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Abstract

In this paper we consider a Cox process driven by a Markov process. We markovize this Cox process by adding supplementary governing component and find the extended generator $\mathcal{A}$ of such constructed process and its domain $D(\mathcal{A})$. As an example we discuss a risk process with Coxian arrival process generated by a diffusion process. For this process we derive Lundberg inequality for the ruin probability.

Keywords: ruin probability, Lundberg inequality, Cox process, extended generator, diffusion process, exponential martingale.

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1 Introduction

Denote by $\mathcal{L}$ the space of measurable functions on a Banach space E. By *extended generator* $\mathcal{A}$ of a process $\{Z(t), t \geq 0\}$ (the notion introduced by Stroock and Varadhan (1969)) we mean

$$\mathcal{A} = \{(g,f) \in \mathcal{L} \times \mathcal{L} : M^g(t) \in \mathcal{M}_{\text{loc}}\} \ ,$$

where $\mathcal{M}_{\text{loc}}$ is a family of local martingales and

$$M^g(t) = g(Z(t)) - \int_0^t f(Z(s))\, ds \ . \tag{1.1}$$

We identify all versions of functions f up to the sets of potential zero and we denote all these versions by $\mathcal{A}g$ if $(g,f) \in \mathcal{A}$. By *domain of the extended generator* $D(\mathcal{A})$ we mean the set of measurable functions $g \in \mathcal{L}$ such that $M^g \in \mathcal{M}_{\text{loc}}$. Although it is rather easy to determine the operator $\mathcal{A}$ (it is a infinitisimal generator if it exists) but it is often quite difficult to characterize the domain $D(\mathcal{A})$. Davis (1993) find the necessary and sufficent conditions for membership of $D(\mathcal{A})$ in case of piecewise deterministic Markov process (PDMP). In most cases it is only possible to determine some subset of the $D(\mathcal{A})$ (for instance in case of diffusion), which is often also denoted by $D(\mathcal{A})$.

We consider a cádlág processes $\{Y(t), t \geq 0\}$ and $\{X(t), t \geq 0\}$ with values in Banach spaces E_Y and E_X respectively. Denote by $\mathcal{A}^X$ the extended generator of process $\{X(t), t \geq 0\}$ and by $D(\mathcal{A}^X)$ its domain. Let $\{Y(t), t \geq 0\}$ be a Cox process driven by $\{X(t), t \geq 0\}$. That is, for a given realization $x(t)$ of $\{X(t), t \geq 0\}$ process $\{Y(t), t \geq 0\}$ has the same law like a given process $\left\{Y^{(x)}(t), t \geq 0\right\}$ with generator $\mathcal{A}^{(x)}$ and domain $D(\mathcal{A}^{(x)})$. Define operator $(\mathcal{A}^Y g)(r)$ in the following way. If the realization of the process $\{X(t), t \geq 0\}$ is $x(t)$, then

$$(\mathcal{A}^Y g)(r) = (\mathcal{A}^{(x)} g)(r) \ . \tag{1.2}$$

By $D(\mathcal{A}^Y)$ we denote a collection of all functions

$$g(r) \in D(\mathcal{A}^{(x)}) \qquad \text{for all } x \in D_{E_X}[0, +\infty). \tag{1.3}$$

In section 2 we prove that under some additional assumptions process $\{(Y(t), X(t)), t \geq 0\}$ has the following extended generator

$$\mathcal{A} = \mathcal{A}^Y \oplus \mathcal{A}^X$$

and $\overline{D(\mathcal{A}^Y) \otimes D(\mathcal{A}^X)} \subset D(\mathcal{A})$, where closure is in a uniform convergence topology.

Bjőrk and Grandell (1988) derived by a *martingale approach* an exponential upper bound for the ruin probability $\psi(u)$ when the occurrence of the claims is described by the Cox process with an intensity process having Markovian piecewise constant realizations. In section 3 we apply our general theory of extended generator of Cox process to get (infinite time) Cramér-Lundberg inequality, when occurrence of the claims is described by the Cox process with general (not necessary Markov) intensity and also the distribution function of the claim size depends on a state $X(t)$ of the governing process at time t. In particular, we consider diffusion intensities and in section 4 we calculate directly upper bound for the intensity being square function of Ornstein-Uhlenbeck process. Similar considerations are in Embrechts *et al* (1993) (finite time non-Markovian intensities), and Grigolionis (1992a).

2 Generator of Cox process

Denote by $\{X(t), t \geq 0\}$ the underlying canonical cádlág process with values in Banach spaces E_X. That is, $\{X(t), t \geq 0\}$ is defined on the following probability space $(\Omega_X = D_{E_X}[0, +\infty), \mathcal{F}^X, \{\mathcal{F}^X_{t+}\}, \mathbb{P}^X)$, where $\mathcal{F}^X_{t+}$ is generated by the sets $\{x \in \Omega_X : x_{|[0,t]} \in A\}$ for $A \in \mathcal{B}(D_{E_X}[0,t])$ and $x_{|[0,t]}$ being a restriction of function x to $[0,t]$. Cádlág process $\{Y(t), t \geq 0\}$ with values in the Banach space E_Y is a Cox process driven by the $\{X(t), t \geq 0\}$. That is, for a given realization $x(t) \in D_{E_X}[0, +\infty)$ of process $\{X(t), t \geq 0\}$ process $\{Y(t), t \geq 0\}$ behaves like a given process $Y^{(x)}(t)$ on probability space $(D_{E_Y}[0, +\infty), \mathcal{F}^Y, \{\mathcal{F}^Y_{t+}\}, \mathbb{P}^{(x)})$. Here, $\mathcal{F}^Y_{t+}$ is generated by the sets $\{y \in \Omega_Y : y_{|[0,t]} \in B\}$ for $B \in \mathcal{B}(D_{E_Y}[0,t])$. We assume that $Y(0) = u$, that is $Y^{(x)}(0) = u$ for all realizations x of $\{X(t), t \geq 0\}$. Let $\Omega = \Omega_Y \times \Omega_X$. By $\mathcal{F}^{(*,X)}_{t+}$ we denote the σ-field which consists of sets $\{(y,x) \in \Omega : x_{|[0,t]} \in A\} = \{(y,x) \in \Omega : x_{|[0,t]} \in A, y_{|[0,t]} \in D_{E_Y}[0,t]\}$. Similarly, by $\mathcal{F}^{(Y,*)}_{t+}$ we mean the σ-field which consists of sets $\{(y,x) \in \Omega : y_{|[0,t]} \in B\}$, where $B \in \mathcal{B}(D_{E_Y}[0,t])$. Let $\mathcal{F}^{(*,X)} = \bigvee_{t \geq 0} \mathcal{F}^{(*,X)}_{t+}$, $\mathcal{F}^{(Y,*)} = \bigvee_{t \geq 0} \mathcal{F}^{(Y,*)}_{t+}$ and $\mathcal{F}^{(Y,X)} = \mathcal{F}^{(Y,*)} \vee \mathcal{F}^{(*,X)}$.

On probability space $(\Omega, \mathcal{F}^{(Y,X)}, \mathbb{P})$ we define canonical process $\{(Y(t), X(t)), t \geq 0\}$ (see Parasarathy (1967), Th. 8.1, p. 147, Brémaud (1981), Grandell (1976) and Grandell (1991)) by the following equality:

$$d\mathbb{P}(y,x) = d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) \; . \tag{2.4}$$

In analysis of extended genarator the most convenient filtration on $(\Omega, \mathcal{F}^{(Y,X)}, \mathbb{P})$ is natural filtration $\{\mathcal{F}^{(Y,X)}_{t+}\}$ of process $\{(Y(t), X(t)), t \geq 0\}$; see Grandell (1991), p. 114. The σ-field $\mathcal{F}^{(Y,X)}_{t+}$ is generated be the sets

$$\begin{aligned} &\{(y,x) \in \Omega : y_{|[0,t]} \in A_1, x_{|[0,t]} \in A_2\} = \\ &\quad = \{y \in \Omega_Y : y_{[0,t]} \in A_1\} \times \{x \in \Omega_X : x_{|[0,t]} \in A_2\} \; , \end{aligned}$$

where $A_1 \in \mathcal{B}(D_{E_Y}[0,t])$ and $A_2 \in \mathcal{B}(D_{E_X}[0,t])$. Thus

$$\mathcal{F}^{(Y,X)}_{t+} = \mathcal{F}^Y_{t+} \times \mathcal{F}^X_{t+} = \mathcal{F}^{(Y,*)}_{t+} \vee \mathcal{F}^{(*,X)}_{t+}$$

We now find the extended generator of process $\{(Y(t), X(t)), t \geq 0\}$. We define $\mathcal{A}^Y$ and $D(\mathcal{A}^Y)$ by (1.2) and (1.3) respectively.

Lemma 2.1 *If $g(r) \in D(\mathcal{A}^Y)$, then*

$$M^{Y,g}(t) = g(Y(t)) - \int_0^t (\mathcal{A}^Y g)(Y(s))\, ds$$

is a local $(\{\mathcal{F}^{(Y,X)}_{t+}\}, \mathbb{P})$-martingale.

Proof. Let $\{\tau_n^{(x)}\}$ be a fundamental sequence of $\{\mathcal{F}^{Y^{(x)}}_{t+}\} = \{\mathcal{F}^Y_{t+}\}$ stopping times for local martingale $\{M^{x,g}(t) = g(Y^{(x)}(t)) - \int_0^t (\mathcal{A}^{(x)} g)(Y^{(x)}(s))\, ds\}$. It can be for instance the jump epochs of process $\left\{Y^{(x)}(t), t \geq 0\right\}$, that is $\tau_n^{(x)}(y) = \inf\{t \geq \tau_{n-1}^{(x)} : y(t) \neq y(t-)\}$. Define $\tau_n^Y(y,x) = \tau_n^{(x)}(y)$. Then

$$\{(y,x) \in \Omega : \tau_n^Y(y,x) \leq t\} = \{(y,x) \in \Omega : \tau_n^{(x)}(y) \leq t\} \in \mathcal{F}^{(Y,*)}_{t+} \subset \mathcal{F}^{(Y,X)}_{t+} \; . \tag{2.5}$$

It means that τ_n^Y is a $\mathcal{F}_t^{(Y,X)}$-stopping time. The sequence of stopping times $\{\tau_n^Y\}$ will be also the fundamental sequence for $\{M^{Y,g}(t), t \geq 0\}$. In fact,

$$\begin{aligned} &\mathbb{E}M^{Y,g}(t \wedge \tau_n^Y) = \mathbb{E}^X \mathbb{E}^{(x)} M^{x,g}(t \wedge \tau_n^Y) \\ &= \mathbb{E}^X g(Y^{(x)}(0)) = g(u) < +\infty , \end{aligned}$$

where $\mathbb{E}^X$ is a expectation with respect to $\mathbb{P}^X$. Thus it suffices to prove that

$$\mathbb{E}[M^{Y,g}(t \wedge \tau_n^Y) | \mathcal{F}_{s+}^{(Y,X)}] = M^{Y,g}(s \wedge \tau_n^Y) \qquad \text{a.e.}$$

for $g \in D(\mathcal{A}^Y)$ and $s \leq t$. That is, that for $A \in \mathcal{F}_{s+}^{(Y,X)}$ we have

$$\int_A M^{Y,g}(t \wedge \tau_n^Y)\, d\mathbb{P} = \int_A M^{Y,g}(s \wedge \tau_n^Y)\, d\mathbb{P} \; . \tag{2.6}$$

It suffices to prove (2.6) for the set $A = A_1 \times A_2$, where $A_1 \in \mathcal{F}_{s+}^Y$ and $A_2 \in \mathcal{F}_{s+}^X$. Let

$$m^{x,g}(t) = g(y(t)) - \int_0^t (\mathcal{A}^{(x)} g)(y(s))\, ds \; .$$

We have

$$\begin{aligned} \int_A M^{Y,g}(t \wedge \tau_n^Y)\, d\mathbb{P} &= \int_A m^{x,g}(t \wedge \tau_n^Y)\, d\mathbb{P}(y,x) = \\ &= \int_{A_2} \int_{A_1} m^{x,g}(t \wedge \tau_n^{(x)})\, d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) \\ &= \int_{A_2} \int_{A_1} m^{x,g}(s \wedge \tau_n^{(x)})\, d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) \\ &= \int_A m^{x,g}(s \wedge \tau_n^Y)\, d\mathbb{P}(y,x) = \int_A M^{Y,g}(s \wedge \tau_n^Y)\, d\mathbb{P} \; , \end{aligned}$$

which completes the proof.

□

Let $\mathcal{A}^X$ be the extended generator of the process $\{X(t), t \geq 0\}$ and $D(\mathcal{A}^X)$ its domain. That is, for $f \in D(\mathcal{A}^X)$

$$M^{X,f}(t) = f(X(t)) - \int_0^t (\mathcal{A}^X f)(X(s))\, ds \tag{2.7}$$

is a local $(\mathcal{F}_{t+}^X, \mathbb{P}^X)$-martingale. Hence it is also a local $(\mathcal{F}_{t+}^{(*,X)}, \mathbb{P})$-martingale and finally a local $(\mathcal{F}_t^{(Y,X)}, \mathbb{P})$-martingale. By $\{\tau_n^X\}$ we denote its fundamental sequence. Let

$$m^{X,f}(t) = f(x(t)) - \int_0^t (\mathcal{A}^X f)(x(s))\, ds \; .$$

We assume that $X(0) = w$.

Lemma 2.2 *Process $\{M^{Y,g}(t) M^{X,f}(t), t \geq 0\}$ is a local $(\mathcal{F}_{t+}^{(Y,X)}, \mathbb{P})$-martingale for $g \in D(\mathcal{A}^Y)$ and $f \in D(\mathcal{A}^X)$.*

Proof. Consider sequence of stopping times $\tau_n = \tau_n^X \wedge \tau_n^Y$. We prove that process $\{M^{Y,g}(t \wedge \tau_n) M^{X,f}(t \wedge \tau_n), t \geq 0\}$ is a martingale. Note that by Lemma 2.1 we have

$$\begin{aligned}
&\mathbb{E} M^{Y,g}(t \wedge \tau_n) M^{X,f}(t \wedge \tau_n) = \mathbb{E}^X M^{X,f}(t \wedge \tau_n) \mathbb{E}^{(x)} M^{x,g}(t \wedge \tau_n) = \\
&= \int_{D_{E_X}[0,+\infty)} m^{X,f}(t \wedge \tau_n) \int_{D_{E_Y}[0,+\infty)} m^{x,g}(t \wedge \tau_n)\, d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) = \\
&= \int_{D_{E_X}[0,+\infty)} m^{X,f}(t \wedge \tau_n) \int_{D_{E_Y}[0,+\infty)} m^{x,g}(0)\, d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) = \\
&= g(u) \mathbb{E}^X M^{X,f}(t \wedge \tau_n) = g(u) f(X(0)) = g(u) f(w) < +\infty \ .
\end{aligned}$$

We now prove the martingale property, that is that

$$\mathbb{E}\left[M^{Y,g}(t \wedge \tau_n) M^{X,f}(t \wedge \tau_n) | \mathcal{F}_{s+}^{(Y,X)}\right] = M^{Y,g}(s \wedge \tau_n) M^{X,f}(s \wedge \tau_n), \qquad \text{a.e.}$$

Note that for $A = A_1 \times A_2 \in \mathcal{F}_{s+}^{(Y,X)}$, where $A_2 \in \mathcal{F}_{s+}^X$ and $A_1 \in \mathcal{F}_{s+}^Y$, by Lemma 3.1 we have

$$\begin{aligned}
&\int_A M^{Y,g}(t \wedge \tau_n) M^{X,f}(t \wedge \tau_n)\, d\mathbb{P} = \int_{A_2} m^{X,f}(t \wedge \tau_n) \int_{A_1} m^{x,g}(t \wedge \tau_n)\, d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) \\
&= \int_{A_2} m^{X,f}(t \wedge \tau_n) \int_{A_1} m^{x,g}(s \wedge \tau_n)\, d\mathbb{P}^{(x)}(y)\, d\mathbb{P}^X(x) = \int_A M^{Y,g}(s \wedge \tau_n) M^{X,f}(t \wedge \tau_n)\, d\mathbb{P} \ .
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\left[M^{Y,g}(t \wedge \tau_n) M^{X,f}(t \wedge \tau_n) | \mathcal{F}_{s+}^{(Y,X)}\right] = \mathbb{E}\left[M^{Y,g}(s \wedge \tau_n) M^{X,f}(t \wedge \tau_n) | \mathcal{F}_{s+}^{(Y,X)}\right] \\
&= M^{Y,g}(s \wedge \tau_n) \mathbb{E}\left[M^{X,f}(t \wedge \tau_n) | \mathcal{F}_{s+}^{(Y,X)}\right] = M^{Y,g}(s \wedge \tau_n) M^{X,f}(s \wedge \tau_n), \qquad \text{a.e.}
\end{aligned}$$

which completes the proof.

□

We now can prove the main theorem.

Theorem 2.1 *Assume that* $\mathcal{A}^Y g \in \mathcal{C}(E_Y)$ *and* $\mathcal{A}^X f \in \mathcal{C}(E_X)$ *for* $g \in D(\mathcal{A}^Y)$ *and* $f \in D(\mathcal{A}^X)$, *where* $\mathcal{C}(E_X)$ *and* $\mathcal{C}(E_Y)$ *are spaces of continuous functions on* E_X *and* E_Y *respectively. Then the process* $\{(Y(t), X(t)), t \geq 0\}$ *has the following extended generator:*

$$\mathcal{A} = \mathcal{A}^Y \oplus \mathcal{A}^X \tag{2.8}$$

and $D(\mathcal{A}^Y) \otimes D(\mathcal{A}^X) \subset D(\mathcal{A})$.

Remark 2.1 The equation (2.8) means that

$$\mathcal{A}(f(z)g(r)) = g(r) \mathcal{A}^X f(z) + f(z) \mathcal{A}^{Y,z} g(r) \ ,$$

where $\mathcal{A}^{Y,z}$ is the operator $\mathcal{A}^Y$, when we put z instead of $X(t)$.

Proof. Note that it suffices to prove that for $g \in D(\mathcal{A}^Y)$ and $f \in D(\mathcal{A}^X)$, process

$$M^{gf}(t) = g(Y(t)) f(X(t)) - \int_0^t (\mathcal{A} f g)(Y(s), X(s))\, ds \tag{2.9}$$

is a local $(\mathcal{F}_{t+}^{(Y,X)}, \mathbb{P})$-martingale. From integration-by-parts formula for semimartingales we have

$$\begin{aligned} & g(Y(t))f(X(t)) = \\ & \quad = \int_0^t f(X(s-))dg(Y(s)) + \int_0^t g(Y(s-))\, df(X(s)) - [f(X), g(Y)]_t \ . \end{aligned}$$

By assumptions of the theorem processes $\{\int_0^t \mathcal{A}^Y g(Y(s))\, ds, t \geq 0\}$ and $\{\int_0^t \mathcal{A}^X f(X(s))\, ds, t \geq 0\}$ are continuous processes of finite variation. Hence

$$H(t) = [f(X), g(Y)]_t = [M^{Y,f}, M^{Y,g}]_t \ .$$

By Jacod and Shiryaev (1987), Th. 4.50 a), p. 53, and Lemma 2.2 this process is the local martingale. Moreover,

$$\begin{aligned} & g(Y(t))f(X(t)) = \\ & \quad = \int_0^t f(X(s-))dg(Y(s)) + \int_0^t g(Y(s-))\, df(X(s)) + H(t) \\ & \quad = \int_0^t f(X(s-))dM^{Y,g}(s) + \int_0^t g(Y(s-))\, dM^{X,f}(s) \\ & \quad + \int_0^t f(X(s))(\mathcal{A}^Y g)(Y(s))\, ds + \int_0^t g(Y(s))(\mathcal{A}^X f)(X(s))\, ds + H(t) \\ & \quad = \int_0^t f(X(s-))dM^{Y,g}(s) + \int_0^t g(Y(s-))\, dM^{X,f}(s) + H(t) \\ & \quad + \int_0^t (\mathcal{A}fg)(Y(s), X(s))\, ds \ . \end{aligned} \tag{2.10}$$

Processes $\{\int_0^t f(X(s-))dM^{Y,g}(s), t \geq 0\}$ and $\{\int_0^t g(Y(s-))\, dM^{X,f}(s), t \geq 0\}$ similarly like $\{H(t), t \geq 0\}$ are local martingales. This completes the proof in view of (2.10).

□

Corollary 2.1 *Let $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ be processes with continuous realizations and*

$$D(\mathcal{A}^X) \subset \mathcal{C}(E_X), \qquad D(\mathcal{A}^Y) \subset \mathcal{C}(E_Y) \ . \tag{2.11}$$

Assume also that if $k_n \to k$, then $\mathcal{A}k_n \to \mathcal{A}k$ in the uniform convergence topology. Then under assumptions of Theorem 2.1 $\overline{D(\mathcal{A}^Y) \otimes D(\mathcal{A}^X)} \subset D(\mathcal{A})$.

Proof. The domain of the full generator is closed under linear combination. Let $k_n \in D(\mathcal{A}^Y) \otimes D(\mathcal{A}^X)$ for $n = 1, 2, \ldots$ and $k_n \to k$ in the uniform convergence topology. Then $\int_0^t \mathcal{A}k_n(y(s), x(s))\, ds \to \int_0^t \mathcal{A}k(y(s), x(s))\, ds$ for all $t \geq 0$ and $x \in \mathcal{C}_{E_X}[0, +\infty)$, $y \in \mathcal{C}_{E_Y}[0, +\infty)$. Moreover,

$$M^n(t) = k_n(Y(t), X(t)) - \int_0^t \mathcal{A}k_n(Y(s), X(s))\, ds$$

are local martingales with localizing sequence $\{\tau_m^n\}_{m=1,2,\ldots}$ each. Note that $M^n(t) \to M(t)$ as $n \to +\infty$ for all realizations of processess $\{Y(t), t \geq 0\}$ and $\{X(t), t \geq 0\}$, where

$$M(t) = k(Y(t), X(t)) - \int_0^t \mathcal{A}k(Y(s), X(s))\, ds \ .$$

Process $\{M(t), t \geq 0\}$ is the local martingale. In fact, define the following sequence of stopping times $\tau_m = \wedge_n \tau_m^n \wedge T_m$, where $T_0 = 0$ and $T_m = \inf\{t \geq T_{m-1} : |Y(t)| > m \wedge |X(t)| > m\}$. Then

$$\mathbb{E}|M(t \wedge \tau_m)| < +\infty \qquad \text{for all } m = 1, 2, \ldots$$

This completes the proof by dominated convergence theorem for conditional expectation.

□

3 Risk process

In this section we consider a canonical *risk reserve process* $\{R(t), t \geq 0\}$:

$$R(t) = u + pt - \sum_{i=1}^{P(t)} U_i \ ,$$

where the rate of income and the probability distribution of the cost of the claim at time t are the functions of $X(t)$. That is, if realizations of the process $\{X(t), t \geq 0\}$ is $x(t) \in D_{E_X}[0, +\infty)$, then for nonnegative continuous function $\lambda : E_X \to \mathbb{R}_+ \cup \{0\}$, we define a non-homogeneous Poisson process $\left\{P^{(x)}(t), t \geq 0\right\}$ with intensity function $\overline{\lambda}(t) = \lambda(x(t))$. The *claim sizes* $U_1, U_2, \ldots$ are i.i.d. and independent of process $\{N(t), t \geq 0\}$. We assume that common distribution function $F_U(x(t), x)$ depends on realization $x(t)$ at time t of the $\{X(t), t \geq 0\}$ and has continuous density $f_U(x(t), x)$ and mean $\nu(x(t))$. Assume that $R(t) \to +\infty$ a.e. In this section we find an upper exponential bound called *Cramér-Lundberg upper bound* for *infinite horizon ruin probability*

$$\psi(u) = \mathbb{P}(\inf_t R(t) < 0) \tag{3.12}$$

with initial reserve $R(0) = u$.

Let $Y(t) = (t, R(t))$ and $E_Y = \mathbb{R}^2$. Then from Davis (1993) we get following lemma.

Lemma 3.1 *The operator $\mathcal{A}^Y$ is given in the following way:*

$$(\mathcal{A}^Y g)(t, r) = \frac{\partial}{\partial t} g(t, r) + p \frac{\partial}{\partial r} g(t, r) + \lambda(X(t)) \int_0^{+\infty} (g(t, r - y) - g(t, r)) f_U(X(t), y)\, dy$$

with domain $D(\mathcal{A}^Y)$ consisting of functions $g(t, r) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ fulfilling the following conditions:

i) function $t \to g(t, pt)$ is absolutely continuous;

ii)

$$\mathbb{E}^{(x)} \left[\sum_{i=1}^{n} |g(\sigma_i, R(\sigma_i)) - g(\sigma_i -, R(\sigma_i -))| \right] < +\infty \tag{3.13}$$

for all $n \geq 1$ and realizations x of process $\{X(t), t \geq 0\}$, where $\{\sigma_i\}$ are moments of jumps of the process $\{R(t), t \geq 0\}$ and the expectation is with respect to $\mathbb{P}^{(x)}$.

Let $h(t,r,x) = f(x)e^{-\delta r}$, where $f \in D(\mathcal{A}^X)$, $\mathcal{A}^X f \in \mathcal{C}(E_X)$ and $f(x) > 0$. Similar derivations like in Rolski *et al* (1999), p. 459, show that if

$$\hat{F}_U(x(t),\delta) < +\infty \tag{3.14}$$

for all realizations $x \in D_{E_X}[0,+\infty)$ of $\{X(t), t \geq 0\}$ and $t \geq 0$, then

$$\mathbb{E}^{(x)}\left[\sum_{i=1}^{n} |\exp\{R(\sigma_i)\} - \exp\{R(\sigma_i -)\}|\right] < +\infty \tag{3.15}$$

for all $n \in N$. In that case $h \in D(\mathcal{A}^Y) \otimes D(\mathcal{A}^X) \subset D(\mathcal{A})$ by Theorem 2.1 and $\mathcal{A} = \mathcal{A}^Y \oplus \mathcal{A}^X$ is the extended generator. Hence by Ethier and Kurtz (1986) and Palmowski (1999) process

$$N(t) = \frac{h(t,R(t),X(t))}{h(0,R(0),X(0))}\exp\{-\int_0^t \frac{\mathcal{A}h(s,R(s),X(s))}{h(s,R(s),X(s))}\,ds\}$$

is the local martingale. Assume that

$$\mathcal{A}h = 0\ , \tag{3.16}$$

that is that function $f(x) > 0$ is a solution of

$$\mathcal{A}^X f(z) - (p\delta - \lambda(z)(\hat{F}_U(z,\delta) - 1))f(z) = 0\ . \tag{3.17}$$

Then process

$$\{N(t) = \frac{h(t,R(t),X(t))}{h(0,R(0),X(0))}, t \geq 0\} \tag{3.18}$$

is a mean-one positive local martingale. Thus from Dellacherie and Meyer (1982), p. 88, $\{N(t), t \geq 0\}$ is a supermartingale. Let $\mathbb{P}^{(0,u,w)}$ be a probability under which $X(0) = w$ and $\mathbb{E}^{(0,u,w)}$ expectation with respect to $\mathbb{P}^{(0,u,w)}$. Choose $\overline{u}$ such that $\overline{u} < +\infty$. Define

$$\tau(u) = \inf\{t \geq 0 : R(t) < 0\}\ .$$

Then

$$\psi(u) = \mathbb{P}(\tau(u) < +\infty)\ .$$

Consider $\tau(u) \wedge \overline{u}$, which is bounded stopping time. By Optional Sampling Theorem we get

$$\begin{aligned}\mathbb{E}^{(0,u,w)}N(0) &\geq \mathbb{E}^{(0,u,w)}[N(\tau(u)\wedge\overline{u})] \\ \geq \mathbb{E}^{(0,u,w)}[N(\tau(u)); \tau(u) \leq \overline{u}] &= \mathbb{E}^{(0,u,w)}[N(\tau(u)) | \tau(u) \leq \overline{u}]\mathbb{P}^{(0,u,w)}(\tau(u) \leq \overline{u})\ .\end{aligned}$$

Provided that

$$\mathbb{E}^{(0,u,w)}[N(\tau(u)) | \tau(u) < +\infty] > 0 \tag{3.19}$$

and letting $\overline{u} \to +\infty$ we get

$$\mathbb{P}^{(0,u,w)}(\tau(u) < +\infty) \leq \frac{\mathbb{E}^{(0,u,w)}N(0)}{\mathbb{E}^{(0,u,w)}[N(\tau(u)) | \tau(u) < +\infty]}\ . \tag{3.20}$$

Note that $N(0) = e^{-\delta u}f(X(0))$ and $R(\tau(u)) < 0$ on $\{\tau(u) < +\infty\}$. Hence assuming (3.19) and $\mathbb{E}N(0) < +\infty$ inequality (3.20) yields Cramér-Lundberg inequality:

$$\psi(u) \leq Ce^{-\delta u}\ ,$$

where

$$C = \int_{-\infty}^{+\infty} \frac{f(w)}{\mathbb{E}^{(0,u,w)}[f(X(\tau(u))) | \tau(u) < +\infty]}\,dF^0(w)$$

and $F^0(x)$ is a distribution function of random variable $X(0)$.

In many cases equation (3.17) is difficult to solve. However, we can still can give few examples, when we can do it.

4 Examples

Example 4.1 (Asmussen and Rolski (1994), Asmussen (1989), Grigolionis (1992b)) Let $\{X(t), t \geq 0\}$ be the stationary Markov process on a state space $\{1, 2, \dots, \ell\}$ with intensity matrix $\boldsymbol{Q} = \mathcal{A}^X$. If $X(t) = i$, then $\lambda(X(t)) = \lambda_i > 0$ and $F_U(X(t), x) = F_{U,i}(x)$. Denote by $\nu_i = \int_0^{+\infty} x \, dF_{U,i}(x)$. Let $\pi = (\pi_{\mathbf{1}}, \dots, \pi_\ell)$ be a stationary distribution of $\{X(t), t \geq 0\}$. Assume a net profit condition

$$p > \sum_{i=1}^{\ell} \pi_i \lambda_i \nu_i \ .$$

Let $\boldsymbol{K}(\delta)$ be diagonal matrix with entries $k_{ii}(\delta) = \lambda_i(\hat{F}_{U,i}(\delta) - 1)$ and

$$f(i) = f_i > 0 \qquad i = 1, \dots, \ell \ .$$

Then vector $\boldsymbol{f} = (f_1, \dots, f_n)$ and δ solve the following equation equivalent to the equation (3.17):

$$(\boldsymbol{Q} + \boldsymbol{K}(\delta) - p\delta\mathbf{I})\mathbf{f} = \mathbf{0} \ . \tag{4.21}$$

In this case assumptions (3.19) and $\mathbb{E}N(0) < +\infty$ are fulfilled. Moreover,

$$\int_{-\infty}^{+\infty} f(w) \, dF^0(w) = \sum_{i=1}^{\ell} \pi_i f_i < +\infty \ .$$

Hence

$$\psi(u) \leq e^{-\delta u} \max_i \frac{1}{f_i} \sum_{j=1}^{\ell} \pi_j f_j \ .$$

Example 4.2 (Bjőrk and Grandell (1998)) Let distribution function of the claim size does not depend on a state $X(t)$ of the governing process, that is $F_U(X(t), x) = F_U(x)$. Consider a stationary Ornstein-Uhlenbeck process $\{X(t), t \geq 0\}$ with generator

$$(\mathcal{A}f)(z) = \frac{1}{2} a \frac{\partial^2}{\partial z^2} f(z) - \alpha z \frac{\partial}{\partial z} f(z)$$

for some constants $a > 0$ and $\alpha \in \mathbb{R}$ and $D(\mathcal{A}) \subset \mathcal{C}^2(\mathbb{R})$. That is, $\{X(t), t \geq 0\}$ is defined by equation

$$dX(t) = \sqrt{a} dB(t) - \alpha X(t) \, dt \ ,$$

where $\{B(t), t \geq 0\}$ is a Brownian motion. In case of stationary Ornstein-Uhlenbeck process $X(0)$ has a normal distribution function $F^0(x)$ with mean zero and variance $\frac{a}{2\alpha}$. Assume that $\lambda(z) = z^2$ and $f(z) = \exp\{Kz^2\}$, for some

$$0 < K < \frac{\alpha}{a} \ . \tag{4.22}$$

Note that in that case $\mathbb{E}N(0) < +\infty$. Under assumption (4.22) we have

$$\int_{-\infty}^{+\infty} f(y) \, dF^0(y) = \frac{\sqrt{2}}{\sqrt{1 - K\frac{a}{\alpha}}} < +\infty \ . \tag{4.23}$$

Note also that $f(y) \geq 1$. Thus condition (3.19) is fulfilled and

$$C \leq \frac{\sqrt{2}}{\sqrt{1 - K\frac{a}{\alpha}}} \ . \tag{4.24}$$

Equation (3.17) is reduced to

$$\frac{1}{2}a(4z^2K^2+2K)-2\alpha z^2K-p\delta+z^2(\hat{F}_U(\delta)-1)=0\ ,$$

which implies that

$$K=\frac{\alpha}{2a}-\sqrt{\frac{\alpha^2}{4a^2}-\frac{\hat{F}_U(\delta)-1}{2a}}<\frac{\alpha}{a} \tag{4.25}$$

and δ is a solution of following equation

$$p\delta=\frac{\alpha-\sqrt{\alpha^2-2a(\hat{F}_U(\delta)-1)}}{2}\ . \tag{4.26}$$

Note that we must have

$$\delta\le\overline{\delta}\ , \tag{4.27}$$

where $\overline{\delta}$ is the solution of $\hat{F}_U(\overline{\delta})=\frac{\alpha^2}{2a}+1$. In this case assumption (3.14) is obviously fulfilled. For exponentially distributed claims with mean value μ we have that $\overline{\delta}=\frac{\alpha^2}{(\alpha^2+2a)\mu}$ and for

$$p<\frac{a}{2\alpha}(\mu+1)+\frac{\alpha}{2}\mu \tag{4.28}$$

there exists solution of (4.26)

$$\delta=\frac{\alpha\mu+p}{2p\mu}\left(1-\sqrt{1-\frac{2\mu(2p\alpha-a\mu)}{(\alpha\mu+p)^2}}\right)\ . \tag{4.29}$$

Note that if (4.28) holds, then assumption (4.27) and hence also (3.14) is fulfilled. Thus, if occurrence of claims are described by the Cox process with the intensity process being the function of stationary Ornstein-Uhlenbeck process and claims are exponentially distributed, then under assumption (4.28) by (4.24) we have the following Cramér-Lundberg inequality

$$\psi(u)\le\frac{\sqrt{2}}{\sqrt{1-K\frac{a}{\alpha}}}e^{-\delta u}\ ,$$

where K is given in (4.25) and δ in (4.29).

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