb{P}\{|W(T_{on})| > x\} \sim \frac{1}{\sqrt{\pi}} 2^{\nu}\Gamma(\nu+\frac{1}{2})\mathbb{P}\{T_{on} > x^2\}.$$

In particular, the right tail of both the distribution of $W(T_{on})$ and of $\sup_{0<t<T_{on}} W(t)$ is regularly varying with index -2ν. This is an important ingredient in the proof of the following result.

Lemma 3. *In the setting of this subsection, we have*

$$\mathbb{P}\{M_1^* > x\} = \mathbb{P}\{M_1 > x\} \sim \mathbb{P}\{\sup_{0<t<T_{on}} W(t) > x\} \text{ and } \mathbb{P}\{X_1 > x\} \sim \mathbb{P}\{W(T_{on}) > x\}.$$

Proof. First, we prove the assertion for M_1. From the definition of M_1 it immediately follows that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{M_1 > x\}}{\mathbb{P}\{\sup_{0<t<T_{on}} W(t) > x\}} \geq 1.$$

To prove an asymptotic upper bound, use $T_{off} < \infty$ to obtain

$$\begin{aligned} M_1 &\leq \sup_{t>0}[W(t) - (t - T_{on})^+] \\ &= \max\{\sup_{0<t<T_{on}}[W(t)], W(T_{on}) + \sup_{t>T_{on}}[W(t) - W(T_{on}) - (t - T_{on})]\} \\ &\leq \sup_{0<t<T_{on}}[W(t)] + \sup_{t>T_{on}}[W(t) - W(T_{on}) - (t - T_{on})]. \end{aligned}$$

Since the increments of the Wiener process are independent, it is clear that the two terms in the last line are independent. Furthermore, the random variable $\sup_{t>T_{on}}[W(t) - W(T_{on}) - (t - T_{on})]$ has an exponential distribution with mean 1/2, see for example [3]. Moreover, the right tail of $\sup_{0<t<T_{on}} W(t)$ is regularly varying with index -2ν, which leads us to

$$\mathbb{P}\{\sup_{0<t<T_{on}}[W(t)] + \sup_{t>T_{on}}[W(t) - W(T_{on}) - (t - T_{on})] > x\} \sim \mathbb{P}\{\sup_{0<t<T_{on}} W(t) > x\}.$$

Consequently,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{M_1 > x\}}{\mathbb{P}\{\sup_{0<t<T_{on}} W(t) > x\}} \leq 1,$$

which implies our first assertion. To prove the second assertion, let $\tilde{W}$ be another standard Wiener process, independent of W, T_{on} and T_{off}. We first prove an asymptotic lower bound. Note that

$$X_1 \stackrel{d}{=} W(T_{on}) + \tilde{W}(T_{off}) - T_{off}.$$

Consequently, for any $y > 0$,

$$\mathbb{P}\{X_1 > x\} \geq \mathbb{P}\{W(T_{on}) > x + y\}\mathbb{P}\{\tilde{W}(T_{off}) - T_{off} > -y\}.$$

The random variable $W(T_{on})$ has a right tail which is regularly varying with index -2ν. In particular, its right tail is long-tailed. This implies that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{X_1 > x\}}{\mathbb{P}\{W(T_{on}) > x\}} \geq \mathbb{P}\{\tilde{W}(T_{off}) - T_{off} > -y\}.$$

The asymptotic lower bound follows now by letting $y \to \infty$. We now turn to the upper bound. Since

$$\mathbb{P}\{\tilde{W}(T_{off}) - T_{off} > x\} \leq \mathbb{P}\{\sup_{t>0}[\tilde{W}(t) - t] > x\} = e^{-2x},$$

the upper bound follows by a similar argument as the one made for M_1. □

Putting everything together, it follows that $\mathbb{P}\{M_1 > x\} \sim 2\mathbb{P}\{X_1 > x\}$, and both tails are regularly varying with index -2ν. Using Theorem 1, this yields

$$\mathbb{P}\{M > x\} \sim \frac{1}{\mathbb{E}\{T_{off}\}} \frac{1}{\sqrt{\pi}} 2^{\nu+1}\Gamma(\nu + \frac{1}{2}) \int_x^\infty \mathbb{P}\{T_{on} > u^2\}\mathrm{d}u.$$

To conclude, it is interesting to note that the tail asymptotics for M in the zero-delayed case discussed here, are regularly varying with index $1 - 2\nu$. This differs significantly from the tail behavior in the delayed (stationary) case, which is regularly varying with index $2-2\nu$, cf. Theorem 4.1 of Zwart *et al.* [29].

4. A model from insurance risk

Consider the following regenerative process $S(t), t \geq 0$, given by

$$S(t) = \sum_{i=1}^{N(t)} U_i - t, \qquad t \geq 0.$$

$N(t), t \geq 0$, is a Cox process with an underlying regenerative process $R(t), t \geq 0$, with renewal epochs $T_i, i \geq 1$. That is, there exists a a nonnegative measurable function $\lambda : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ with the following property: for a realization $r(t), t \geq 0$ of the process $R(t), t \geq 0$, the process $N(t), t \geq 0$, has the same law as a non-homogeneous Poisson process with intensity $\overline{\lambda}(t) = \lambda(r(t))$ at time $t \geq 0$. A detailed discussion of Cox processes and their impact on risk theory can be found in Grandell [16], Rolski *et al.* [23], Bjőrk and Grandell [6], Schmidli [25], and Asmussen *et al.* [4]. The claim sizes $U_i, i \geq 1$, are i.i.d. r.v.'s independent of the process $N(t), t \geq 0$, with a common non-lattice distribution function F_U. Let x be the initial reserve and assume that $S(t) \to -\infty$ a.s. as $t \to +\infty$. The process $x - S(t), t \geq 0$, is known as the surplus process, and we say that ruin occurs if this process hits 0. The *infinite horizon ruin probability* is then given by

$$\psi(x) = \mathbb{P}\{M > x\}.$$

4.1. **Applications of Theorem 4.** We first focus on the Cramér case covered by Theorem 4. That is, we assume that there exists a constant $\kappa > 0$ such that

$$\mathbb{E}\{e^{\kappa S(T_1)}\} = \mathbb{E}\exp\left\{\int_0^{T_1} \lambda(R(s))\mathrm{d}s(\hat{m}_U(\kappa) - 1) - \kappa T_1\right\} = 1 \tag{16}$$

with $\hat{m}_U(\theta) = \mathbb{E}\{e^{\theta U}\}$. In risk theory, κ is called the adjustment coefficient. In addition, we assume that

$$m = \mathbb{E}\left\{\hat{m}'_U(\kappa)\int_0^{T_1} \lambda(R(s))\mathrm{d}s - T_1\right\}\exp\left\{\int_0^{T_1} \lambda(R(s))\mathrm{d}s(\hat{m}_U(\kappa) - 1) - \kappa T_1\right\} < \infty,$$

and that

$$\mathbb{E}\exp\left\{\int_0^{T_1} \lambda(R(s))\mathrm{d}s(\hat{m}_U(\kappa) - 1)\right\} < \infty.$$

From Theorem 4 we derive the exponential asymptotics

$$\psi(x) \sim Ke^{-\kappa x}. \tag{17}$$

Note that the constant K can be bounded from above by

$$\begin{aligned} \overline{K} &= \frac{1}{\kappa m}\mathbb{E}\exp\left\{\kappa \sum_{i=1}^{N(T_1)} U_i\right\} \\ &= \frac{1}{\kappa m}\mathbb{E}\exp\left\{\int_0^{T_1} \lambda(X(s))\mathrm{d}s(\hat{m}_U(\kappa) - 1)\right\}. \qquad (18) \end{aligned}$$

An important special case is the Bjőrk-Grandell model, where $\lambda(x) = x$, and $R(s) = L_i$ for $\sum_{j=1}^{i-1} \sigma_j \leq s < \sum_{j=1}^{i} \sigma_j$, with $(L_i, \sigma_i), i \geq 1$, an i.i.d. sequence of vectors with positive components. In this particular case, the adjustment coefficient $\kappa > 0$ is a solution of the equation

$$\mathbb{E} \exp \{\sigma L(\hat{m}_U(\kappa) - 1) - \kappa\sigma\} = 1.$$

The upper bound $\overline{K}$ of K satisfies $\overline{K} = \frac{1}{\kappa m} \mathbb{E} \exp \{\sigma L(\hat{m}_U(\kappa) - 1)\}$, with $m = \mathbb{E}\{(\hat{m}_U'(\kappa)\sigma L - \sigma) \exp \{\sigma L(\hat{m}_U(\kappa) - 1) - \kappa\sigma\}\}$. Existence of the exponential asymptotics was first established by Schmidli [25]. A different upper bound for K was derived in [23], Theorem 12.5.3.

We now discuss another example of a Cox process where the intensity process is described by a diffusion process. A motivating example comes from vehicle insurance, where the intensity of the claim arrivals may depend on the density of vehicles insured. The latter can randomly change in time due to the variability of the number of inhabitants, or market share within an area. A functional of a diffusion process seems flexible enough to take the stochastic variability of the intensity process into account; we refer to Schmock [26] for further discussion and motivation. One can use the exponential asymptotics given by (17) to obtain an exponential approximation of the ruin probability in this model. We consider the example of an Ornstein-Uhlenbeck process $R(t), t \geq 0$, which starts at 0 and has generator

$$(\mathbf{A}f)(x) = \frac{1}{2}\frac{d^2}{dx^2} f(x) - bx\frac{d}{dx} f(x),$$

for some $b > 0$. The domain of this generator contains all functions f in $\mathcal{C}^2(\mathbf{R})$. In addition, we assume that $\lambda(x) = x^2 + k$ for $k \geq 0$. In Palmowski [22] it is shown that if $\mathbb{E}\{U\} < (2b)/(1 + 2bk)$, then $S(t)$ tends to $-\infty$ a.s. as $t \to \infty$; the inequality $\mathbb{E}\{U\} < (2b)/(1 + 2bk)$ will be assumed from now on. Note that $R(t), t \geq 0$, is a regenerative process with regeneration epochs $T_{n+1} = \inf\{t \geq I_n : R(t) = 0\}$, where $I_n = \inf\{t \geq T_n : |R(t)| = 1\}, n \geq 0$.

Let $\eta > 0$. Under the twisted law Q of R with respect to the martingale

$$M(t) = \exp\left\{-\frac{\eta^2 - b^2}{2}\int_0^t R^2(s)\,\mathrm{d}s - \frac{\eta - b}{2}(R^2(t) - R^2(0) - t)\right\}, t \geq 0$$

one obtains an Ornstein-Uhlenbeck process with parameter η (see Leblanc *et al.* [24]). Writing $Z = \int_0^{T_1} \lambda(R(s))\mathrm{d}s = \int_0^{T_1} \left(R^2(s) + k\right) \mathrm{d}s$, $\xi = \hat{m}_U(\kappa) - 1$, and applying the twisted law with $\eta = \sqrt{b^2 - 2\xi}$, we obtain the following equation for the adjustment coefficient κ: $\mathbb{E}_Q\{e^{-(\hat{\eta}+\kappa)T_1}\} = 1$, where $\hat{\eta} = -(b - \eta)/2 - \xi k$. Hence $\hat{\eta} + \kappa = 0$ and κ solves the following equation:

$$\kappa = \frac{b - \sqrt{b^2 - 2(\hat{m}_U(\kappa) - 1)}}{2} + (\hat{m}_U(\kappa) - 1)k.$$

If the claims are exponentially distributed with mean φ and $k = 0$ then for $\varphi < 2b$ we have

$$\kappa = \frac{b\varphi + 1}{2\varphi}\left(1 - \sqrt{1 - \frac{2\varphi(2b - \varphi)}{(b\varphi + 1)^2}}\right).$$

To the best of our knowledge, this is the first exact expression for the adjustment coefficient in a Cox model driven by a diffusion process. In addition, one can obtain an explicit expression for the upper bound $\overline{K}$ of K. Note that T_1 is the sum of the exit time from interval $[-1, 1]$ and the (independent) first passage time into the negative half-line for the Ornstein-Uhlenbeck process starting from 1. Write $D_{-\zeta}$ for the parabolic cylinder function and set

$$S(\zeta, x, y) = \frac{\Gamma(\zeta)}{\pi} e^{(x^2+y^2)/4} \left(D_{-\zeta}(-x)D_{-\zeta}(y) - D_{-\zeta}(x)D_{-\zeta}(-y)\right).$$

Using similar arguments as in Palmowski [22] and results from Borodin and Salminen [7] (p. 429 and p. 434) we obtain

$$\overline{K} = \frac{1}{\kappa m}\mathbb{E}\{e^{\xi Z}\} = \frac{1}{\kappa m}\mathbb{E}_Q\{e^{-\hat{\eta}T_1}\} = \frac{1}{\kappa m}H(\hat{\eta})$$

if $\hat{\eta}/\eta > -2$, where

$$H(s) = \frac{S(s/\eta, \sqrt{2\eta}, 0) + S(s/\eta, 0, -\sqrt{2\eta})}{S(s/\eta, \sqrt{2\eta}, -\sqrt{2\eta})} \frac{e^{\eta/2} D_{-\frac{s}{\eta}}(\sqrt{2\eta})}{D_{-\frac{s}{\eta}}(0)}$$

for $m = \partial/\partial\kappa\, H(\hat{\eta} + \kappa)$.

4.2. **An application of Theorem 2.** Consider again the Bjőrk-Grandell model with $\mathbb{P}\{L = \lambda_i\} = p_i > 0$ for $i = 0, 1, \ldots, d$. We assume that $\lambda_0 > \lambda_1 > \ldots > \lambda_d > 0$ and that $\lambda_0 \mathbb{E}\{U\} < 1$. Moreover, let

$$\mathbb{P}\{\sigma_0 > x\} := \mathbb{P}\{\sigma > x \mid L = \lambda_0\} \sim x^{-\alpha_0} l_0(x)$$

for a slowly varying function l_0, and let there exists a solution $\nu_0 > 0$ of the equation $\phi_0(\theta) = \lambda_0(\hat{m}_U(\theta) - 1) - \theta = 0$. We assume that $\hat{m}_U(\theta)$ is finite in a neighborhood of ν_0. Define also $\phi_i(\theta) = \lambda_i(\hat{m}_U(\theta) - 1) - \theta$. Writing $\tau(x) = \inf\{t \geq 0 : S(t) > x\}$ for the ruin time we derive

$$\mathbb{P}\{M_1 > x\} = \mathbb{P}\{\tau(x) < \sigma\} = \sum_{i=0}^{d} \mathbb{P}\{\tau(x) < \sigma | L = \lambda_i\} p_i.$$

Our main goal is to determine the tail behavior of M_1, and to verify that the conditions of Theorem 2 are satisfied.

Proposition 1. *In the setting of this subsection, it holds that*

$$\mathbb{P}\{M_1 > x\} \sim p_0 \mathbb{P}\{\sigma_0 > x/\phi_0'(\nu_0)\} \frac{-\phi_0'(0)}{\phi_0'(\nu_0)} \mathrm{e}^{-\nu_0 x}.$$

Proof. We first focus on the tail behavior of $\mathbb{P}\{M_1 > x \mid L = \lambda_0\}$. A crucial observation is that, conditionally upon $L = \lambda_0$, the risk process evolves according to a standard compound Poisson process with rate λ_0 up to time σ_0. Let $\mathbb{P}_0\{\cdot\}$ be the probability measure under which the intensity equals λ_0 (we define $\mathbb{P}_i\{\cdot\}$ similarly). Then $\mathbb{P}\{M_1 > x \mid L = \lambda_0\} = \mathbb{P}_0\{\tau(x) < \sigma_0\}$ such that σ_0 is independent of $(S(t))$ under $\mathbb{P}_0\{\cdot\}$. We now state two important results, which directly follow from results of Höglund [19]. Define $\hat{d} = 1/\phi_0'(\nu_0)$ and let $\epsilon > 0$. Corollary 2.3 of [19] implies that

$$\mathbb{P}_0\{\tau(x) < (\hat{d} + \epsilon)x\} \sim \mathbb{P}_0\{\tau(x) < \infty\} \sim \frac{-\phi_0'(0)}{\phi_0'(\nu_0)} \mathrm{e}^{-\nu_0 x}. \tag{19}$$

Moreover, for every $\epsilon \in (0, \hat{d})$ there exists a constant $\delta > 0$ such that

$$\mathbb{P}_0\{\tau(x) < (\hat{d} - \epsilon)x\} = \mathrm{o}(\mathrm{e}^{-(\nu_0 + \delta)x}). \tag{20}$$

Therefore, from (19),

$$\mathbb{P}_0\{\tau(x) < \sigma_0\} \geq \mathbb{P}_0\{\tau(x) < (\hat{d} + \epsilon)x\} \mathbb{P}\{\sigma_0 > (\hat{d} + \epsilon)x\}.$$

Combining this with the fact that $\mathbb{P}\{M_1 > x\} \geq p_0 \mathbb{P}_0\{\tau(x) < \sigma_0\}$, we obtain

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{M_1 > x\}}{p_0 \mathbb{P}\{\sigma_0 > x/\phi_0'(\nu_0)\} \frac{-\phi_0'(0)}{\phi_0'(\nu_0)} \mathrm{e}^{-\nu_0 x}} \geq \left(\frac{\hat{d}}{\hat{d} + \epsilon}\right)^{\alpha_0}.$$

The proof of the lower bound is now completed by letting $\epsilon \to 0$.

For the upper bound, observe that

$$\mathbb{P}\{M_1 > x\} \leq p_0 \mathbb{P}_0\{\tau(x) < \sigma_0\} + \sum_{i=1}^{d} \mathbb{P}_i\{\tau(x) < \infty\}. \tag{21}$$

Since $\lambda_i > \lambda_0$ for $i \geq 1$ and since the moment generating function of U is finite in a neighborhood of ν_0, the quantity $\nu_i = \sup\{s : \phi_i(s) \leq 0\}$ is strictly larger than ν_0 for $i \geq 1$. From Lundberg's inequality, we obtain that

$$\sum_{i=1}^{d} \mathbb{P}_i\{\tau(x) < \infty\} \leq \sum_{i=1}^{d} \mathrm{e}^{-\nu_i x} = \mathrm{o}(\mathrm{e}^{-(\nu_0 + \eta)x}), \tag{22}$$

for an appropriate choice of $\eta > 0$. In addition, observe that

$$\mathbb{P}_0\{\tau(x) < \sigma_0\} \leq \mathbb{P}_0\{\tau(x) \leq (\hat{d}-\epsilon)x\} + \mathbb{P}_0\{\infty > \tau(x) > (\hat{d}-\epsilon)x\}\mathbb{P}\{\sigma_0 > (\hat{d}-\epsilon)x\}. \tag{23}$$

So from (19)-(20), and by substituting the bounds (23) and (22) into (21) we obtain that

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{M_1 > x\}}{p_0\mathbb{P}\{\sigma_0 > x/\phi_0'(\nu_0)\}\frac{-\phi_0'(0)}{\phi_0'(\nu_0)}\mathrm{e}^{-\nu_0 x}} \leq \left(\frac{\hat{d}}{\hat{d}-\epsilon}\right)^{\alpha_0}.$$

The proof of the upper bound is now completed by letting $\epsilon \to 0$. □

We now derive the behavior of the ruin probability $\psi(x)$ as $x \to \infty$. Since $\lambda_i < \lambda_0$ for $i \geq 1$, it follows that $\phi_i(\nu_0) < 0$ if $i \geq 1$. Consequently,

$$\mathbb{E}\{\mathrm{e}^{\nu_0 X_1}\} = \mathbb{E}\{\mathrm{e}^{\nu_0 S(\sigma)}\} = \sum_{i=0}^{d} p_i\mathbb{E}\{\mathrm{e}^{\nu_0 S(\sigma)} \mid L = \lambda_i\} = \sum_{i=0}^{d} p_i\mathbb{E}\{\mathrm{e}^{\phi_i(\nu_0)\sigma} \mid L = \lambda_i\} < 1. \tag{24}$$

Applying Theorem 2 yields

$$\psi(x) = \mathbb{P}\{M > x\} \sim C_R x^{-\alpha_0} l_0(x)\mathrm{e}^{-\nu_0 x},$$

for $C_R = \frac{-p_0\phi_0'(0)(\phi_0'(\nu_0))^{\alpha_0-1}}{1-\mathbb{E}\{\mathrm{e}^{\nu_0 S(\sigma)}\}}$, where $\mathbb{E}\{\mathrm{e}^{\nu_0 S(\sigma)}\}$ is given by (24).

5. Concluding remarks

We have examined the tail behavior of the supremum of a process of which the increment process is regenerative. We identified four different regimes, all exhibiting qualitatively different behavior of $\mathbb{P}\{M > x\}$ as $x \to \infty$. Our results focus on the zero-delayed case. It is not difficult to extend our results to the delayed case, using the representation $M = \max\{M_0, X_0 + M_{zd}\}$, with M_{zd} a random variable independent of (M_0, X_0), having the same distribution of M as in the zero-delayed case, where $M_0 = \sup_{t\leq T_0} S(t)$ and $X_0 = S(T_0)$.

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