# Cyclic Queueing Networks with Subexponential Service Times

H. Ayhan[1,4] Z. Palmowski[2] S. Schlegel[3]

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**Abstract**

For a $K$-stage cyclic queueing network with $N$ customers and general service times we provide bounds on the $n^{\text{th}}$ departure time from each stage. Furthermore, we analyze the asymptotic tail behavior of cycle times and waiting times given that at least one service time distribution is subexponential.

KEYWORDS: CYCLE TIME, CLOSED NETWORK, SUBEXPONENTIAL ASYMPTOTICS, HARRIS MARKOV CHAIN

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## 1 Introduction

We consider a cyclic $K$-stage ($K \geq 2$) queueing system as shown in Figure 1. There is a single server at each station $i$ ($i = 1, \ldots, K$) and the service discipline at all stations is First Come First Served (FCFS). The capacity of the buffer between any two consecutive stations is infinite. There are $N$ customers in the system, who cyclically visit station 1 to station $K$. We assume that at time zero there are $N_i$ customers in front of station $i$, $i = 1, \ldots, K$. Hence, $\sum_{i=1}^{K} N_i = N$. Service times at station $i$ are independent and identically distributed random variables $\{B_n^i\}$ with distribution function $B_i(\cdot)$, and the sequences $\{B_n^1\}, \ldots, \{B_n^K\}$ are assumed to be mutually

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[1] School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0205, USA

[2] Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

[3] Department of Biometry and Medical Documentation, University of Ulm, 89069 Ulm, Germany

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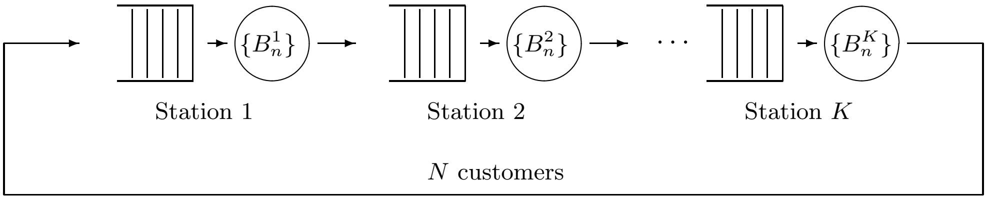

Figure 1: Cyclic queueing network with $K$ stations

independent. Furthermore, we assume that there exists a subexponential distribution $F(\cdot)$ $(F \in \mathcal{S})$ and there exist constants $c_i \in [0, \infty)$ with $\sum_{i=1}^{K} c_i > 0$ such that for all $i = 1, \ldots, K$

$$\lim_{x \to \infty} \frac{\overline{B}_i(x)}{\overline{F}(x)} = c_i \, .$$

The tail behavior of the cycle time (the time between successive departures of the same customer from a given station $i$, $i = 1, \ldots, K$) and tail behavior of waiting time at station $i$ $(i = 1, \ldots, K)$ seen by an arriving customer are main objects of our interest. Relatively few mathematical methods are available for treating these quantities in closed queueing networks. The reason for this is that a customer passing through the system experiences the whole space-time structure of the network state process. One can use general methods for closed Gordon-Newell networks [23], Laplace transform techniques [11, 14, 29], reversibility arguments [15], standard embedded Markov chain techniques [18], or the theory of point processes [19]. All of these results are valid under the assumption that service times are exponential random variables. Boxma [11] derives an expression for the Laplace-Stieltjes transform of the distribution of the stationary cycle time in a two-stage cyclic queueing network with one exponential and one general station. This result reveals a surprising phenomenon: in general, the distribution of the cycle time depends on the order in which two stations are visited. As Boxma [11] noted, this is because of the dependence between successive response times at two queues. As far as the open networks with subexponential service times are concerned, Baccelli, Schlegel and Schmidt [8] consider the tail behavior of stationary response times in irreducible stochastic event graphs. In a similar paper, Huang and Sigman [24] focus on the asymptotics of sojourn times and queue lengths in tandem queues and split-match queues. More recently Baccelli and Foss [5] compute upper and lower bounds for the tail asymptotics of the stationary maximal dater in more general monotone-separable stochastic networks. Moreover, they obtain exact asymptotics for various special cases of these networks. Baccelli, Foss and Lelarge [6] compute the exact tail asymptotics of stationary response times for both irreducible and reducible open stochastic event graphs under the assumptions that the arrival process is a renewal process and the service times have subexponential distributions.

In this paper we first provide upper and lower bounds on the $n^{\text{th}}$ departure epoch from any given station (for similar results in open networks see e.g. [8]). Then we obtain the subexponential asymptotics of the $n^{\text{th}}$ cycle time at station $i$ which is denoted by $C_n^i$. In particular, we have

$$\mathbb{P}(C_n^i > x) \sim N \sum_{\ell=1}^{K} c_\ell \overline{F}(x) \quad \text{as } x \to \infty \tag{1}$$

where $f(x) \sim g(x)$ as $x \to \infty$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$ for two functions $f(x)$ and $g(x)$. Note that the cycle time asymptotics is the same for each station. This is different for the cycle time distribution, which indeed depends on the station where the cycle starts; see [11]. Next we study the tail behavior of the distribution of the $n^{\text{th}}$ waiting time at station $i$ for $i = 1, \dots, K$. Using Harris recurrence methods we prove that if the service times at any of the stations have infinite support, then there exists a random variable $C^i$ such that

$$C_n^i \xrightarrow{\mathcal{D}} C^i \tag{2}$$

holds as $n \to \infty$ where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. The sequence $C_n^i$ couples with $C^i$ in finite time for all initial conditions $N_1, N_2, \dots, N_K$ such that $\sum_{i=1}^{K} N_i = N$. Using (1) and (2) we obtain the subexponential asymptotics for $C^i$:

$$\mathbb{P}(C^i > x) \sim N \sum_{\ell=1}^{K} c_\ell \overline{F}(x) \quad \text{as } x \to \infty. \tag{3}$$

Results similar to the one in (3) are also obtained for stationary waiting times. To the best of our knowledge, our results are the first of this kind for cyclic queues with more than two stations and non-exponential service times.

The paper is organized as follows. In Section 2, we introduce the notation and derive upper and lower bounds on the $n^{\text{th}}$ departure time from station $i$. We obtain the subexponential asymptotics for the $n^{\text{th}}$ cycle time $C_n^i$ in Section 3. Section 4 is devoted to the asymptotics of the $n^{\text{th}}$ waiting time at station $i$. In Section 5, we prove (3) and also obtain the results on tail behavior of stationary waiting times. Finally, in the appendix we recall some properties of subexponential distributions used in this paper.

# 2 Preliminaries

In this section, we introduce the notation used throughout the paper and develop upper and lower bounds on the $n^{\text{th}}$ departure time from station $i \in \{1, \dots, K\}$ denoted by $X_n^i$.

Location of the customers in the network at time 0 is called the initial marking. We will see that the stationary regime does not depend on this initial marking. Without loss of generality we assume that no customers are being served at time 0.

We use the following notation

$$[j] = \begin{cases} K & \text{if } j \bmod K = 0, \\ j \bmod K & \text{if } j \bmod K \neq 0, \end{cases}$$

where $K$ is the number of stations as defined above. Furthermore, we use the symbol $\oplus$ for maximization, and the symbol $\otimes$ for addition. Thus, we write $a \oplus b$ for $\max\{a, b\}$, and $a \otimes b$ (or shortly $ab$) for $a + b$ where $a$ and $b$ are real numbers. This is the standard (max,+) algebra notation (see [4] for more details on this formalism). Note that as in conventional algebra $\otimes$ has precedence over $\oplus$. Even though cyclic tandem queue is an example of a (max,+) linear system, general (max,+) linear systems and (max,+) algebra are beyond the scope of this paper. We use these symbols simply for notational convenience. Throughout our developments we set $X_n^i = 0$ and $B_n^i = 0$ for all $n \leq 0$ and $i \in \{1, \dots, K\}$. One can easily see that in any cyclic queueing network for all $i \in \{1, \dots, K\}$

$$X_n^i = B_n^i X_{n-1}^i \oplus B_n^i X_{n-N_i}^{[i-1]}. \tag{4}$$

The following proposition provides an upper bound on $X_n^i$ for $i \in \{1, \dots, K\}$.

**Proposition 2.1** *For all $i \in \{1, \dots, K\}$,*

$$X_n^i \leq \bigotimes_{u=0}^{K-1} \bigotimes_{r=1}^{n-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]} = U_n^i$$

*with the convention that $\otimes$ over an empty set is 0.*

*Proof* First assume that $n > N$ and $n \bmod N = 0$ and apply the following service mechanism. First the server at station $[i+1]$ serves all the $N_{[i+1]}$ customers while the other servers remain idle. Then the server at station $[i+2]$ serves $N_{[i+1]} + N_{[i+2]}$ customers while the other servers remain idle. Servers at successive stations work in this sequential manner until the $n^{\text{th}}$ customer departs from station $i$. Under this service mechanism, the departure time of the $n^{\text{th}}$ customer from station $i$ is equal to $U_n^i$ which is clearly greater than $X_n^i$ since the servers work sequentially. Next assume that $n > N$ and $n \bmod N \neq 0$. In this case, add $n \bmod N$ customers to $N_{[i+1]}$ and apply the sequential service mechanism described above but eliminate the $n \bmod N$ additional customers after their first service at station $i$. The departure time of the $n^{\text{th}}$ customer from station $i$ is again equal to $U_n^i$. Finally, suppose that $n < N$. In this case, apply the sequential service mechanism starting from the station where the $n^{\text{th}}$ departing customer (from station $i$) is at time 0 but at that station only serve this customer ($n^{\text{th}}$ departing customer from station $i$) and the ones in front of him. □

The next proposition provides a lower bound on $X_n^i$ for $i \in \{1, \ldots, K\}$.

**Proposition 2.2** *For all $i \in \{1, \ldots, K\}$,*

$$X_n^i \geq \bigoplus_{u=0}^{K-1} \bigotimes_{r=1}^{n-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]} = L_n^i$$

*with the convention that $\oplus$ over an empty set is $-\infty$ and $\otimes$ over an empty set is 0.*

*Proof* The proof follows from the observation that until $X_n^i$ at least $(n - \sum_{q=0}^{u-1} N_{[i-q]}) \oplus 0$ customers must have departed from station $[i-u]$ for all $u = 0, \ldots, K-1$. □

Propositions 2.1 and 2.2 can also be obtained from the explicit representation of $X_n^i$ given in [2].

# 3 Cycle Times

Recall that $C_n^i$ denotes the $n^{\text{th}}$ cycle time at station $i \in \{1, \ldots, K\}$. By a cycle time we mean the time between two successive departures of the same customer from a given station. Thus, the $n^{\text{th}}$ cycle time at station $i$ is computed as

$$C_n^i = X_{N+n}^i - X_n^i. \tag{5}$$

The next proposition provides the tail asymptotics of the $n^{\text{th}}$ cycle time.

**Proposition 3.1** *For $i \in \{1, \ldots, K\}$ and $n \geq N - N_{[i+1]}$, we have*

$$\mathbb{P}(C_n^i > x) \sim N \sum_{k=1}^{K} c_k \overline{F}(x) \quad \text{as } x \to \infty. \tag{6}$$

*Proof* We first derive an upper bound for $C_n^i$. Define the set of service times $\mathcal{J}$ as

$$\mathcal{J} = \bigcup_{u=0}^{K-1} \bigcup_{r=n+1-\sum_{q=0}^{u-1} N_{[i-q]}}^{n+N-\sum_{q=0}^{u-1} N_{[i-q]}} \{B_r^{[i-u]}\}.$$

Note that at least one of the service times in $\mathcal{J}$ must be in progress at any time in the interval $[X_n^i, X_{n+N}^i]$ and there is no other service time (other than those in $\mathcal{J}$) that could take place in this time interval. Then from (5)

$$C_n^i \leq \bigotimes_{u=0}^{K-1} \bigotimes_{r=n+1-\sum_{q=0}^{u-1} N_{[i-q]}}^{n+N-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]}. \tag{7}$$

Thus, from Corollary 6.1, for all $n \geq N - N_{[i+1]}$

$$\limsup_{x\to\infty} \frac{\mathbb{P}(C_n^i > x)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\mathbb{P}(\bigotimes_{u=0}^{K-1} \bigotimes_{r=1}^{N} B_r^{[i-u]} > x)}{\overline{F}(x)} = N \sum_{k=1}^{K} c_k. \tag{8}$$

Next we obtain a lower bound on $C_n^i$. Note that all service times in $\mathcal{J}$ take place within the time interval $[X_{n-N}^i, X_{n+N}^i]$. Let

$$T_n^i = X_n^i - X_{n-N}^i$$

and observe that the service times that occur in the interval from $X_{n-N}^i$ to $X_n^i$ do not have an effect on $C_n^i$. Hence,

$$\begin{aligned} C_n^i &\geq \bigoplus_{u=0}^{K-1} \bigoplus_{r=n+1-\sum_{q=0}^{u-1} N_{[i-q]}}^{n+N-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]} - T_n^i \\ &\geq \bigoplus_{u=0}^{K-1} \bigoplus_{r=n+1-\sum_{q=0}^{u-1} N_{[i-q]}}^{n+N-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]} - \bigotimes_{u=0}^{K-1} \bigotimes_{r=n-N+1-\sum_{q=0}^{u-1} N_{[i-q]}}^{n-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]} \end{aligned}$$

where the second inequality follows since from the definition of $T_n^i$ and (7), we have

$$T_n^i \leq \bigotimes_{u=0}^{K-1} \bigotimes_{r=n-N+1-\sum_{q=0}^{u-1} N_{[i-q]}}^{n-\sum_{q=0}^{u-1} N_{[i-q]}} B_r^{[i-u]}.$$

Note that the upper bound on $T_n^i$ is independent of the service times in set $\mathcal{J}$. Therefore, from Lemmas 6.3, 6.4 and Corollary 6.1, for $n \geq N - N_{[i+1]}$,

$$\begin{aligned} \liminf_{x\to\infty} \frac{\mathbb{P}(C_n^i > x)}{\overline{F}(x)} &\geq \liminf_{x\to\infty} \frac{\mathbb{P}(\bigoplus_{u=0}^{K-1} \bigoplus_{r=N+1}^{2N} B_r^{[i-u]} - \bigotimes_{u=0}^{K-1} \bigotimes_{r=1}^{N} B_r^{[i-u]} > x)}{\overline{F}(x)} \\ &= N \sum_{k=1}^{K} c_k \end{aligned} \tag{9}$$

which together with (8) completes the proof for the asymptotics of the $n^{\text{th}}$ cycle time at station $i$.

$\square$

Since the convergence in (8) and (9) is uniform in $n$, one can conclude the uniformity of convergence in $n$ in Proposition 3.1.

**Remark 3.1** The cycle time asymptotics is the same for each station. Thus, it does not matter at which station a cycle starts. This is different for the cycle time distribution, for which indeed one gets different results depending on where the cycle starts; see [11].

**Remark 3.2** Let $c_{i_0} > 0$ and $c_i = 0$ for all $i \neq i_0$. Then

$$\mathbb{P}(C_n^i > x) \sim N c_{i_0} \overline{F}(x) \sim N \overline{B}_{i_0}(x) \quad \text{as } x \to \infty.$$

Hence, the asymptotics depends only on the service time distribution with the heaviest tail.

## 4 Waiting Times

Let $W_n^i$ denote the $n^{\text{th}}$ waiting time at station $i \in \{1, \ldots, K\}$. Thus, $W_n^i$ is the time from the arrival of the $n^{\text{th}}$ customer at station $i$ to the beginning of his service at this station and it is computed as

$$W_n^i = \left( X_{n-1}^i - X_{n-N_i}^{[i-1]} \right) \oplus 0 \; . \tag{10}$$

The next proposition provides the tail asymptotics of the $n^{\text{th}}$ waiting time.

**Proposition 4.1** *For $i \in \{1, \ldots, K\}$ with $B_i \in \mathcal{S}$ and $n \geq N$, we have*

$$\mathbb{P}(W_n^i > x) \sim (N-1)\overline{B_i}(x) \sim (N-1) c_i \overline{F}(x) \quad \text{as } x \to \infty. \tag{11}$$

*Proof* We again start by obtaining an upper bound on $W_n^i$. Define the set of service times $\mathcal{K}$ as

$$\mathcal{K} = \bigcup_{r=n-N+1}^{n-1} \{B_r^i\}.$$

Note that at least one of the service times in $\mathcal{K}$ must be in progress on station $i$ at anytime in the time interval $[X_{n-N_i}^{[i-1]}, X_{n-1}^i]$ if $X_{n-1}^i \geq X_{n-N_i}^{[i-1]}$ (otherwise $W_n^i = 0$) and there is no other service time (other than those in $\mathcal{K}$) that could take place on station $i$ in this time interval. Then from (10)

$$W_n^i \leq \bigotimes_{r=n-N+1}^{n-1} B_r^i.$$

Thus, from Corollary 6.1, for $n \geq N$

$$\limsup_{x\to\infty} \frac{\mathbb{P}(W_n^i > x)}{\overline{B_i}(x)} \leq \limsup_{x\to\infty} \frac{\mathbb{P}(\bigotimes_{r=1}^{N-1} B_r^i > x)}{\overline{B_i}(x)} = (N-1)\ . \tag{12}$$

Next we obtain a lower bound on $W_n^i$. Note that all service times in $\mathcal{K}$ take place on station $i$ in the time interval $[X_{n-N}^i, X_{n-1}^i]$. Moreover, completed service times that take place in the interval from $X_{n-N}^i$ to $X_{n-N_i}^{[i-1]}$ do not have an effect on $W_n^i$ (in order to see this note that $(n-N)^{\text{th}}$ customer departing from station $i$ is the same as the $(n-N_i)^{\text{th}}$ customer departing from station $[i-1]$). It follows from (4) that $X_{n-N}^i \geq X_{n-N-N_i}^{[i-1]}$. As in the proof of Proposition 3.1, let

$$\begin{aligned} T_{n-N_i}^{[i-1]} &= X_{n-N_i}^{[i-1]} - X_{n-N-N_i}^{[i-1]} \\ &\leq \bigotimes_{u=0}^{K-1} \bigotimes_{r=n-N_i-N+1-\sum_{q=0}^{u-1} N_{[i-1-q]}}^{n-N_i-\sum_{q=0}^{u-1} N_{[i-1-q]}} B_r^{[i-1-u]} \\ &= \bigotimes_{u=0}^{K-2} \bigotimes_{r=n-N_i-N+1-\sum_{q=0}^{u-1} N_{[i-1-q]}}^{n-N_i-\sum_{q=0}^{u-1} N_{[i-1-q]}} B_r^{[i-1-u]} \otimes \bigotimes_{r=n-2N+1}^{n-N} B_r^i \end{aligned}$$

where the upper bound on $T_{n-N_i}^{[i-1]}$ follows from the proof of Proposition 3.1. Note that the upper bound on $T_{n-N_i}^{[i-1]}$ is independent of the service times in $\mathcal{K}$. Then

$$\begin{aligned} W_n^i &\geq \bigoplus_{r=n-N+1}^{n-1} B_r^i - T_{n-N_i}^{[i-1]} \\ &\geq \bigoplus_{r=n-N+1}^{n-1} B_r^i - \Big( \bigotimes_{u=0}^{K-2} \bigotimes_{r=n-N_i-N+1-\sum_{q=0}^{u-1} N_{[i-1-q]}}^{n-N_i-\sum_{q=0}^{u-1} N_{[i-1-q]}} B_r^{[i-1-u]} \otimes \bigotimes_{r=n-2N+1}^{n-N} B_r^i \Big) \end{aligned}$$

and employing Lemma 6.3 and Corollary 6.1, for $n \geq N$, we have

$$\begin{aligned} \liminf_{x\to\infty} \frac{\mathbb{P}(W_n^i > x)}{\overline{B_i}(x)} &\geq \liminf_{x\to\infty} \frac{\mathbb{P}(\bigoplus_{r=N+1}^{2N-1} B_r^i - \Big(\bigotimes_{u=0}^{K-2} \bigotimes_{r=1}^{N} B_r^{[i-1-u]} \otimes \bigotimes_{r=1}^{N} B_r^i\Big) > x)}{\overline{B_i}(x)} \\ &= (N-1). \end{aligned} \tag{13}$$

Since $\overline{B_i}(x) \sim c_i \overline{F}(x)$ as $x \to \infty$, the assertion follows. □

Since the convergence in (12) and (13) is uniform in $n$, one can conclude the uniformity of convergence in $n$ in Proposition 4.1.

## 5 Stationary Results

In this section, we assume that the service time distribution at any one of the stations, say station $i_0$, has infinite support. Clearly, this assumption holds for subexponential distributions. We observe the state of the cyclic network at the time of the departures. Note that we consider the departures from all the stations in the network and number the departure times according to the order that they happen. Let

$$\mathbf{Y}(k) = (Y_1(k), \ldots, Y_K(k), Y_{K+1}(k), \ldots, Y_{2K}(k))$$

be the system state vector, where $Y_\ell(k)$ and $Y_{K+\ell}(k)$ ($\ell = 1, \ldots, K$) represent, respectively, the number of customers and the elapsed service time at station $\ell$ at the time of the $k^{\text{th}}$ departure. Hence, if $Y_\ell(k) = 0$, then $Y_{K+\ell}(k) = 0$. Moreover, let $\mathbf{Y}(0) = (N_1, \ldots, N_K, 0, \ldots, 0)$ which is consistent with the initial condition defined in Sections 1 and 2. Note that $\{\mathbf{Y}(k), k = 0, 1, 2 \ldots\}$ is a Markov chain. Let $R = \{\mathbf{Y}^0\}$ be the one point set such that $\mathbf{Y}^0$ is a $\{0, 1, \ldots, N\}^K \times \mathbb{R}^K$ vector fulfilling

$$Y^0_{i_0+1} = 1, \quad Y^0_{i_0} = N - 1, \quad Y^0_\ell = 0 \quad (\ell \neq i_0, i_0 + 1), \quad Y^0_{K+\ell} = 0 \quad (\ell = 1, \ldots, K)\,.$$

The event $R$ means that at a departure epoch from station $i_0$ the departing customer leaves all the other $N - 1$ customers behind at station $i_0$. Note that for any $m > 0$ there exists $k > m$ such that $Y(k) = Y^0$ if

$$X^{i_0}_{n+1} > X^{[i_0-1]}_{n+N-N_{i_0}}$$

for some $n$ such that $(n - N)K > m$. Hence, the set $R$ is a regeneration set such that, for any initial marking,

$$\mathbb{P}(\mathbf{Y}(k) \in R) \geq \mathbb{P}(X^{i_0}_{n+1} > X^{[i_0-1]}_{n+N-N_{i_0}}) \geq \mathbb{P}(B^{i_0}_{n+1} > X^{[i_0-1]}_{n+N-N_{i_0}} - X^{[i_0-1]}_{n+1-N_{i_0}})\,,$$

where the last step follows from (4). Similar to (7), we have

$$X^{[i_0-1]}_{n+N-N_{i_0}} - X^{[i_0-1]}_{n+1-N_{i_0}} \leq \bigotimes_{r=n+2-N_{i_0}}^{n+N-N_{i_0}} B^{[i_0-1]}_r \otimes \bigotimes_{u=1}^{K-1} \bigotimes_{r=n+2-N_{i_0}-\sum_{q=0}^{u-1} N_{[i_0-1-q]}}^{n+N-N_{i_0}-\sum_{q=0}^{u-1} N_{[i_0-1-q]}} B^{[i_0-1-u]}_r\,,$$

and therefore

$$\begin{aligned} &\mathbb{P}(X^{i_0}_{n+1} > X^{[i_0-1]}_{n+N-N_{i_0}}) \\ &\geq \mathbb{P}\left( B^{i_0}_{n+1} > \bigotimes_{r=n+2-N_{i_0}}^{n+N-N_{i_0}} B^{[i_0-1]}_r \otimes \bigotimes_{u=1}^{K-1} \bigotimes_{r=n+2-N_{i_0}-\sum_{q=0}^{u-1} N_{[i_0-1-q]}}^{n+N-N_{i_0}-\sum_{q=0}^{u-1} N_{[i_0-1-q]}} B^{[i_0-1-u]}_r \right) > 0 \end{aligned} \tag{14}$$

since the service time distribution at station $i_0$ has infinite support. Moreover, the sequence of events $\{B^{i_0}_{n+1} > \bigotimes_{r=n+2-N_{i_0}}^{n+N-N_{i_0}} B^{[i_0-1]}_r \otimes \bigotimes_{u=1}^{K-1} \bigotimes_{r=n+2-N_{i_0}-\sum_{q=0}^{u-1} N_{[i_0-1-q]}}^{n+N-N_{i_0}-\sum_{q=0}^{u-1} N_{[i_0-1-q]}} B^{[i_0-1-u]}_r\}$ constitute a stationary sequence (in $n$) for all $n \geq N-1$ and they are $(N-1)$-dependent (see also [22] for more on Harris ergodic Markov chains). Note also that in this case the inter-regeneration times have finite mean. Thus, $\{\mathbf{Y}(k), k=0,1,2\ldots\}$ is a Harris ergodic Markov chain and has a unique stationary distribution. Namely, from (14) and Theorem 3.6 and Proposition 3.13 of [1] (see also [3, 4, 9]) we have the following theorem.

**Theorem 5.1** *If service time distribution at any one of the stations has infinite support, then there exists a random vector* $\mathbf{Y}$ *such that the distribution of* $\{\mathbf{Y}(k), k=0,1,2\ldots\}$ *converges in total variation to the distribution of* $\mathbf{Y}$*. The Markov chain* $\{\mathbf{Y}(k), k=0,1,2\ldots\}$ *couples with* $\mathbf{Y}$ *in finite time for any initial marking* $N_1, N_2, \ldots, N_K$*.*

**Remark 5.1** Theorem 5.1 is a special case of Theorem 5 in [21], where sufficient conditions are given for the stability (i.e. the existence of a limiting distribution) for closed Jackson-type queueing networks having non-lattice service time distributions. The reader is encouraged to refer to [21] for a more general proof. See also [7, 10, 30].

Note that the $n^{\text{th}}$ cycle time at station $i$, $C_n^i = X_{N+n}^i - X_n^i$, can be expressed in terms of $\mathbf{Y}(M_n)$, where $M_n$ is the time the of $n^{\text{th}}$ departure from station $i$, and the completed services taking place in the time interval $[X_n^i, X_{n+N}^i)$. At the $n^{\text{th}}$ departure epoch from station $i$, the number of completed services since time 0 at the individual stations depends on the initial marking. Each station $l$ has completed at least $\max\{n-N_i, 0\} \geq \max\{n-N, 0\}$ and at most $n+N-N_i-1 \leq n+N-1$ services. We define the following sequence of random vectors for $n \geq N$

$$\mathbf{B}^\ell(n) = (B_{n-N+1}^\ell, B_{n-N+2}^\ell, \ldots, B_{n+2N-1}^\ell), \quad \ell = 1, 2, \ldots, K.$$

Since $\oplus$ and $\otimes$ are continuous operators, we have

$$C_n^i = f_1(\mathbf{Y}(M_n), \mathbf{B}^1(n), \ldots, \mathbf{B}^K(n))$$

for $n \geq N$ and some continuous function $f_1 : \{0, 1, \ldots, N\}^K \times \mathbb{R}^{3NK} \to \mathbb{R}_+$. Note that $(n-N)K \leq M_n \leq (n+N)K$. Thus, since convergence in total variation implies weak convergence, from the continuous mapping theorem and Theorem 5.1 we obtain the following result.

**Theorem 5.2** *If service time distribution at any one of the stations has infinite support, then there exists a random variable* $C^i = f_1(\mathbf{Y}, \mathbf{B}^1, \ldots, \mathbf{B}^K)$ *such that* $C_n^i$ *converges in distribution to* $C^i$ *as* $n \to \infty$*. The sequence* $C_n^i$ *couples with* $C^i$ *in finite time for any initial marking* $N_1, N_2, \ldots, N_K$*.*

Since the convergence in Proposition 3.1 is uniform in $n$, for all $n \geq N - N_{[i+1]}$ and any $\varepsilon > 0$ there exists $\mathcal{L}$ (which does not depend on $n$) such that for all $x > \mathcal{L}$ we have

$$(1-\varepsilon)\, N \sum_{k=1}^{K} c_k\, \overline{F}(x) \ \leq \mathbb{P}(C_n^i > x) \leq (1+\varepsilon)\, N \sum_{k=1}^{K} c_k\, \overline{F}(x)\ ,$$

for $i \in \{1, \ldots, K\}$. Taking limits as $n \to \infty$ and $\varepsilon \downarrow 0$ we obtain the following corollary.

**Corollary 5.1** *If service time distribution at any one of the stations has infinite support, then for all $i \in \{1, \ldots, K\}$ we have that*

$$\mathbb{P}(C^i > x) \sim N \sum_{k=1}^{K} c_k\, \overline{F}(x) \quad \textit{as } x \to \infty.$$

Note also that

$$W_n^i = f_2(\mathbf{Y}(M_n), \mathbf{B}^1(n), \ldots, \mathbf{B}^K(n))$$

for some continuous function $f_2 : \{0, 1, \ldots, N\}^K \times \mathbb{R}^{3NK} \to \mathbb{R}_+$. Thus, from Theorem 5.1 and Proposition 4.1 we can obtain the tail asymptotics of the distribution of the stationary waiting time at station $i$ (denoted by $W^i$).

**Corollary 5.2** *If service time distribution at any one of the stations has infinite support, then for any $i \in \{1, \ldots K\}$ with $B_i \in \mathcal{S}$, we have*

$$\mathbb{P}(W^i > x) \sim (N-1)\overline{B_i}(x) \sim (N-1)c_i\, \overline{F}(x) \quad \textit{as } x \to \infty.$$

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## 6 Appendix

In this appendix we recall some properties of the class $\mathcal{S}$ of subexponential distributions introduced in [16].

### 6.1 Subexponential distributions

As before, we write $f(x) \sim cg(x)$ to express that $\lim_{x\to\infty} f(x)/g(x) = c$ for two functions $f(x), g(x)$, and some constant $c \geq 0$.

**Definition 6.1** *A distribution function $F$ on $\mathbb{R}_+ = [0, \infty)$ with $F(x) < 1$ for all $x > 0$ is called subexponential ($F \in \mathcal{S}$) if*

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x)\,,$$

*where $\overline{F}(x) = 1 - F(x)$ and $F^{*2}$ denotes the convolution $F * F$.*

The class $\mathcal{S}$ has some very useful properties. Those which are used in this paper are the following ones.

**Lemma 6.1** *Let $F$ and $G$ be two distribution functions on $\mathbb{R}_+$ and assume that there exists a constant $c \in (0, \infty)$ with $\overline{G}(x) \sim c\overline{F}(x)$. Then, $F \in \mathcal{S}$ if and only if $G \in \mathcal{S}$.*

**Lemma 6.2** *Let $F, G$, and $H$ be distribution functions on $\mathbb{R}_+$ such that $F \in \mathcal{S}$, $\overline{G}(x) \sim c_1\overline{F}(x)$ and $\overline{H}(x) \sim c_2\overline{F}(x)$, where $c_i \in [0, \infty)$ for $i = 1, 2$ and $c_1 + c_2 > 0$. Then,*

$$\overline{G * H}(x) \sim (c_1 + c_2)\overline{F}(x)\ .$$

The results stated in Lemmas 6.1 and 6.2 are well-known. For a proof see for example [17] and [20].

**Lemma 6.3** *Let $X$ and $Y \geq 0$ be independent random variables with distribution functions $F_X \in \mathcal{S}$ and $F_Y$, respectively. Then*

$$\mathbb{P}(X - Y > x) \sim \mathbb{P}(X > x) \quad \text{as } x \to \infty.$$

For the proof of Lemma 6.3, see [27]. We further need the following lemma which is standard.

**Lemma 6.4** *Let $F$ and $G_1, \ldots, G_n$, $n \geq 1$, be distribution functions on $\mathbb{R}_+$ such that $\overline{G}_i(x) \sim c_i\overline{F}(x)$ as $x \to \infty$; $c_i \geq 0$. Then,*

$$1 - \prod_{i=1}^{n} G_i(x) \sim \sum_{i=1}^{n} c_i\,\overline{F}(x)\ .$$

The following corollary follows from Lemma 6.2 by induction.

**Corollary 6.1** *Let* $F \in \mathcal{S}$ *and let* $F_1, \ldots, F_n$, $n \geq 1$, *and* $G_1, \ldots, G_m$, $m \geq 1$, *be distribution functions on* $\mathbb{R}_+$ *such that* $\overline{F}_i(x) \sim c_i \overline{F}(x)$ *with* $c_i > 0$, $1 \leq i \leq n$, *and* $\overline{G}_i(x) = o(\overline{F}(x))$ *for* $1 \leq i \leq m$. *Then,*

$$\overline{F_1 * \ldots * F_n * G_1 * \ldots * G_m}(x) \sim \sum_{i=1}^{n} c_i \, \overline{F}(x) \, .$$

# References

[1] Asmussen, S. (1987) *Applied Probability and Queues*, John Wiley and Sons, New York.

[2] Ayhan, H., Palmowski, Z. and Schlegel, S. (2002) "Subexponential asymptotics of cycle time in closed networks, "EURANDOM Report no 024.

[3] Baccelli, F. and Brémaud, P. (1992) *Elements of Queueing Theory*, Springer-Verlag, Berlin.

[4] Baccelli, F., Cohen, G., Olsder, G.J. and Quadrat, J.-P. (1992) *Synchronization and Linearity: An Algebra for Discrete Event Systems*, Springer Verlag and Sons, Chichester.

[5] Baccelli, F., and Foss, S. (2001) "Moments and tails in monotone-separable stochastic networks," *Rapport de Recherche de l'INRIA-Rocquencourt 4197.*

[6] Baccelli, F., Foss, S. and Lelarge, M. (2003) "Asymptotics of Subexponential Max Plus Networks: the Stochastic Event Graph Case", *Rapport de Recherche de l'INRIA-Rocquencourt 4952*, to appear *Queueing Systems.*

[7] Baccelli, F., Foss, S. and Mairesse, J. (1996) "Stationary ergodic Jackson networks: results and counter-examples," In *Stochastic Networks: Theory and Applications* (edited by F.P. Kelly, S. Zachary and I. Ziedins), Oxford University Press, 281–307.

[8] Baccelli F., Schlegel S. and Schmidt V. (1999) "Asymptotics of stochastic networks with subexponential service times," *Queueing Systems* **33**, 205–232.

[9] Borovkov, A. (1984) *Asymptotic Methods in Queueing Theory*, John Wiley and Sons, New York.

[10] Borovkov, A. (1986) "Limit theorems for service networks I," *J. Theory Probab. Appl.*, **31(3)**, 413–427.

[11] Boxma, O.J. (1983) “The cyclic queue with one general and one exponential server,” *Adv. Appl. Prob.*, **15**, 857–873.

[12] Boxma, O.J. (1984) “Joint distribution of sojourn time and queue length in the M/G/1 queue with (in)finite capacity,” *European Journal of Operations Research* **16**, 246–256.

[13] Boxma, O.J. (1988) “Sojourn times in cyclic queues - the influence of the slowest server,” *Computer Performance and Reliability*, 13–24, G. Iazeolla, P. Courtois and O. Boxma (eds.), North-Holland Publishing Co., Amsterdam.

[14] Boxma, O.J. and Donk, P. (1982)“ On response time and cycle time distribution in a two-stage cyclic queue,” *Performance Evaluation*, **2**, 181–194.

[15] Boxma, O.J., Kelly, F.P. and Konheim, A.G. (1984) “The product form for sojourn time distributions in cyclic exponential queues,” *J.Assoc.Comput.Mach.*, **31**, 128–133.

[16] Chistyakov, V.P. (1964) “A theorem on sums of independent, positive random variables and its applications to branching processes,” *Theory Prob. Appl.*, **9**, 640–648.

[17] Cline, D.B.H. (1986) “Convolution tails, product tails and domains of attraction,” *Prob. Theory Rel. Fields*, **72**, 529–557.

[18] Chow, W.M. (1980) “The cycle time distribution of exponential cyclic queues,” *J.Assoc.Comput.Mach.*, **27**, 281–286.

[19] Daduna, H. and Szekli, R. (1999) “Monotonicity and dependence properties of sojourn and cycle times in closed networks,” Manuscript.

[20] Embrechts, P. and Goldie, C.M. (1982) “On convolution tails,” *Stoch. Proc. Appl.*, **13**, 263–278.

[21] Foss, S. (1991) “Ergodicity of queueing networks,” *Siberian Mathematical Journal*, **32(4)**, 184–203.

[22] Foss, S. and Kalashnikov, V. (1991) “Regeneration and renovation in queues,” *Queueing Systems* **8(3)**, 211–224.

[23] Gordon, W.J. and Newell, G.F. (1967) “Closed queueing networks with exponential servers,” *Operations Research* **15**, 252–267.

[24] Huang, T. and Sigman, K. (1999) “Steady state asymptotics for tandem, split-match and other feedforward queues with heavy tailed service,” *Queueing Systems* **33**, 233–259.

[25] Klüppelberg, C. (1988) “Subexponential distributions and integrated tails,” *J. Appl. Probab.*, **25**, 132 - 141.

[26] Morozov, E. (1990) “Regeneration of a closed queueing network,” *Journal of Mathematical Sciences* **69(4)**, 1186–1192.

[27] Pitman, E.J.G. (1980) “Subexponential distribution functions,” *J. Austral. Math. Soc.*, (Series A), **29**, 337–347.

[28] Resing, J.A.C., R.E. de Vries, G. Hooghiemstra, M.S. Keane and Olsder, G.J. (1990) “Asymptotic behaviour of random discrete event systems,” *Stoch. Proc. Appl.*, **36**, 195–216.

[29] Schassberger, R. and Daduna, H. (1983) “The time for a round trip in a cycle of exponential queues,” *J.Assoc.Comput.Mach.*, **30**, 146–150.

[30] Sigman, K. (1989) “Notes on the stability of closed networks,” *J. Appl. Probab.*, **26(3)**, 678–682.