Oszacowania prawdopodobieństwa przepelnienia bufora
w kolejkowych modelach przepływu

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Chapter 1

Introduction

In this dissertation we say that a random variable $X^*$ admits an exponential two-sided bounds if for some $C_*, C^* > 0$ and $\eta > 0$

$$C_*e^{-\eta x} \leq P(X > x) \leq C^*e^{-\eta x}, \quad x \geq 0,$$  \hspace{1cm} (1.0.1)

when of course in (1.0.1) we look for the best possible constants $C_*$, $C^*$. In the theory of queues Kingman [64] and Ross [109] derived upper exponential bounds for the steady-state waiting time in the GI/GI/1 queue. Independently in context of risk theory two-sided exponential bounds were derived by Taylor [120]. Kingman [64] used martingale inequalities to prove this result. A prototype of method used here is in Asmussen [6], where an exponential change of measure was used to get exponential bounds for the ruin probability in the classical risk model.

In fluid models a counterpart of the steady-state waiting time is the steady-state buffer content $X^*$, which in many cases has the representation

$$X^* \overset{d}{=} \sup_{t \geq 0} \int_0^t \left( \sum_{k=1}^N f_{Z_k(s)} - c \right) ds,$$  \hspace{1cm} (1.0.2)

where $\{Z^k(t), \ t \geq 0\}$ $(k = 1, \ldots, N)$ are a finite state, stationary and ergodic processes and $c$ is a constant. Although in this dissertation we consider infinite capacity buffer, the probability $P(X^* > x)$ may serve as an approximation for the finite buffer overflow probability. In this dissertation we study two-sided exponential bounds for $X^*$.

Our motivation to consider exponential bounds in the fluid models takes its origins in papers of Anick et al [1] and Kosten [72], in which the classical AMS fluid model and a dominant eigenvalue technique was considered. In the AMS fluid model, environment processes $\{Z^k(t), \ t \geq 0\}$ $(k = 1, \ldots, N)$ are 0–1 Markov processes. The main aim of this work is to generalize their results to more general fluid models. To deal with more general environment processes we develop the exponential change of measure technique and prove a Girsanov type transformation theorem, which seems to be of own interest. We also apply this results to find an exponential upper bound for the ruin probability by a risk processes with a Coxian arrival process governed by a diffusion.
In Section 1.1 we present fluid models considered in this dissertation. Ruin theory for a risk process with Coxian arrival process governed by a diffusion is developed in Section 1.2. The organization of this dissertation is given in Section 1.3.

1.1 Fluid model

A stochastic fluid model is a stochastic system, wherein the input is described as a continuous flow of fluid that enters into a single buffer according to a randomly varying rate and leave with a constant rate. Such the models are motivated as approximations to discrete manufacturing systems, high speed data networks, etc (see e.g. Elwalid and Mitra [44]).

In this dissertation we consider fluid models in which the fluid enters the buffer from $N$ independent sources. The traffic generated from $k$th ($k = 1, \ldots, N$) source is driven by a stationary, ergodic environment process $\{Z^k(t), t \in \mathbb{R}\}$. That is, when the environment process is in state $m$, the fluid is generated into the buffer at a rate $r^k_m$. We assume that $\{Z^k(t), t \geq 0\}$ are càdlàg processes. The fluid is removed then from the buffer by a channel with constant capacity $c$; see Figure 1.

**Figure 1.**

\[ \text{Figure 1.} \]

Thus $\{\sum_{k=1}^{N} r^k_{Z^k(t)}, t \geq 0\}$ is an input process. The amount of the fluid at time
\(t\), denoted by \(X(t)\), is governed by the equation:

\[
\frac{dX(t)}{dt} = \begin{cases} 
\sum_{k=1}^{N} r_{kZ^k(t)} - c, & \text{for } X(t) > 0 \\
\left(\sum_{k=1}^{N} r_{kZ^k(t)} - c\right)^+, & \text{for } X(t) = 0
\end{cases},
\]

where \((x)^+ = \max\{x, 0\}\). We assume the following stability condition:

\[
d = \mathbb{E}\sum_{k=1}^{N} r_{Z^k(t)} - c < 0,
\]

that is that drift \(d\) is negative. Then the steady-state buffer content defined by

\[
\mathbb{P}(X^* > x) = \lim_{t\to+\infty} \mathbb{P}(X(t) > x)
\]

is a finite a.s. random variable.

Following Borovkov [19], Theorem 1, page 44 (see Theorem 2.1 from Kulkarni and Rolski [75] concerning fluid theory), r.v. \(X^*\) has the following representation:

\[
X^* = \sup_{t \geq 0} \int_{0}^{t} \left(\sum_{k=1}^{N} r_{Z^k(s)} - c\right) ds.
\]

Consider cádlág reversed version \(\{Z^k(t), t \geq 0\}\) of the environment process \(\{Z^*k(t), t \geq 0\}\). Then

\[
X^* = \sup_{t \geq 0} \int_{0}^{t} \left(\sum_{k=1}^{N} r_{Z^k(s)} - c\right) ds.
\]

In this dissertation we will study the function \(\Psi(x)\) defined by

\[
\Psi(x) = \mathbb{P}(X^* > x) = \mathbb{P}(\sup_{t \geq 0} \int_{0}^{t} \left(\sum_{k=1}^{N} r_{Z^k(s)} - c\right) ds > x),
\]

which is called the buffer overflow probability. We consider only the case, when \(X^*\) admits exponential bounds, which is not true in general. For instance, if \textit{on} time in the single \textit{on-off} fluid model has a subexponential distribution function, then exponential bounds for \(\Psi(x)\) are impossible; see e.g. Rolski \textit{et al} [107], Boxma [21]. However, in this dissertation we will not consider this case.

The function \(\Psi(x)\) is known only for few fluid models, in particular for the AMS fluid models. However in practice, even for these models, computations are frequently cumbersome. Therefore most of results concern different asymptotics of \(\Psi(x)\). For exponential type fluid models the following types of results can be studied:

- \textit{The logarithmic asymptotics:}

  \[
  \lim_{x \to +\infty} \frac{1}{x} \log \Psi(x) = -\eta \quad \text{for some } 0 < \eta < +\infty;
  \]

  see Botvich and Duffield [20], Duffield [40], Palmowski and Rolski [101] and Duffield and O’Connel [39] for results for the AMS and \textit{on-off} fluid model.
In a few models one can obtain the exact asymptotics of function \( \Psi(x) \) for \( x \to +\infty \)
\[
\Psi(x) = C e^{-\eta x} + o(e^{-\eta x}) ,
\]
where prefactor \( C \) is known. However, for general fluid models such results seems to be difficult to prove, in particular computing constant \( C \); see Anick \textit{et al} [1], Mitra [90] and [91], McDonald and Qian [88].

For majority of fluid models we can only find the exponential upper bound for \( \Psi(x) \)
\[
\Psi(x) \leq C^* e^{-\eta x} ,
\]
where constant \( C^* \) and adjustment coefficient \( \eta \) are given explicitly, or two-sided exponential bounds
\[
C_1 e^{-\eta x} \leq \Psi(x) \leq C^* e^{-\eta x} .
\] (1.1.3)

In next subsections we briefly present main results concerning fluid models, which we will deal with in this dissertation.

### 1.1.1 AMS fluid model

The following model is already classical in the theory of fluid models and plays a similar role as M/M/1 system in queueing theory. Anick \textit{et al} [1] consider \( N \) sources modulated by \( 0 - 1 \) alternating Markov processes \( \{Z^k(t), t \geq 0\} \). If \( Z^k(t) \) is in state \textit{on} \((Z^k(t) = 1)\), then it generates fluid at rate \( r^k_1 = r \). If \( Z^k(t) \) is in state \textit{off} \((Z^k(t) = 0)\), then it generates no fluid into the buffer, that is \( r^k_0 = 0 \). The amount of time that process \( \{Z^k(t), t \geq 0\} \) spends in the \textit{on} state has exponential distribution function \( F^k_{\text{on}}(\cdot) = F^k_{\text{off}}(\cdot) = \text{Exp}(\alpha) \) with mean \( \tau^k_{\text{on}} = \alpha^{-1} \) and \textit{off}-time has exponential distribution function \( F^k_{\text{off}}(\cdot) = \text{Exp}(\beta) \) with mean \( \tau^k_{\text{off}} = \beta^{-1} \). Let
\[
p = \frac{\tau^k_{\text{on}}}{\tau^k_{\text{on}} + \tau^k_{\text{off}}} = \frac{\beta}{\alpha + \beta}
\]
be the probability that a source is in state \textit{on}. Assume stability condition (1.1.1)
\[
d = Npr - e < 0
\] (1.1.4)
and a natural condition
\[
Nr > c ,
\]
otherwise the buffer is always empty in the steady-state. Then by (1.1.2) the buffer overflow probability has the following representation:
\[
\Psi(x) = \mathbb{P}(\sup_{t \geq 0} \int_0^t (rY(s) - c) \, ds > x) ,
\] (1.1.5)
where $Y(t)$ is a number of active souces at time $t$. Note that $\{Y(t), t \geq 0\}$ is a continuous time Markov chain (abbreviated by CTMC) with infinitesimal matrix

$$
\mathcal{A} = \begin{pmatrix}
-N\beta & N\beta & 0 & 0 & 0 \\
\alpha & -\alpha - (N-1)\beta & (N-1)\beta & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(N-1)\alpha & -(N-1)\alpha - \beta & \beta & N\alpha & -N\alpha
\end{pmatrix}.
$$

Thus the stationary number of on sources is

$$
p_j = \binom{N}{j} p^j (1-p)^{N-j} \quad (j = 0, \ldots, N).
$$

For the AMS fluid model we can get all types of results concerning function $\Psi(x)$. For instance, Anick et al [1], Mitra [90] proved that

$$
\Psi(x) = \sum_{i=1}^{m-} a_i e^{z_i x}, \quad x \geq 0, \quad (1.1.7)
$$

for certain constants $a_i$ and

$$
\Re(z_{m-}) < \Re(z_{m-1}) \leq \ldots \leq \Re(z_1) = z_1 < 0,
$$

which can be found by solving a given system of equations. From (1.1.7) we have also

$$
\Psi(x) = a_1 e^{z_1 x} + o(1) \quad \text{for } x \rightarrow +\infty.
$$

The system of equations for $a_i$ and for $z_i$ is very cumbersome to solve. Therefore there are considered miscellaneous asymptotics. Let $\eta = -z_1$. Then by (1.1.7) we have logarithmic asymptotics

$$
\lim_{x \rightarrow +\infty} \frac{1}{x} \log \Psi(x) = -\eta. \quad (1.1.8)
$$

We can also look for constants $C_*$ and $C^*$ fulfilling

$$
C_* e^{-\eta x} \leq \Psi(x) \leq C^* e^{-\eta x}. \quad (1.1.9)
$$

This is how we can derive (1.1.8) and (1.1.9) using the exponential change of measure.

Consider canonical process $\{Y(t), t \geq 0\}$, that is $Y(\omega, t) = \omega(t)$, on stochastic basis $(D[0, +\infty), \mathcal{F}, \{\mathcal{F}_t^Y\}_{t \geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t^Y\}$ denotes the right-continuous natural filtration. Probability measure $\mathbb{P}$ is chosen in such a way, that matrix $\mathcal{A}$ given in (1.1.6) is an infinitesimal generator of Markov process $\{Y(t), t \geq 0\}$. Let $\mathbf{h}$ be a positive vector. By Ethier and Kurtz [48], the process

$$
N(t) = \frac{h_Y(t)}{h_i} e^{-\int_0^t \frac{(\mathcal{A}\mathbf{h})_{Y(s)}}{h_{Y(s)}} ds} \quad (1.1.10)
$$

is a martingale with respect to $\mathbb{P}^i$, where $\mathbb{P}^i$ denote the underlying measure $\mathbb{P}$ such that $Y(0) = i$. Let $\mathbb{P}_{Y}^i = \mathbb{P}_{Y_t}^{i+}$. From Theorem 3.1.4 we have that there exists
unique probability measure $\bar{P}$ such that
\[
\frac{d\bar{P}^i_t}{d\bar{P}^i_t} = N(t) ,
\] (1.1.11)
where $\bar{P}^i_t = \bar{P}^i_{\tau^i_t}$. Define
\[
\tau(x) = \inf\{t \geq 0 : \int_0^t (rY(s) - c) \, ds > x \} .
\]
By Jacod and Shiryaev [58], Theorem 3.4(ii), page 153 (see Lemma 3.1.8) and (1.1.5) we have that
\[
\Psi(x) = \bar{P}(\tau(x) < + \infty) = \sum_{i=1}^{N} p_i \bar{P}^i(\tau(x) < + \infty)
\]
\[
= \sum_{i=1}^{N} p_i \bar{E}^i (N(\tau(x))^{-1}; \tau(x) < + \infty) ,
\]
(1.1.12)
where $\bar{E}^i$ is the expectation with respect to $\bar{P}^i$. Let
\[
\theta = \frac{\alpha}{\beta N r - c} ,
\]
which is greater than 1 by stability condition (1.1.4). Choosing
\[
h_i = \theta^i
\]
we have
\[
\mathcal{A}h = -\eta \Delta h ,
\]
(1.1.13)
where $\Delta = \text{diag}\{r_i - c\}$ and
\[
\eta = \frac{N (\alpha + \beta) c - \beta N r}{c (N r - c)} .
\]
(1.1.14)
Hence by (1.1.10)
\[
N(t) = \frac{h Y(t)}{h_i} e^{-\eta \int_0^t (rY(s) - c) \, ds}
\]
under the $\bar{P}^i$. By (1.1.12) we have then
\[
\Psi(x) = e^{-\eta x} \sum_{i=1}^{N} p_i h_i \bar{E}^i [h_Y^{-1}; \tau(x)]; \tau(x) < + \infty] .
\]
(1.1.15)
We now use Girsanov type transformation Theorem 3.2.2, called in this dissertation the perturbation theorem. Consider perturbation of the generator $\mathcal{A}$ of the Markov process $\{Y(t), t \geq 0\}$, defined in the following way:
\[
\tilde{\mathcal{A}}g = \mathcal{A}g + \frac{<h, g>_{\mathcal{A}}}{h} ,
\]
where \( < h, g >_{\mathcal{A}} = \mathcal{A}(hg) - g\mathcal{A}h - h\mathcal{A}g \). Then on the new probability stochastic basis \((D[0, +\infty), \mathcal{F}, \{\mathcal{F}_t^Y\}, \mathbb{P})\) defined by (1.11), process \(\{Y(t), t \geq 0\}\) is a Markov process with generator \(\mathcal{A}\). From this we get that on the new probability basis process, \(\{Y(t), t \geq 0\}\) is a CTMC with new parameters

\[
\tilde{\beta} = \theta \beta, \quad \tilde{\alpha} = \theta^{-1} \alpha .
\]

Then the drift \(\tilde{d}\) is positive:

\[
d = \tilde{\mathbb{E}}(rY(t) - c) = r \frac{\tilde{\beta}}{\tilde{\alpha} + \tilde{\beta}} - c > 0 .
\]

Thus

\[
\mathbb{P}^i(\tau(x) < +\infty) = 1
\]

and by (1.15) we get

\[
\Psi(x) = e^{-\eta x} \sum_{i=1}^{N} p_i h_i \mathbb{E}^i[\tau(\tau(x))] .
\]

Clearly, at time \(\tau(x)\) process \(\{Y(t), t \geq 0\}\) can only be in state \(i\) such that \(ri > c\). Hence a lower bound for \(\mathbb{E}^i[\tau(\tau(x))]\) is \(1/\max_{i \geq \varphi} h_i\) and an upper bound is \(1/\min_{i \geq \varphi} h_i\). This yields the two-sided exponential bounds

\[
C_s e^{-\eta x} \leq \Psi(x) \leq C_e e^{-\eta x} ,
\]

where constant \(\eta\) is given in (1.14) (see Theorem 6.2.2).

### 1.1.2 On-off fluid model

Similar considerations as in the AMS fluid model can be done for more general fluid models. The high-speed networks are expected to handle a wide variety of traffic, that is a cell may carry one of different types of information: voice, video, data, etc. We assume that each source produces a single class of traffic but different sources may produce traffic belonging to distinct classes. In other words, environment processes \(\{Z^k(t), t \in \mathbb{R}\} (k = 1, \ldots, N)\) may have different law. Such a case we call a heterogeneous one. Thus, to generalize the AMS fluid model we can for instance consider heterogeneous \(0 - 1\) Markov environment processes. We can also make further generalization and consider general alternating renewal \(0 - 1\) environment processes \(\{Z^k(t), t \geq 0\} (k = 1, \ldots, N)\), where \(N\) is the number of all sources. Such a model we call on-off fluid model and was considered in Palmowski and Rolski [101].

In the on-off fluid model the on and off times of process \(\{Z^k(t), t \geq 0\}\) are i.i.d. random variables having generic distribution functions \(F_{\text{on}}^k(\cdot)\) and \(F_{\text{off}}^k(\cdot)\) with means \(\tau_{\text{on}}^k\) and \(\tau_{\text{off}}^k\) respectively. If \(Z^k(t)\) is in state on \((Z^k(t) = 1)\), then fluid is transmitted from \(k\)th source at rate \(r^k\). If \(Z^k(t)\) is in the state off \((Z^k(t) = 0)\), then there is
no fluid generated from the $k$th source ($r^k_0 = 0$). To get the stationary and ergodic process \( \{ Z^k(t), t \geq 0 \} \) we must assume that \( (\tau^k_{on} + \tau^k_{off}) < +\infty \). Let

\[
p^k = \frac{\tau^k_{on}}{\tau^k_{on} + \tau^k_{off}}.
\]

Recall that our main aim is to derive exponential bounds for the buffer overflow probability $\Psi(x)$ given in (1.1.2).

Note that process \( \{ Z^k(t), t \geq 0 \} \) is not Markovian, however with the age component $S^k(t)$ a process \( \{ w^k(t) = (Z^k(t), S^k(t)), t \geq 0 \} \) is a piecewise deterministic Markov process. For this class of processes Davis [31] computed extended generator and its domain. That is, it can be found operator $A^k$ such that process

\[
M^k(t) = g(w^k(t)) - \int_0^t (A^k g)(w^k(s)) \, ds, \quad t \geq 0
\]

is a local martingale. The set of functions $g(x) = (g(0,x), g(1,x))$ for which it holds is called the domain of extended generator and is denoted by $D(A^k)$. For positive function $h^k(\cdot, \cdot) \in D(A^k)$ we define process

\[
N^k(t) = \frac{h^k(w^k(t))}{h^k(w(0))} e^{-\int_0^t \frac{(A^k h^k)(w^k(s))}{h^k(w^k(s))} \, ds},
\]

From Lemma 3.2.1 process \( \{ N^k(t), t \geq 0 \} \) is the local martingale. If $\inf_x h^k(i,x) > 0$ and $h^k(i, \cdot) \in C^1_b(\mathbb{R}_+)$ ($i = 0, 1$), then \( \{ N^k(t), t \geq 0 \} \) is in fact a true martingale. By independence of sources process

\[
N(t) = \prod_{k=1}^N N^k(t), \quad t \geq 0 \tag{1.1.16}
\]

is also the martingale. Choose now functions $h^k(\cdot, \cdot)$ ($k = 1, \ldots, N$) in such a way that

\[
(A^k h^k)(i, x) = -\eta (r^k_i - c^k) h(i, x) \tag{1.1.17}
\]

for some constants $\eta > 0$ and $c^k > 0$ fulfilling

\[
\sum_{k=1}^N c^k = c, \tag{1.1.18}
\]

where $c$ is output rate. This is a counterpart of (1.1.13). Then we have

\[
N(t) = \prod_{k=1}^N \frac{h^k(w^k(t))}{h^k(w(0))} e^{-\eta \int_0^t \sum_{k=1}^N c^k h^k(x) - c \, ds}. \tag{1.1.19}
\]

Now to get exponential bounds for $\Psi(x)$ we can proceed like in the AMS fluid model.
- We define new probability measure $\bar{\mathbb{P}}$ by
\[
\frac{d\bar{\mathbb{P}}_{t}^{(i, \bm{x})}}{d\mathbb{P}_{t}^{(i, \bm{x})}} = N(t) ,
\]
where $i = (i^1, \ldots, i^N)$ ($i^k \in \{0, 1\}$), $\bm{x} = (x^1, \ldots, x^N) \in \mathbb{R}^N$ and $\mathbb{P}^{(i, \bm{x})}$ is an underlying probability measure $\mathbb{P}$ for which $\bm{w}^k(0) = (i^k, x^k)$.

- We use Jacod and Shiryaev [58], Theorem 3.4(ii), page 153 (Lemma 3.1.8) and (1.1.19) to get following representation
\[
\mathbb{P}(\tau(x) < +\infty) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \int_{\mathbb{R}^N} \pi^k_j(y^k)dy^k \bar{\mathbb{E}}^\mathbb{P}^{(j, \bm{y})} [N(\tau(x))^{-1}; \tau(x) < +\infty] ,
\]
where $j = (j^1, \ldots, j^N)$, $\bm{y} = (y^1, \ldots, y^N)$ and
\[
\pi^k_1(y^1)dy^1 = p^k \frac{\hat{F}_{on}^k(x^k)}{\tau_{on}^k} dy, \quad \pi^k_0(y^1)dy^1 = p^k \frac{\hat{F}_{off}^k(x^k)}{\tau_{off}^k} dy ,
\]
give a stationary distribution of process \(\{\bm{w}^k(t), t \geq 0\}\).

- We apply our perturbation Theorem 3.2.2 to identify the parameters of processes \(\{Z^k(t), t \geq 0\}\) after the exponential change of measure and prove that the drift changes from negative to positive.

- Finally, we estimate
\[
\bar{\mathbb{E}}[h^1(\bm{w}^1(\tau(x))) \cdot \ldots \cdot h^N(\bm{w}^N(\tau(x)))]^{-1}
\]
to get lower and upper exponential bounds (see Theorem 6.1.1).

The constant $\eta$ in the exponent is a solution of systems of equations given in (1.1.17) and (1.1.18). This system is equivalent to the following:
\[
\begin{aligned}
\hat{F}_{on}^k(\eta(p^k - c^k))\hat{F}_{off}^k(-\eta c^k) = 1 \\
\sum_{k=1}^{N} c^k = c,
\end{aligned}
\]
where $c^k \geq 0$ and $\hat{F}(\delta)$ denotes the moment generating function of distribution function $F(x)$. This system of equations we call the Basic System of Nonlinear Equations (abbreviated by BSNE). The conditions for the existence of a unique solution of the BSNE are discussed in Section 5.5. It appears that the class of phase-type distributions for on-time and off-time is sufficient to get the solution of the BSNE and get the exponential bounds for $\Psi(x)$ (see Corollary 5.5.4). It may happen that the BSNE has no solution. This is the case for example, when $\hat{F}_{on}^k(\delta) = +\infty$ for some $k = 1, \ldots, N$ and all $\delta > 0$. Then we get nonexponential
asymptotics for the buffer overflow probability $\Psi(x)$; see Rolski et al. [107], Boxma [21]. This case is not considered in this dissertation. The second scenario is when

$$\delta^* = \min_k \sup \{ \delta \geq 0 : \hat{F}^k_{on}(\delta) < +\infty \}$$

and BSNE has also no solution. However, then we can still get an exponential upper bound

$$\Psi(x) \leq C^* e^{-\eta^* x},$$

where constant $C^*$ is given explicitly and $\eta^*$ is the biggest $\delta$ fulfilling

$$\hat{F}^k_{on}(\delta(x^k - c^k)) \hat{F}_{off}(-\delta c^k) \leq 1$$

$$\sum_{k=1}^N c^k = c ;$$

(1.1.21)

see Theorem 5.6.3. We give also an example (Theorem 5.6.2) of a single on-off fluid model such that

$$\Psi(x) = x^{-\sigma} e^{-\eta^* x}$$

for $\sigma > 1$. Thus in general, lower exponential bounds for $\Psi(x)$ are impossible in this case.

### 1.1.3 SMP fluid model

The necessity of considering semi-Markov environment processes $\{Z^k(t), t \geq 0\}$ ($k = 1, \ldots, N$) (abbreviated by SMP) follows from analysing of tandem fluid models (Sections 6.3 and 6.4). Firstly, we find a condition (called non-anticipativity) under which reversed version $\{\bar{Z}^k(t), t \geq 0\}$ of an SMP $\{Z^{*k}(t), t \geq 0\}$ is also an SMP. Under this condition we can generalize all results concerning on-off fluid model, in particular exponential bounds for the buffer overflow probability $\Psi(x)$ (see Theorem 5.3.1). In Section 4.6 we convert constants $C_*, C^*$ and $\eta$ in the bounds (1.1.3) of $\Psi(x)$ in terms of original processes $\{Z^{*k}(t), t \geq 0\}$. By enlarging the state space of general SMP $\{Z^{*k}(t), t \geq 0\}$ we construct SMP fulfilling assumption of non-anticipativity. Thus we can also get exponential bounds for the buffer overflow probability, when the environment processes are general semi-Markov processes (see Theorem 4.6.7).

### 1.2 Bjöck-Grandell model

In the following model, to get exponential bounds, we have to combine existing separately theories for PDMP and diffusion processes. Consider risk process $\{R(t), t \geq 0\}$ with a Coxian arrival process with a intensity function governed by a diffusion in such a way that the arrival intensity is a function of the diffusion process. The main trouble in finding exponential upper bound for the ruin probability in this model is to determine extended generator of $\{R(t), t \geq 0\}$ and its domain.

Define canonical Markov diffusion process $\{X(t), t \geq 0\}$ having the following extended generator

$$(A^d f)(z) = \frac{1}{2} a(z) \frac{\partial^2}{\partial z^2} f(z) + b(z) \frac{\partial}{\partial z} f(z),$$

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where $\inf_z a(z) > 0$ and measurable functions $a, b \in C(\mathbb{R})$ fulfill following conditions

$$a(z) \leq L(1 + z^2), \quad |b(z)| \leq L(1 + |z|)$$

for some constant $L$. Then the family of functions $C^2(\mathbb{R})$ are included in the domain $D(A^2)$.

Provided that realization of process $\{X(t), t \geq 0\}$ is $x(t) \in C(\mathbb{R}_+)$, then for a given nonnegative continuous function $\lambda : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ we define a non-homogeneous Poisson process $\{P(x)(t), t \geq 0\}$ with intensity function $\lambda(t) = \lambda(x(t))$. Then the risk process $\{R(t), t \geq 0\}$ is equal to

$$R(t) = R(x)(t) = u + pt - \sum_{i=1}^{P(x)(t)} U_i . \quad (1.2.1)$$

That is, $u$ is a initial reserve, $p$ is a constant premium rate, arrival epochs of claims are given by process $\{P(x)(t), t \geq 0\}$ and claims sizes $U_1, U_2, \ldots$ are i.i.d. random variables independent of $\{P(x)(t), t \geq 0\}$ with common density function $f_U(x)$. Moreover, we must assume that $\lim_{t \to +\infty} R(t) = +\infty$ a.s.

Denote by $D(\mathcal{A}^{(X)})$ a collection of all functions $g(t, r) \in C^1(\mathbb{R}^2)$ fulfilling

$$\mathbb{E} \left[ \sum_{i=1}^{n} g(\sigma_i, R(\sigma_i)) - g(\sigma_i -, R(\sigma_i -)) \right] < +\infty \quad (1.2.2)$$

for all $n \geq 1$, where $\{\sigma_i\}$ are moments of jumps of the process $\{R(t), t \geq 0\}$. For families of functions $A$ and $B$ we define new new family $A \otimes B = \{f(x)g(y) : f \in A \quad \text{and} \quad g \in B\}$. In Chapter 7 we prove the following theorem.

**Theorem 1.2.1** The Markov process $\{(t, R(t), X(t)), t \geq 0\}$ has the extended generator:

$$(\mathcal{A}k)(t, w, z) =$$

$$= \frac{1}{2} a(z) \frac{\partial^2}{\partial z^2} k(t, w, z) + b(z) \frac{\partial}{\partial z} k(t, w, z)$$

$$+ \frac{\partial}{\partial t} k(t, w, z) + p \frac{\partial}{\partial w} k(t, w, z) + \lambda(z) \int_0^{+\infty} (k(t, w - y, z) - k(t, w, z)) f_U(y) \, dy$$

and $D(\mathcal{A}^{(X)}) \otimes C^2(\mathbb{R}) \subset D(\mathcal{A})$, where $D(\mathcal{A})$ is a domain of the generator $\mathcal{A}$.

This theorem is used to get upper exponential bound for the ruin probability

$$\mathbb{P}(\inf_{t \geq 0} R(t) < 0) .$$

This upper bound is directly calculated for Ornstein-Uhlenbeck process as the governing diffusion process, $\lambda(x) = x^2$ and exponential distributed claim sizes.
1.3 Organization of the dissertation

This dissertation is organized as follows.

Chapter 2 contains preliminaries presenting unified theory of martingales and stochastic integrals (Section 2.2), Markov processes and extended generators (Section 2.3). These concepts are necessary for the understanding of the main body of this dissertation, in particular the proof of the perturbation theorem. In the preliminaries there are also introduced some notations needed in further chapters.

The foundations for the exponential change of measure are considered in Chapter 3. Thus in Section 3.2 we will prove a perturbation theorem giving the form of the extended generator after the exponential change of measure. This theorem we then apply to piecewise deterministic Markov processes (Section 3.3) and also to diffusion processes (Section 3.4) to obtain the classical Cameron-Martin-Girsanov Theorem.

In Chapter 4 we give exponential bounds for the buffer overflow probability in the single source fluid models driven by semi-Markov process. Chapter 5 generalizes this result to N source case.

In Chapter 6 we apply results of Chapter 5 to more specific fluid models: on-off fluid model (Sections 6.1.1 and 6.1.2), Markovian and the AMS fluid models (Sections 6.2 and 6.2.2), and finally to tandem fluid models (Sections 6.3 and 6.4).

In Chapter 7 we calculate the extended generator and its domain of the risk process with the Coxian arrival process with intensity driven by a diffusion process. We use this result to get a Crámer-Lundberg type inequality for the ruin probability in this risk model.

Each chapter is divided into sections, with consecutive labelling within each of equations and statements (like Definitions, Lemmas, Propositions, Theorems, Corollaries, Remarks) for each section. We will also use the statement to which we will refer only by the label without the parenthesis in contrast to labelling of equations. In such statements we give mainly known facts or some comments. The symbol \( \Box \) denotes the end of a proof.
Chapter 2

Preliminaries

2.1 General

Here are some standard notations and general definition to be used in this dissertation.

2.1.1 The ordered pair \((\Omega, \mathcal{F})\) consisting of a set \(\Omega\) and a \(\sigma\)-field \(\mathcal{F}\) of subsets of \(\Omega\) is called a \textit{measurable space}. Let \((\Omega, \mathcal{F})\) and \((E, \mathcal{G})\) be two measurable spaces. A mapping \(f\) of \(\Omega\) into \(E\) is called \(\mathcal{F}|\mathcal{G}\)-measurable if

\[ f^{-1}(A) \in \mathcal{F} \quad \text{for all } A \in \mathcal{G}. \]

2.1.2 Space \(E\) is called \textit{Borel} if it is a Borel subset of some Polish space (complete, metrizable and separable space). See Bertsekas and Shreve [11], Definition 7.7, page 108, for further properties of Borel spaces.

2.1.3 \textit{Stochastic basis} is a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a \textit{filtration} \(\{\mathcal{F}_t\}_{t \geq 0}\), where \(\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t\) is the smallest \(\sigma\)-field generated by all subset from \(\mathcal{F}_t\) \((t \geq 0)\). Here, \textit{filtration} means an increasing family of \(\sigma\)-fields. By \textit{natural filtration} of process \(\{X(t), t \geq 0\}\) we mean \(\mathcal{F}_t^X = \sigma(\{X(s), s \leq t\})\).

2.1.4 We define filtration \(\{\mathcal{F}_t\_\}\) by \(\mathcal{F}_t\_ = \bigcap_{s \geq t} \mathcal{F}_s\). The family \(\{\mathcal{F}_t\}\) is said to be \textit{right-continuous} if \(\mathcal{F}_t = \mathcal{F}_t\_\) for all \(t \geq 0\). For example the most often used filtration in this dissertation is \(\{\mathcal{F}_t^X\}\), which is right-continuous.

2.1.5 The stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) is said to satisfy \textit{usual conditions} if \(\sigma\)-field \(\mathcal{F}\) is complete, filtration \(\{\mathcal{F}_t\}\) is right-continuous and if every \(\mathcal{F}_t\) contains all \(\mathbb{P}\)-null sets of \(\mathcal{F}\). It is always possible to "complete" a given stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) as follows. Let \(\mathcal{N}^\mathbb{P}\) denotes the set of all \(\mathbb{P}\)-null sets and \(\mathcal{F}_t^\mathbb{P}\) is the smallest \(\sigma\)-field that contains \(\mathcal{F}_t\) and \(\mathcal{N}^\mathbb{P}\). Let \(\mathcal{F}^\mathbb{P} = \bigvee_{t \geq 0} \mathcal{F}_t^\mathbb{P}\). Stochastic basis \((\Omega, \mathcal{F}^\mathbb{P}, \{\mathcal{F}_t^\mathbb{P}\}, \mathbb{P})\) is called the \textit{augmentation} of \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) and satisfy usual conditions.
2.1.6 Let $E$ be Borel space and $\Omega \subset E^{[0, +\infty)}$ be the space of elementary events. By $\mathcal{G}_t$ we denote the $\sigma$-field generated by the sets \{ $\omega \in \Omega : (\omega(t_1), \omega(t_2), \ldots, \omega(t_n)) \in B$ \} for all $t_1, \ldots, t_n \leq t$ and $B \in \mathcal{B}(E^n)$, where $\mathcal{B}(E^n)$ is the $\sigma$-field of $E^n$ Borel sets. It is called the $\sigma$-field of cylindrical sets of interval $[0, t]$. By $\mathcal{G} = \bigvee_{t \geq 0} \mathcal{G}_t$ we denote the $\sigma$-field of cylindrical sets of $\Omega$.

2.1.7 By $D_E[0, +\infty)$ we denote the space of $E$-valued right-continuous functions on $[0, +\infty)$ which admit finite left-hand limits. We call them cadlag functions. If $E = \mathbb{R}$, then we will drop the subscript $E$.

2.1.8 By $C_E[0, +\infty)$ we denote the space of $E$-valued continuous functions on $[0, +\infty)$. If $E = \mathbb{R}$, then we will drop the subscript $E$.

2.1.9 If $E$ is an open subset of some Polish space, then by $\mathcal{C}(E)$ and $\mathcal{C}_b(E)$ we denote respectively the space of continuous real functions and bounded continuous real functions on a space $E$. By $\mathcal{C}_b^k(E)$ we denote the space of continuously differentiable bounded real functions on a space $E$ and by $\mathcal{C}^k(E)$ collection of all continuous functions, which have continuous derivatives up to $k$.

2.1.10 An $E$-valued stochastic process is a family $\{X(t), t \geq 0\}$ of mappings from $\Omega$ to $E$. Let $\Omega$ be a function space. Typically in this dissertation $\Omega$ is the space $D_E[0, +\infty)$ or $C_E[0, +\infty)$, where $E$ is a Borel space. In this dissertation there will be considered only canonical process $\{X(t), t \geq 0\}$. That is, $X(\omega, t) = \omega(t)$ denote the position of $\omega \in \Omega$ at time $t \geq 0$. We set $\mathcal{F}_t = \mathcal{F}_t^X = \sigma \{X(s) : s \leq t\}$. Thus $\mathcal{F}_t^X$ is the smallest $\sigma$-field over $\Omega$ with respect to which all the maps $\omega \rightarrow X(\omega, s) = \omega(s)$ are measurable for $0 \leq s \leq t$ and $\mathcal{F}_t^X = \mathcal{G}_t$ by 2.1.6. In this case probability measure $\mathbb{P}$ will be called the law of the process $\{X(t), t \geq 0\}$.

2.1.11 If $\Omega = D_E[0, +\infty)$ or in other words all trajectories of process $\{X(t), t \geq 0\}$ are cadlag functions, then we call such a process by cadlag process. For real-valued cadlag process we define two other processes $\{X(t-), t \geq 0\}$ and $\{\Delta X(t), t \geq 0\}$ by

\[
X(0-) = X(0), \quad X(t-) = \lim_{s \uparrow t} X(s) \quad \text{for} \ t > 0,
\]

\[
\Delta X(t) = X(t) - X(t-).
\]

2.1.12 $E$-valued processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are called indistinguishable if set $\{X \neq Y\} = \{ \omega, t \in \Omega : X(\omega, t) \neq Y(\omega, t)\}$ is an evanescent set, i.e. $\mathbb{P} (\omega : \exists t \ X(\omega, t) \neq Y(\omega, t)) = 0$. If processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are indistinguishable, then $\mathbb{P}(X(t) = Y(t)) = 1$ for all $t \geq 0$. We say then, that process $\{X(t), t \geq 0\}$ is a modification of process $\{Y(t), t \geq 0\}$. 

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We now give some facts concerning space of cádlág and continuous functions.

2.1.13 Let $E$ be a Borel state space. Consider space $D_E[0, +\infty)$ equipped with the Skorokhod topology (see Jacod and Shiryaev [58], page 292, for definition). Then $D_E[0, +\infty)$ is a Borel space (see Jacod and Shiryaev [58], Theorem 1.14, page 292, and Ethier and Kurtz [48], Theorem 5.6, page 121) and by Jacod and Shiryaev [58] Lemmas 1.45 and 1.46, page 299, $\sigma$-field $G$ of cylindrical sets on $D_E[0, +\infty)$ is a Borel $\sigma$-field of $D_E[0, +\infty)$. Similarly, space of continuous processes $C_E[0, +\infty)$ is Borel and $B(C_E[0, +\infty)) = G$ (see Jacod and Shiryaev [58], page 289). Moreover, if the state space $E$ is Polish, then $D_E[0, +\infty)$ and $C_E[0, +\infty)$ are Polish spaces.

We now present some facts concerning measure theory.

2.1.14 The $\sigma$-field $B$ is said to be separable if there exists a denumerable class $D \subset B$ such that $D$ generates $B$. Filtration $\{F_t\}$ is separable if each $\sigma$-field $F_t$ ($t \geq 0$) is separable. We say that $\sigma$-fields $B_1$ and $B_2$ are $\sigma$-isomorphic if there exists an one-to-one measurable map from $B_1$ onto $B_2$ preserving the countable union operation and the complementation.

A measurable space $(\Omega, B)$ with $B$ separable is called standard if there exists a complete separable metric space $W$ such that $\sigma$-fields $B$ and $B_W$ are $\sigma$-isomorphic, where $B_W$ is a $\sigma$-field of Borel sets of $W$.

2.1.15 If $\Omega$ is a Borel space and $B$ is the class of Borel subsets of $\Omega$, then $(\Omega, B)$ is a standard measurable space; see e.g. Parathasarathy [102], Theorem 2.2, page 133.

2.1.16 An atom of a measurable space $(\Omega, B)$ is a set $A_0 \in B$ such that relations $A \subset A_0$, $A \in B$ imply that $A = A_0$ or $A = \emptyset$.

2.1.17 Consider general filtration $\{F_t\}$ on $(\Omega, F)$, where $F = \bigvee_{t \geq 0} F_t = \sigma\{\cup_{t \geq 0} F_t\}$. A family of measures $\{\mathbb{P}_t\}$ defined on $(\Omega, F_t)$ is said to be consistent if for $s \leq t$

$$\mathbb{P}_t(A) = \mathbb{P}_s(A) \quad \text{for all } A \in F_s.$$ 

The following theorem plays an important role in the change of measure technique; see Parathasarathy [102], Theorem 4.2, page 143.

**Theorem 2.1.1** If

(i) $(\Omega, F_t)$ is a standard measurable space for all $t \geq 0$,

(ii) $\cap_{n \geq 1} A_n \neq \emptyset$ for any sequence of sets $A_1 \supseteq A_2 \supseteq \ldots$ such that $A_n$ is an atom of $F_{t_n}$ and $t_1 < t_2 < \ldots$,
then there exists a unique measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}|_{\mathcal{F}_t} = \mathbb{P}_t$, i.e. $\mathbb{P}(A) = \mathbb{P}_t(A)$ for all $A \in \mathcal{F}_t$.

2.1.18 An optional time is the extended random variable $\tau : \mathcal{F} \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\{\tau < t\} \in \mathcal{F}_t$ for each $t \geq 0$. By stopping time we mean the extended random variable $\tau$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. If $T$ and $S$ are two stopping times, then $S \wedge T$ is also stopping time; see Jacod and Shiryaev [58], page 5.

2.1.19 Process $\{X(t), t \geq 0\}$ is adapted if $X(t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$. Optional $\sigma$-field $\mathcal{O}$ is the $\sigma$-field on $\Omega \times \mathbb{R}_+$ that is generated by all adapted càdlàg processes. Predictable $\sigma$-field $\mathcal{P}$ is the $\sigma$-field on $\Omega \times \mathbb{R}_+$ that is generated by all adapted left-continuous processes. A process that is $\mathcal{O}$-measurable or $\mathcal{P}$-measurable is called optional or predictable respectively. In particular every continuous process is predictable. Process $\{X(t), t \geq 0\}$ is progressively measurable if the mapping $(s, \omega) \rightarrow X(s, \omega)$ is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable. By Dellacherie and Meyer [32], Theorem 15, page 89, every predictable or optional process is progressive.

2.1.20 Real-valued process $\{X(t), t \geq 0\}$ is locally bounded if exist stopping times $\tau_n \uparrow +\infty$ and constants $k_n$ such that $\{\|X(t)\|, t \geq 0\}$ is bounded by $k_n$ on stochastic interval $[[0, \tau_n]]$, where

$$[[0, \tau_n]] = \{(\omega, t) : t \in \mathbb{R}_+, 0 \leq t \leq \tau_n(\omega)\}.$$ 

2.1.21 Every predictable càdlàg process is locally bounded (see the proof in Dellacherie and Meyer [33], page 221).

2.1.22 A real-valued stochastic process $\{A(t), t \geq 0\}$ is said to be of finite variation if it is càdlàg, adapted and $A(0) = 0$ and each path $t \rightarrow A(\omega, t)$ has a finite variation over each finite interval $[0, t]$, i.e.

$$\text{Var}(\omega, t) = \sup \sum_{i=1}^{n} \left| A(\omega, t_i) - A(\omega, t_{i-1}) \right| < +\infty,$$  \hspace{1cm} (2.1.1)

where the supremum is taken over all partitions $0 \leq t_0 < t_1 < \ldots < t_n = t$.

2.1.23 If $\int_0^t X(s) \, ds < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\{X(t), t \geq 0\}$ is $\mathcal{F}_t$-progressive, then $\{\int_0^t X(s) \, ds, t \geq 0\}$ is $\mathcal{F}_t$-adapted; see Brémaud [23], page 33. By 2.1.19 this is a case when $\{X(t), t \geq 0\}$ is predictable or optional.
2.1.24 We define stopped process \( \{ X^\tau(t), t \geq 0 \} \) at time \( \tau \) by

\[
X^\tau(t) = X(\tau \land t).
\]

2.1.25 Let \( n \geq 1 \) be a natural number. Suppose that for each \( i \in I = \{1, \ldots, n\} \), \((\Omega^{(i)}, \mathcal{F}^{(i)}, \mathbb{P}^{(i)})\) is a probability space (we do not assume completeness). Let \( \Omega \) denote the Cartesian product

\[
\Omega = \times_{i \in I} \Omega^{(i)}
\]

with typical element \( x = (x^{(i)} : i \in I) \). The product \( \sigma \)-fields \( \mathcal{F} = \times_{i \in I} \mathcal{F}^{(i)} \) is defined to be the smallest \( \sigma \)-field under which each projection \( \pi^{(i)} : \Omega \to \Omega^{(i)} \) with \( \pi^{(i)}(x) = x^{(i)} \) is measurable. Then by Williams [122], Theorem II.9, page 45, there exists a unique measure \( \mathbb{P} \) on \((\Omega, \mathcal{F})\) such that whenever \( A^{(i)} \in \mathcal{F}^{(i)} \) for all \( i \)

\[
\mathbb{P} \left( \prod_i A^{(i)} \right) = \prod_i \mathbb{P}^{(i)}(A^{(i)}).
\]

(2.1.2)

We denote measure \( \mathbb{P} \) by \( \mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n \) and call it product measure of measures \( \mathbb{P}^{(i)} \). Note that if we have \( n \) processes \( \{ X^{(i)}(t), t \geq 0 \} \) (i = 1, \ldots, n) each defined on stochastic basis \((\Omega^{(i)}, \mathcal{F}^{(i)}, \{ \mathcal{F}^{(i)}_t \}, \mathbb{P}^{(i)})\), then on \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \mathbb{P})\), where \( \mathcal{F}_t = \times \mathcal{F}^{(i)}_t \), we can define process \( \{ (X^{(1)}(t), \ldots, X^{(n)}(t)), t \geq 0 \} \) and by (2.1.2) on this product stochastic basis processes \( \{ X^{(i)}(t), t \geq 0 \} \) are independent.

2.1.26 By shift operators \( \{ \theta_t \} \) we mean a family of operators \( \theta_t : \Omega \to \Omega \) fulfilling the following conditions:

(i) \( (t, \omega) \to \theta_t \omega \) is \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})\)-measurable,

(ii) \( \theta_t \) is bijective for all \( t \geq 0 \),

(iii) \( \theta_t \circ \theta_s = \theta_{t+s} \) for all \( t, s \in \mathbb{R} \); in particular \( \theta_0 \) is identity and \( \theta_{t}^{-1} = \theta_{-t} \).

2.2 Martingales and stochastic integrals

2.2.1 Process \( \{ M(t), t \geq 0 \} \) is a martingale (resp. supermartingale) on a stochastic basis \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \mathbb{P})\) if and only if it is integrable, càdlàg and adapted process fulfilling

\[
\mathbb{E}(M(s) | \mathcal{F}_t) = M(t) \quad \text{for} \ s \geq t, \quad \text{(resp. } \mathbb{E}(M(s) | \mathcal{F}_t) \leq M(t)) \).
\]

The family of martingales we denote by \( \mathcal{M} \).
2.2.2 Similarly to Jacod and Shiryaev [58] we somehow depart from standard conventions in this definition. Namely, the stochastic basis is not assumed to satisfy usual conditions and instead of this we assume that the martingale is a càdlàg process. Such a definition is very useful in further chapters because we cannot assume usual conditions in the construction of a new probability measure in (3.1.1). Nevertheless, the all properties of martingale remain true.

2.2.3 Moreover, every martingale with respect to filtration \( \{ \mathcal{F}_t \} \) is a also martingale with respect to augmented filtration \( \{ \mathcal{F}_t^\mathbb{P} \} \). Further, if we drop the assumption of right-continuity in the definition of the martingale, then every martingale with respect to the augmented filtration has a càdlàg modification (see Dellacherie and Meyer [33], Theorem VI.4, page 69).

2.2.4 According to our definition every martingale is right-continuous, thus by Dellacherie and Meyer [33], Theorem VI.2, page 67, every martingale with respect to filtration \( \{ \mathcal{F}_t \} \) is also a martingale with respect to \( \{ \mathcal{F}_{t+} \} \). Furthermore, if we take martingale \( \{ M(t), t \geq 0 \} \) with respect to augmented filtration \( \{ \mathcal{F}_t^\mathbb{P} \} \), which is adapted to filtration \( \{ \mathcal{F}_t \} \), then \( \{ M(t), t \geq 0 \} \) is a also martingale with respect to filtration \( \{ \mathcal{F}_t \} \).

Optional Sampling Theorem (see Karatzas and Shreve [61], Theorem 3.22, page 19, or Williams [122] Theorem 53.1, page 79, and also Dellacherie and Meyer [33], page 74) says that:

**Theorem 2.2.1** If \( \{ M(t), t \geq 0 \} \) is a martingale (supermartingale) with a last element \( M(+\infty) \) and \( \tau_1 \leq \tau_2 \) are two optional times of the filtration \( \{ \mathcal{F}_t \} \), then we have

\[
\mathbb{E}(M(\tau_2)|\mathcal{F}_{\tau_1^+}) = (\leq)M(\tau_1) \quad \text{a.s.}
\]

The theorem applies to bounded optional times \( \tau_1 \) and \( \tau_2 \) without the further assumption of existence the last element.

2.2.5 By square-integrable martingale we mean martingale \( \{ M(t), t \geq 0 \} \) such that

\[
\sup_{t \geq 0} \mathbb{E}M^2(t) < +\infty.
\]

2.2.6 If \( \mathcal{C} \) is a certain class of processes, we denote \( \mathcal{C}_{\text{loc}} \) the localized class defined as such: a process \( \{ X(t), t \geq 0 \} \) belongs to \( \mathcal{C}_{\text{loc}} \) if and only if there exists an increasing sequence \( \{ \tau_n \} \) of stopping times such that \( \lim_{n \to +\infty} \tau_n = +\infty \) a.s. and that each stopped process \( \{ X^{\tau_n}(t), t \geq 0 \} \) belongs to \( \mathcal{C} \). Such a sequence is called a fundamental sequence. In particular using this definition we define local martingale and locally square-integrable martingale.

The following fact gives condition under which a local martingale is a martingale.
2.2.7 Ethier and Kurtz [48], page 92, state that if process \( \{X(t), t \geq 0\} \) is a local martingale and \( \mathbb{E} \sup_{s \leq t} |X(s)| < +\infty \) for each \( t > 0 \), then \( \{X(t), t \geq 0\} \) is a martingale.

2.2.8 By Jacod and Shiryaev [58], Corollary 3.16, page 32, (see also Elliot [43], Lemma 11.39) every predictable local martingale of finite variation is equal to constant up to an evanescent set.

2.2.9 A semimartingale \( \{X(t), t \geq 0\} \) is a real-valued process of the form

\[
X(t) = X(0) + M(t) + A(t),
\]

where \( X(0) \) is finite-valued variable and \( \mathcal{F}_0 \)-measurable, process \( \{M(t), t \geq 0\} \) is a local martingale such that \( M(0) = 0 \) and process \( \{A(t), t \geq 0\} \) is a process of finite variation starting at 0. Note that every semimartingale is a càdlàg process.

2.2.10 For locally square integrable martingale \( \{M(t), t \geq 0\} \) we define a predictable process \( \{< M >_t, t \geq 0\} \) of finite variation by

\[
M^2(t) - < M >_t \in \mathcal{M}_{\text{loc}}.
\]

It will be called quadratic variation. By Liptser and Shiryaev [84], Theorem 2, page 41, every local martingale \( \{M(t), t \geq 0\} \) has a unique decomposition

\[
M(t) = M^c(t) + M^d(t),
\]

where \( \{M^c(t), t \geq 0\} \) is a continuous local martingale such that \( M^c(0) = 0 \). By Elliot [43], Lemma 10.15, page 103, every continuous local martingale is square integrable. Thus for a local martingale \( \{M(t), t \geq 0\} \) we can define process \( \{< M^c >_t, t \geq 0\} \). If \( \{X(t), t \geq 0\} \) is a semimartingale, then by Jacod and Shiryaev [58], Proposition 4.27, page 45, there exists unique up to indistinguishability continuous local martingale \( \{X^c(t), t \geq 0\} \) with \( X(0) = 0 \), such that any decomposition \( X(t) = X(0) + M(t) + A(t) \) meets \( M^c(t) = X^c(t) \) (up to indistinguishability again). Process \( \{X^c(t), t \geq 0\} \) is called the continuous martingale part of \( \{X(t), t \geq 0\} \). Then we define quadratic variation of semimartingale \( \{X(t), t \geq 0\} \) by

\[
[X]_t = < X^c >_t + \sum_{s \leq t} (\Delta X(s))^2.
\]

If \( \{X(t), t \geq 0\} \) and \( \{Y(t), t \geq 0\} \) are two semimartingales, then by \([X,Y]_t\) we denote quadratic covariation, which is define by

\[
[X,Y]_t = \frac{1}{4} ([X,Y, X+Y] - [X-Y, X-Y]).
\]

We will use also notation \([X(t), Y(t)]_t\) and \( < X >_t \) instead \([X,Y]_t\) and \( < X >_t \), respectively.
2.2.11 Following Jacod and Shiryaev [58], Theorem 4.31, page 46 (see also Dellacherie and Meyer [33], Theorem VIII.3, page 313 and VIII.8, page 320), for a given stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) we define stochastic integral

\[
\left\{ \int_0^t Y(s) \, dX(s), t \geq 0 \right\},
\]

where \(\{Y(t), t \geq 0\}\) is a locally bounded predictable process and process \(\{X(t), t \geq 0\}\) is a semimartingale. Note that similarly like in Jacod and Shiryaev [58] in definition of stochastic integral we do not assume that stochastic basis fulfills usual conditions. Moreover, there is a certain invariance property of the stochastic integral under a filtration change. Namely, from Dellacherie ad Meyer [33], Theorem VIII.13, page 322, if process \(\{X(t), t \geq 0\}\) is also a semimartingale with respect to filtration \(\{\mathcal{F}^P_t\}\), then every locally bounded predictable process with respect to \(\{\mathcal{F}_t\}\) also has these properties relative to \(\{\mathcal{F}^P_t\}\) and the stochastic integrals calculated for \(\{\mathcal{F}_t\}\) and for \(\{\mathcal{F}^P_t\}\) are indistinguishable.

2.2.12 Let \(\{X(t), t \geq 0\}\) be a real-valued semimartingale and \(f\) a class \(C^2(\mathbb{R})\) function. Then \(\{f(X(t)), t \geq 0\}\) is a semimartingale and we have:

\[
f(X(t)) = f(X(0)) + \frac{d}{dx} f(X(t^-)) + \frac{1}{2} \int_0^t \frac{d^2}{dx^2} f(X(s^-)) \, d <X^c>_s
\]

\[+ \sum_{s \leq t} \left[ f(X(s)) - f(X(s^-)) - \frac{d}{dx} f(X(s^-)) \Delta X(s) \right];
\]

see Jacod and Shiryaev [58], Theorem 4.57, page 57.

2.2.13 By Liptser and Shiryaev [84], Corollary 1, page 99, we have the following integration-by-parts formula for semimartingales. If \(\{X(t), t \geq 0\}\) and \(\{Y(t), t \geq 0\}\) are semimartingales, then

\[
X(t)Y(t) = \int_0^t X(s^-) \, dY(s) + \int_0^t Y(s^-) \, dX(s) + [X,Y]_t.
\]

2.2.14 Jacod and Shiryaev [58], Proposition 4.49 d), page 52 (see also Liptser and Shiryaev [84], Proposition 2.1.3, page 75), state following. If process \(\{X(t), t \geq 0\}\) is a semimartingale, \(\{Y(t), t \geq 0\}\) is a process of finite variation and \(\{X(t), t \geq 0\}\) or \(\{Y(t), t \geq 0\}\) is continuous, then

\[
[X,Y] = 0. \tag{2.2.1}
\]
2.2.15 From Dellacherie and Meyer [33], Theorem VIII.3.e, page 314, if process \( \{X(t), t \geq 0\} \) is the local martingale and càdlàg process \( \{Y(t), t \geq 0\} \) is predictable (hence by 2.1.21 also locally bounded), then the stochastic integral
\[
\{ \int_0^t Y(s) \, dX(s), t \geq 0 \}
\]
is the local martingale (see also Jacod and Shiryaev [58], 4.34 b), page 47).

2.3 Markov process

2.3.1 A stochastic process \( \{X(t), t \geq 0\} \) on stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) with values in a space \(E\) is called Markov process if for any Borel set \(B \in \mathcal{B}(E)\) and any \(t \geq s \geq 0\)
\[
\mathbb{P}(X(t) \in B|\mathcal{F}_s) = \mathbb{P}(X(t) \in B|X(s))
\]

2.3.2 A Markov family is an adapted process \( \{X(t), t \geq 0\} \) on some space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\})\), together with a family of probability measures \(\{\mathbb{P}^x\}_{x \in E}\), such that
(i) for each \(F \in \mathcal{F}\) the mapping \(x \to \mathbb{P}^x(F)\) is universally measurable,
(ii) \(\mathbb{P}^x(X(0) = x) = 1\) for each \(x \in E\),
(iii) for any nonnegative measurable functions \(f_j\) we have
\[
\mathbb{E}^x \left[ \prod_{j=1}^n f_j(X(t + t_j))|\mathcal{F}_t \right] = \mathbb{E}^{X(t)} \left[ \prod_{j=1}^n f_j(X(t_j)) \right],
\]
where \(\mathbb{E}^x\) is the expectation with respect to \(\mathbb{P}^x\).

Property (iii) we call a \(\mathbb{P}^x\)-Markov property.

2.3.3 Denote by \(\mathcal{L}\) the space of measurable functions on a space \(E\). Let \(\{X(t), t \geq 0\}\) be a Markov process on stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) with values in \(E\). We define the extended generator \(\mathcal{A}\) of process \(\{X(t), t \geq 0\}\) by
\[
\mathcal{A} = \left\{(g, f) \in \mathcal{L} \times \mathcal{L} : M^p(t) \in \mathcal{M}_{\text{loc}} \right\},
\]
where
\[
M^p(t) = g(X(t)) - \int_0^t f(X(s)) \, ds \tag{2.3.1}
\]
is called Dynkin’s local martingale and function \(s \to f(X(s))\) is integrable \(\mathbb{P}\)-a.s. on \([0, t]\) for all \(t \geq 0\). A set \(A\) is said to have potential zero if
\[
\int_0^{+\infty} \mathbb{I}_A(X(s)) \, ds = 0 \quad \mathbb{P}\text{-a.s.} \tag{2.3.2}
\]
Note that $\mathcal{A}$ is single valued up to the set of potential zero. In fact, let $(g, f_1) \in \mathcal{A}$ and $(g, f_2) \in \mathcal{A}$. Then $M(t) = \int_0^t (f_1(X(s)) - f_2(X(s))) \, ds$ would be a predictable local martingale whose sample paths are continuous and of finite variation. By 2.2.8, such the martingale is identically equal to a constant, which is zero $\mathbb{P}$-a.s. in this case since $M(0) = 0$. Therefore we must have $f_1(x) = f_2(x)$ for every $x \in E$ except possibly on some set $A \in \mathcal{B}(E)$ fulfilling (2.3.2). The process $\{X(t), t \geq 0\}$ 'spends no time' in $A$, so the process $\{M^x(t), t \geq 0\}$ of (2.3.1) does not depend of the values $f(x)$ for $x \in A$, and function $f$ is unique up to sets of potential zero. For the rest of this dissertation we will identify all versions of functions $f$ and we denote all these versions by $\mathcal{A}g$ if $(g, f) \in \mathcal{A}$.

By $D(\mathcal{A})$ we denote the set of measurable functions $g \in \mathcal{L}$ such that $\mathcal{A}g \in \mathcal{L}$, $\int_0^t \mathcal{A}g(X(s)) \, ds < +\infty \, \mathbb{P}$-a.s. for all $t \geq 0$ and

$$M^g(t) = g(X(t)) - \int_0^t \mathcal{A}g(X(s)) \, ds \quad (2.3.3)$$

is a local martingale. $D(\mathcal{A})$ is called the domain of the extended generator $\mathcal{A}$. If process (2.3.3) is a martingale, then operator $\mathcal{A}$ is called full generator of Markov process $\{X(t), t \geq 0\}$. If we restrict the domain of the extended or full generator to a subset $D \subset D(\mathcal{A})$, then to avoid increasing the number of notations from that moment we denote this subset also by $D(\mathcal{A})$.

**Remark 2.3.1** For a deeper understanding of the above definition of the extended generator $\mathcal{A}$ let us consider the classical definition of the infinitesimal generator. The idea is to substrat from a process its drift to get a martingale. If for some measurable bounded functions $g, f \in \mathcal{L}$

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E} [g(X(t)) - g(x) | X(0)] = f(x)$$

in the uniform convergence sense, then Dynkin formula (see Williams [122], III.21, page 127) assures that process (2.3.1) is a martingale. Therefore the domain of the infinitesimal generator is contained in the domain of the extended generator and, the extended generator restricted to the domain of the infinitesimal generator and the infinitesimal generator coincide.

**2.3.4** If $\{X(t), t \geq 0\}$ is càdlàg or continuous process, then by measurability of $\mathcal{A}g$ and 2.1.19 we get that process $\{\mathcal{A}g(X(t)), t \geq 0\}$ is progressive. Thus by 2.1.23 process (2.3.3) is $\mathcal{F}_t^X$-adapted. Assume that process (2.3.3) is a local martingale with respect to augmented filtration $\{\mathcal{F}_t^P\}$ and fundamental sequence of stopping times for local martingale (2.3.3) consists of $\mathcal{F}_t^X$-stopping times. Then by 2.2.4 it is also a local martingale with respect to filtration $\{\mathcal{F}_t^X\}$. 

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Chapter 3

Exponential change of measure

Recently in many cases a technique of the exponential change of measure is used. In the theory of large deviation such the technique is called twisting; see for instance Schwartz and Weiss [112], page 75, Ridder [104]. It is also used in the theory of fluid model; see e.g., Palmowski and Rolski [100], Palmowski and Rolski [101], Kulkarni and Rolski [75], Asmussen [1], in ruin theory; see for instance Dassios and Embrecht [29], Schmidli [111], in fast simulation; see Asmussen [7], Ridder [105], and in population genetic theory; see paper of Ethier and Kurtz [49], Fukushima and Stroock [50]. Frequently we want to know the generator of the process under the new probability measure. Although there are cases when computing new generator is straightforward, e.g. in Asmussen [2], however a unified theory in many situations simplify calculations.

In Section 3.1 we present general theory concerning the change of measure. In Section 3.2 we prove the perturbation Theorem 3.2.2. In Sections 3.3 and 3.4 we apply the perturbation theorem to piecewise deterministic Markov process and to diffusion process respectively.

3.1 Change of measure

In this section we present a general theory allowing computing the generator of the process under the new probability measure. Consider the setup given in 2.1.3 and 2.1.10. That is, $E$ is a Borel space. A space of elementary events $\Omega \subset E^{[0, +\infty)}$ is a space of càdlàg functions $D_{\mathcal{E}}[0, +\infty)$ or a space of continuous functions $C_{\mathcal{E}}[0, +\infty)$. We consider canonical process $\{X(t), t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, thus $X(\omega, t) = \omega(t)$ (see 2.1.10). Then we take filtration generated by process $\{X(t), t \geq 0\}$: $\mathcal{F}_t = \mathcal{F}_t^X =\sigma\{X(s), s \leq t\} = \mathcal{G}_t$, where $\mathcal{G}_t$ is a $\sigma$-field of cylindrical sets on an interval $[0, t]$. We define $\mathcal{F}$ by $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^X = \bigvee_{t \geq 0} \mathcal{G}_t = \mathcal{G}$. Consider on $(\Omega, \mathcal{F})$ the family of probability measures $\{\mathbb{P}_1\}$, which is consistent with respect to $\{\mathcal{F}_t^X\}$ (see definition of consistent measures in 2.1.17). We now state another version of Kolmogorov theorem (see also Stroock [117], Theorem 4.2, page 106, and Stroock and Varadhan [118], Theorem 1.3.5, page 34).

**Theorem 3.1.1** Let $\{\mathbb{P}_1\}$ be the family of consistent probability measures on $(\Omega, \mathcal{F})$ with respect to filtration $\mathcal{F}_t^X$. Then there exists one and only one probability measure
\[ \mathbb{P} \text{ on } (\Omega, \mathcal{F}, \{\mathcal{F}_t^X\}) \text{ such that} \]
\[ \mathbb{P}_t = \mathbb{P}_{\mathcal{F}_t^X}, \quad t \geq 0. \]

Proof. To prove this theorem we check that conditions (i) and (ii) in Theorem 2.1.1 are fulfilled in our setup. First we prove that condition (i) of Theorem 2.1.1 holds. By 2.1.13 spaces \( D_E[0, +\infty) \) and \( C_E[0, +\infty) \) are Borel and filtration \( \{\mathcal{F}_t^X\} = \{\mathcal{G}_t\} \) is the family of Borel \( \sigma \)-fields of interval \([0, t]\). Thus by 2.1.15 \((\Omega, \mathcal{F}_t^X)\) are standard measurable spaces for all \( t \geq 0 \). We now prove that also condition (ii) of 2.1.1 is fulfilled. Note that an atom of \((\Omega, \mathcal{F}_t^X)\) (see 2.1.16) has following form
\[ \{\omega \in \Omega : \omega(s) = x(s) \quad \text{for all } s \leq t\}, \]
where \( x(t) \) is certain function from \( \Omega \). We denote this set by \( \{x\}_t \). Thus \( A_n = \{x_n\}_{t_n} \).
By assumption \( A_m \supseteq A_n \) for \( m \leq n \), we have that functions \( x_n \) and \( x_m \) are the same on interval \([0, t_m] \subseteq [0, t_n]\). Thus there exists function \( x \) such that \( x(t) = x_n(t) \) for all \( t \leq t_n \). Moreover, \( \bigcap_{n \geq 1} A_n = \{x\} \neq \emptyset \). This completes the proof of this theorem.

\[ \square \]

Remark 3.1.1 By Theorem 2.1.1 and 2.1.14 to get this assertion on a Borel space of functions \( \Omega \) it suffices to consider a separable filtration \( \{\mathcal{F}_t\} \), \( \sigma \)-isomorphic with some Borel \( \sigma \)-field of a separable metric space. Note that filtration \( \{\mathcal{F}_t^P\} \) does not fulfill this assumption because it is not separable filtration.

The main technique used in this dissertation is the change of measure technique. Every two processes having paths in the space \( \Omega \) we represent by probability measures \( \mathbb{P} \) and \( \mathbb{P}^\prime \). Thus one could study conditions for the Radon-Nikodym derivative \( d\mathbb{P}/d\mathbb{P} \) to exist. However, as shown by the example given in this section this setup is too restrictive: typically, the parameters of the two processes can be reconstructed from infinite path, and \( \mathbb{P} \) and \( \mathbb{P}^\prime \) are not absolutely continuous. The interesting concept is therefore to look for absolute continuity only on finite intervals (possible random). That is, we define the family of probability measures \( \{\mathbb{P}_t\}_{t \geq 0} \) by Radon-Nikodym derivative
\[ \frac{d\mathbb{P}_t}{d\mathbb{P}} = M(t), \quad (3.1.1) \]
where \( M(0) = 1, \mathbb{P}_t = \mathbb{P}_{\mathcal{F}_t^X} \) and \( \{M(t), t \geq 0\} \) is a nonnegative \( \mathcal{F}_t^X \)-martingale on \((\Omega, \mathcal{F}, \{\mathcal{F}_t^X\}, \mathbb{P})\). Hence, formally we define
\[ \mathbb{P}_t(A) = \mathbb{E}[M(t); A] \quad \text{for each } A \in \mathcal{F}_t^X. \quad (3.1.2) \]

Lemma 3.1.2 The family of measures \( \{\mathbb{P}_t, \mathcal{F}_t^X\} \) is consistent.

Proof. Let \( A \in \mathcal{F}_s^X \) and \( s \leq t \). Then also \( A \in \mathcal{F}_t^X \) and by the martingale property we have
\[ \mathbb{P}_t(A) = \mathbb{E}[M(t); A] = \mathbb{E} \left[ \mathbb{E}[M(t)1_A|\mathcal{F}_s^X] \right] \]
\[ = \mathbb{E} \left[ 1_A \mathbb{E}[M(t)|\mathcal{F}_s^X] \right] = \mathbb{E}[M(s); A] = \mathbb{P}_s(A). \]
Thus by Theorem 3.1.1 and Lemma 3.1.2 we have the following corollary.

**Corollary 3.1.3** Let process \( \{X(t), t \geq 0\} \) be defined on stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t^X\}_{t \geq 0}, \mathbb{P})\) by \( X(\omega, t) = \omega(t) \), where \( \Omega \) is a space of càdlàg functions \( D_E[0, +\infty) \) or continuous functions \( C_E[0, +\infty) \) with values in a Borel space \( E \). Then there exists one and only one probability measure \( \hat{\mathbb{P}} \) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t^X\}_{t \geq 0})\) such that \( \hat{\mathbb{P}}_t = \hat{\mathbb{P}}{\big|}_{\mathcal{F}_t^X} \) and \( \hat{\mathbb{P}}_t \) fulfills (3.1.1).

The above statement can be wrong with filtration \( \{\mathcal{F}_t\} \) bigger than previously considered \( \{\mathcal{F}_t^X\} \) as we present in the following example. That is we find probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) for which there is no probability measure \( \hat{\mathbb{P}} \) defined on \( \mathcal{F} \), such that \( \hat{\mathbb{P}} \) restricted to any \( \mathcal{F}_t \) agrees with \( \hat{\mathbb{P}}_t \), where \( \hat{\mathbb{P}}_t \) is defined by (3.1.1). Consider the space \( \Omega = C[0, +\infty) \) of real continuous functions on \([0, +\infty)\) which are \( 0 \) at 0, and \( \mathbb{P} \) to be the Wiener measure, that is the probability measure which makes the canonical process \( \{B(t), t \geq 0\} \) a standard Brownian motion. For filtration \( \{\mathcal{F}_t\} \) we take the usual \( \mathbb{P} \) - augmentation of natural filtration \( \mathcal{F}_t^B \), that is \( \{\mathcal{F}_t\} = \{\mathcal{F}_t^B\} \). By Karatzas and Shreve [61], Corollary 5.2, page 192, process

\[
N(t) = \exp\{B(t) - \frac{1}{2}m^2t\}
\]

defines \( \mathcal{F}_t^B \) - martingale, hence by 2.2.4 also \( \mathcal{F}_t^B \)-martingale, where \( m \neq 0 \). From Theorem 3.1.1 we can construct new probability measure \( \hat{\mathbb{P}} \) on \( \{\mathcal{F}_t^B\} \) defined by (3.1.1) (see also Rogers and Williams [106], Theorem 38.9, page 82). By Cameron-Martin-Girsanov Theorem (see Theorem 3.4.2) process \( \{B(t) - m, t \geq 0\} \) under probability measure \( \hat{\mathbb{P}} \) is a Brownian motion relative to \( \{\mathcal{F}_t^B\} \). Hence

\[
\hat{\mathbb{P}} \left( \lim_{t \to +\infty} \frac{1}{t} B(t) = m \right) = 1 \tag{3.1.3}
\]

and

\[
\mathbb{P} \left( \lim_{t \to +\infty} \frac{1}{t} B(t) = m \right) = 0 \ . \tag{3.1.4}
\]

Since \( \mathbb{P} \) - null event \( \{\lim_{t \to +\infty} \frac{1}{t} B(t) = m\} \) is in \( \mathcal{F}_t^B \) for every \( t \geq 0 \), \( \hat{\mathbb{P}} \) and \( \hat{\mathbb{P}}_t \) cannot agree on \( \mathcal{F}_t^B \), because \( \hat{\mathbb{P}}_t \) is absolutely continuous with respect to \( \mathbb{P}_t \). We cannot expect to make \( \{B(t) - m, t \geq 0\} \) a \( \hat{\mathbb{P}} \)-Brownian motion relative to \( \{\mathcal{F}_t^B\} \), because every \( \mathcal{F}_t^B \) contains the whole future of \( \{B(t), t \geq 0\} \) under \( \hat{\mathbb{P}} \) (compare with \( \sigma \)-field \( \mathcal{F}_t^B = \mathcal{G}_t \); see also Remark 3.1.1).

In this dissertation we will frequently use the following passage time

\[
\tau(x) = \inf\{t > 0 : X(t) > x\},
\]

for a stochastic process \( \{X(t), t \geq 0\} \). Unfortunately \( \tau(x) \) is not a stopping time with respect to filtration \( \{\mathcal{F}_t^X\} \) (see 2.1.18). However, \( \tau(x) \) is a stopping time with respect to filtration \( \{\mathcal{F}_t^{X+}\} \) (see 2.1.4). Therefore we are interested in proving the existence of probability measure \( \hat{\mathbb{P}} \) fulfilling \( \hat{\mathbb{P}}{\big|}_{\mathcal{F}_t^X} = \hat{\mathbb{P}}_t \).
From 2.2.4 every martingale with respect to \( \{ \mathcal{F}^X_t \} \) is a martingale with respect to \( \{ \mathcal{F}^X_{t+} \} \). Thus by the martingale property the family of probability measures \( \{ \hat{\mathbb{P}}_t \} \) is \( \{ \mathcal{F}^X_{t+} \} \)-consistent. Note that \( \mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}^X_t = \bigvee_{t \geq 0} \mathcal{F}^X_{t+} \). The following theorem builds the setup used later in this dissertation.

**Theorem 3.1.4** Let \( \{ X(t), t \geq 0 \} \) be a canonical stochastic process on \( (\Omega, \mathcal{F}, \{ \mathcal{F}^X_{t+} \}_{t \geq 0}, \mathbb{P}) \), where \( \Omega \) is a space of càdlàg functions \( D_E[0, +\infty) \) or a space of continuous functions \( C_E[0, +\infty) \) on a Borel space \( E \). Define the family of probability measures \( \{ \hat{\mathbb{P}}_t \} \) on \( (\Omega, \mathcal{F}^X_{t+}) \) by

\[
\frac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t} = M(t), \quad t \geq 0,
\]

where \( \{ M(t), t \geq 0 \} \) is a nonnegative \( \mathcal{F}^X_t \)-martingale and \( \mathbb{P}_t = \mathbb{P}_{\mathcal{F}^X_{t+}} \). Then there exists one and only one probability measure \( \hat{\mathbb{P}} \) on \( (\Omega, \mathcal{F}, \{ \mathcal{F}^X_{t+} \}_{t \geq 0}) \) such that

\[
\hat{\mathbb{P}}_t = \hat{\mathbb{P}}_{\mathcal{F}^X_{t+}}.
\]

**Proof.** Let \( A \in \mathcal{F}^X_{t+} \). Then \( A \in \mathcal{F}^X_{t+} \) for every \( n \geq 1 \). Since \( \mathcal{F}^X_t \subseteq \mathcal{F}^X_{t+} \), probability measure \( \mathbb{P}_t \) can be also defined on \( \mathcal{F}^X_t \) and by Corollary 3.1.3 there exists a unique probability measure \( \hat{\mathbb{P}} \) such that \( \hat{\mathbb{P}}_t = \hat{\mathbb{P}}_{\mathcal{F}^X_t} \). Hence

\[
\hat{\mathbb{P}}_{\mathcal{F}^X_{t+}}(A) = \hat{\mathbb{P}}_{t+1}(A)
\]

and by the consistency property of \( \{ \hat{\mathbb{P}}_{\mathcal{F}^X_{t+}} \} \) with respect to filtration \( \{ \mathcal{F}^X_t \} \) and \( \{ \hat{\mathbb{P}}_t \} \) with respect to filtration \( \{ \mathcal{F}^X_{t+} \} \) we get that

\[
\hat{\mathbb{P}}_{\mathcal{F}^X_{t+}}(A) = \hat{\mathbb{P}}_t(A).
\]

The proof is completed.

\[\square\]

**Definition 3.1.5** We say that there is fulfilled the standard setup for change of measure if

(i) stochastic basis \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \mathbb{P}) \) has right-continuous filtration \( \{ \mathcal{F}_t \} \),

(ii) there exists a unique probability measure \( \hat{\mathbb{P}} \) defined by (3.1.1).

**Remark 3.1.2** Note that by Theorem 3.1.4 conditions of the standard setup are fulfilled if we take for a filtration \( \{ \mathcal{F}_t \} \) the right-continuous filtration \( \{ \mathcal{F}^X_{t+} \} \), that is canonical process \( \{ X(t), t \geq 0 \} \) is defined on \( (\Omega, \mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t, \{ \mathcal{F}^X_{t+} \}, \mathbb{P}) \), where \( \Omega \) is the space of càdlàg functions \( D_E[0, +\infty) \) or the space of continuous functions \( C_E[0, +\infty) \) on a Borel space \( E \) and \( X(\omega, t) = \omega(t) \).
Remark 3.1.3 If the standard setup is fulfilled, then each measure \( \hat{\mathbb{P}}_t \) is absolutely continuous with respect a measure \( \mathbb{P}_t \) and we denote this by \( \hat{\mathbb{P}}_t \ll \mathbb{P} \). Note that it does not mean that probability measure \( \hat{\mathbb{P}} \) is absolutely continuous with respect to probability measure \( \mathbb{P} \), that is \( \hat{\mathbb{P}} \ll \mathbb{P} \). In fact, taking previous example with natural filtration \( \{ \mathcal{F}_t \} \) from Theorem 3.1.1 we can construct measure \( \hat{\mathbb{P}} \), so that \( \hat{\mathbb{P}} \) restricted to any \( \mathcal{F}_t \) agrees with \( \mathbb{P}_t \), but from (3.1.3) and (3.1.4) we get that \( \hat{\mathbb{P}} \) is not absolutely continuous with respect to \( \mathbb{P} \). If we want to have absolutely continuous measure \( \hat{\mathbb{P}} \) with respect to \( \mathbb{P} \), then we must put additionally assumption on martingale \( \{ M(t), t \geq 0 \} \) (see Jacod and Shiryaev [58], Theorem 3.5, page 154):

\[
\hat{\mathbb{P}}(\sup_t M(t) < +\infty) = 1
\]  

(3.1.5)

or

\[
\{ M(t), t \geq 0 \}
\]  

is uniformly integrable martingale.  

(3.1.6)

Note that if \( \hat{\mathbb{P}} \ll \mathbb{P} \), then we have also (see Jacod and Shiryaev [58], Theorem 3.5, page 154)

\[
\hat{\mathbb{P}}(\inf_t M(t) > 0) = 1 .
\]  

(3.1.7)

We would say that probability measure \( \hat{\mathbb{P}} \) and \( \mathbb{P} \) are equivalent if \( \hat{\mathbb{P}} \ll \mathbb{P} \) and \( \mathbb{P} \ll \hat{\mathbb{P}} \).

Another interesting problem is to investigate what characteristics of a stochastic process defined on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \mathbb{P}) \) remain the same after the change of measure and what assumptions we should make then on martingale \( \{ M(t), t \geq 0 \} \). The crucial characteristic is a Markov property. Consider a Markov family \( \{ X(t), t \geq 0 \} \) defined on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \{ \mathbb{P}^x \}_{x \in E}) \), where the state space \( E \) is a Borel space. Assume that standard setup is fulfilled. We find conditions on martingale \( \{ M(t), t \geq 0 \} \) to ensure us that process \( \{ X(t), t \geq 0 \} \) remains Markov family with respect to probability measures \( \{ \mathbb{P}^x \}_{x \in E} \). To do this we need a concept of a multiplicative functional. Let \( \{ \theta_t \} \) be a family of shift operators (see definition in 2.1.26).

Definition 3.1.6 Process \( \{ M(t), t \geq 0 \} \) is a multiplicative functional if it is

(i) right-continuous,

(ii) adapted to filtration \( \{ \mathcal{F}_t \} \),

(iii) fulfills condition

\[
M(\omega, t + s) = M(\omega, t) \cdot M(\theta_t \omega, s)
\]  

for all \( s, t \geq 0 \) and \( \omega \in \Omega \),

(iv) \( \mathbb{E}(M(t)) \leq 1 \).

If \( \mathcal{F}_t = \mathcal{F}_t^X \), then the precise meaning of (iii) is the following: being \( \mathcal{F}_t^X \) - measurable, \( M(t) \) has the form

\[
M(\omega, t) = f(\{ X(\omega, u), 0 \leq u \leq t \})
\]
for some measurable mapping \( f : \Omega \to \mathbb{R} \), and then

\[
M(\theta_r \omega, s) = f \left( \{ X(t+u), 0 \leq u \leq s \} \right).
\]

Then, we have the following theorem proved by Kunita and Watanabe [81], Proposition 3 and 5 (see also Asmussen [4], Dynkin [42]).

**Theorem 3.1.7** Let \( \{ X(t), t \geq 0 \} \) be a Markov family on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \{ \mathbb{P}^r \}) \). We define new probability measures \( \{ \tilde{\mathbb{P}}^r \} \) by (3.1.1), where martingale \( \{ M(t), t \geq 0 \} \) is a nonnegative multiplicative functional for which \( \mathbb{E}(M(0)) = 1 \). If the standard setup is fulfilled, then on the new probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \{ \tilde{\mathbb{P}}^r \}) \) process \( \{ X(t), t \geq 0 \} \) is a Markov family.

**Proof.** It suffices to prove the \( \tilde{\mathbb{P}}^r \)-Markov property of process \( \{ X(t), t \geq 0 \} \) (see Definition 2.3.2 (iii)). For \( A \in \mathcal{F}_t \), measurable functions \( f_j \geq 0 \) and \( t_1 \leq t_2 \leq \ldots \leq t_n \) we have:

\[
\mathbb{E}^r \left( \prod_{j=1}^{n} f_j(X(t+t_j)); A \right) = \mathbb{E}^r \left( \prod_{j=1}^{n} f_j(X(t+t_j)) M(t+t_j); A \right) = \mathbb{E}^r \left( \prod_{j=1}^{n} f_j(X(t+t_j)) M(t); A \right) \]

by multiplicativity of martingale \( \{ M(t), t \geq 0 \} \)

\[
= \mathbb{E}^r \left( \mathbb{E}^{X(t)} \left( \prod_{j=1}^{n} f_j(X(t_j)) M(t_n) \right) M(t); A \right) = \mathbb{E}^r \left( \mathbb{E}^{X(t)} \left( \prod_{j=1}^{n} f_j(X(t_j)) \right); A \right),
\]

where by \( \mathbb{E}^x \) and \( \mathbb{E}^{x^r} \) we denote respectively the expectation with respect to \( \mathbb{P}^r \) and \( \tilde{\mathbb{P}}^r \).

In this section we prove some other useful propositions concerning the change of measure of Markov process \( \{ X(t), t \geq 0 \} \).

**3.1.1** We first find a condition on martingale \( \{ M(t), t \geq 0 \} \) to make measure \( \tilde{\mathbb{P}}_t \) and \( \mathbb{P}_t \) equivalent for each \( t \geq 0 \). That is, when equality \( \mathbb{P}_t(A) = \mathbb{E}[M(t); A] = 0 \) yields \( \mathbb{P}_t(A) = 0 \) for \( A \in \mathcal{F}_t \). Obviously it suffices to assume that \( \mathbb{P}(M(t) > 0) = 1 \) for each \( t \geq 0 \).

Under 3.1.1 by Radon-Nikodym theorem there exists a density process which equal by Billingsley [14], page 427, to process \( \{ M(t)^{-1}, t \geq 0 \} \):

\[
\frac{d\tilde{\mathbb{P}}_t}{d\mathbb{P}_t} = M(t)^{-1}.
\]

From Jacod and Shiryaev [58], Theorem 3.4, page 153, process \( \{ M(t)^{-1}, t \geq 0 \} \) is also a martingale. Moreover, \( \mathbb{E}(M(t)^{-1}) = \mathbb{E}(M(t)M(t)^{-1}) = 1 \). Hence \( \mathbb{P} \) is a probability measure. Then by Jacod and Shiryaev [58], Theorem 3.4(iii), page 153, we have the following useful lemma.

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Lemma 3.1.8 Assume that standard setup is fulfilled. Let \( \tau \) be a stopping time. Then
\[
P(A) = \hat{\mathbb{E}} \left[ M(\tau)^{-1}; A \right]
\]
for \( A \subset \{ \tau < +\infty \} \) and \( A \in \mathcal{F}_\tau \).

3.2 Perturbation Theorem

Let \( \{ X(t), t \geq 0 \} \) be a Markov process defined on space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, P) \) fulfilling the standard setup. Let \( \mathcal{A} \) be the extended generator of \( \{ X(t), t \geq 0 \} \), that is process
\[
M^2(t) = g(X(t)) - \int_0^t \langle \mathcal{A}g(X(s)) \rangle \, ds
\]
is a local martingale, where \( \mathcal{A}g \) is a measurable function and \( \int_0^t \mathcal{A}g(X(s)) \, ds < +\infty \) \( P \)-a.s. for all \( t \geq 0 \). The set of all measurable functions \( g : E \rightarrow \mathbb{R} \) for which it holds is called the domain \( D(\mathcal{A}) \) (see 2.3.3). If we restrict the domain \( D(\mathcal{A}) \) to some subset, then from that moment we also denote this subset by \( D(\mathcal{A}) \). To avoid difficulties in the proof of the perturbation theorem, in this section we restrict the domain of this operator \( D(\mathcal{A}) \) to measurable functions \( g : E \rightarrow \mathbb{R} \) such that for each path \( \omega \in \Omega \)
\[
(\text{A}.1) \quad \int_0^t |\mathcal{A}g(X(\omega, s))| \, ds < +\infty \quad \text{for all } t \geq 0.
\]
We also need the positive function \( h \in D(\mathcal{A}) \) fulfilling
\[
(\text{A}.2) \quad \int_0^t \frac{\mathcal{A}h(X(\omega, s))}{h(X(\omega, s))} \, ds < +\infty \quad \text{for all } t \geq 0 \text{ and } \omega \in \Omega.
\]

3.2.1 Note that by (\text{A}.1) process \( \{ \int_0^t \mathcal{A}g(X(s)) \, ds, t \geq 0 \} \) is a continuous process of finite variation. Thus process \( \{ g(X(t)), t \geq 0 \} \) is a semimartingale. In particular, by the fact that martingale \( \{ M^2(t), t \geq 0 \} \) is càdlàg (see definition of martingale in 2.2.1) this process is also càdlàg.

Remark 3.2.1 Note that if canonical process \( \{ X(t), t \geq 0 \} \) is càdlàg or continuous and state space \( E \) is Polish, then for continuous function \( g \in \mathcal{C}(E) \) we have
\[
\sup_{s \leq t} |g(X(\omega, s))| < +\infty \quad \text{for all } t \geq 0 \text{ and } \omega \in \Omega.
\]
In fact, by Ethier and Kurtz [48], page 152, if \( E \) is a Polish space and canonical \( \{ X(t), t \geq 0 \} \) is càdlàg (in particular continuous), then for each \( \omega \in D_E[0, +\infty) \) and each \( t \geq 0 \) there exists compact set \( A \) such that \( X(\omega, s) \in A \) for all \( 0 \leq s \leq t \). Thus by Maurin [87], Corollary II.6, page 46, we get needed assertion.

Thus to get assumptions (\text{A}.1) and (\text{A}.2) it suffices to assume that \( h \in \mathcal{C}(E) \) and restrict the domain of the extended generator to the family of functions \( g \in D(\mathcal{A}) \) such that \( \mathcal{A}g \in \mathcal{C}(E) \) or that \( |\mathcal{A}g| \leq g_A \in \mathcal{C}(E) \). One can also assume that
\[
\inf_x h(x) > 0
\]
and consider the the domain of the extended generator restricted to the family of bounded functions \( g \in B(E) \) such that \( \mathcal{A}g \in B(E) \). This is the most often assumptions made by Ethier and Kurtz [48].

For a positive function \( h \in D(\mathcal{A}) \) such that \( gh \in D(\mathcal{A}) \) for all \( g \in D(\mathcal{A}) \), define process

\[
N(t) = \frac{h(X(t))}{h(X(0))} \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right).
\]

By 3.2.1, process \( \{N(t), t \geq 0\} \) is càdlàg. The following result is similar to Ethier and Kurtz [48] Proposition 2.3, page 175.

**Lemma 3.2.1** Under assumptions (A.1) and (A.2) process \( \{M^h(t), t \geq 0\} \) is a local martingale if and only if process \( \{N(t), t \geq 0\} \) given in (3.2.2) is a local martingale.

**Proof.** Assume that process \( \{M^h(t), t \geq 0\} \) is a local martingale. We prove that then process \( \{N(t), t \geq 0\} \) is a local martingale. Using integration-by-parts formula (see 2.2.13) we have:

\[
dN(t) = \frac{1}{h(X(0))} \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \, dh(X(t))
- \frac{h(X(t_+)) (\mathcal{A}h)(X(t))}{h(X(0))} \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \, dt
+ \frac{1}{h(X(0))} \left[ d \left( h(X(t)), \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \right) \right]_t.
\]

From (A.2) process \( \{\int_0^t \frac{A h(X(s))}{h(X(s))} \, ds, t \geq 0\} \) is continuous of finite variation and by 2.2.15 we have that

\[
\left[ h(X(t)), \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \right]_t = 0.
\]

Hence

\[
dN(t) = \frac{1}{h(X(0))} \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \, dh(X(t))
- \frac{1}{h(X(0))} (\mathcal{A}h)(X(t)) \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \, dt
= \frac{1}{h(X(0))} \exp \left( - \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \, dM^h(t),
\]

(3.2.3)

Thus \( \{N(t), t \geq 0\} \) is a local martingale by 2.1.19 and 2.2.15.

Now, assume that \( \{N(t), t \geq 0\} \) is a local martingale. By (3.2.3) we have also that

\[
dM^h(t) = h(X(0)) \exp \left( \int_0^t \frac{(\mathcal{A}h)(X(s))}{h(X(s))} \, ds \right) \, dN(t),
\]

which completes the proof in the view of 2.2.15.
3.2.2 Assume that \( h \in D(\mathcal{A}) \) and (A.2) holds. Then by Lemma 3.2.1 process \( \{ N(t), t \geq 0 \} \) is a local martingale. We ask for conditions when process \( \{ N(t), t \geq 0 \} \) is a true martingale, which is needed in the sequel. Note that by 2.2.7, if we replace condition (A.2) by
\[
h \in B(E), \quad \frac{\mathcal{A}h}{h} \in B(E),
\]
then \( \{ N(t), t \geq 0 \} \) is a martingale. Note also that if
\[
\inf_x h(x) > 0
\]
and
\[
h, \quad \mathcal{A}h \in B(E),
\]
then \( \{ N(t), t \geq 0 \} \) is a martingale too. This is the assumptions most often used in Ethier Kurtz [48].

For a given function \( h \in D(\mathcal{A}) \), consider the perturbation of generator \( \mathcal{A} \) in the following sense:
\[
\hat{\mathcal{A}}g = \mathcal{A}(gh) - g\mathcal{A}h = Ag + \frac{< h, g >_\mathcal{A}}{h},
\]
where \( < h, g >_\mathcal{A} = \mathcal{A}(hg) - h\mathcal{A}g - g\mathcal{A}h \) and \( gh \in D(\mathcal{A}) \) for each \( g \in D(\mathcal{A}) \). In the proof of the main theorem of this section we need also following condition:
\[
(A.3) \quad \int_0^t |\hat{\mathcal{A}}g(X(\omega, s))| ds < +\infty \quad \text{for all } t \geq 0 \text{ and } \omega \in \Omega.
\]

We are now in the position to state the main result of this chapter called a perturbation theorem.

**Theorem 3.2.2** Consider Markov process \( \{ X(t), t \geq 0 \} \) on stochastic basis \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})\) fulfilling the standard setup. Let \( \mathcal{A} \) be the extended generator of this process with the domain \( D(\mathcal{A}) \) restricted to the functions fulfilling conditions (A.1) and (A.3). Let \( h \in D(\mathcal{A}) \) be a positive function fulfilling (A.2) for which process \( \{ N(t), t \geq 0 \} \) given in (3.2.2) is a martingale and \( gh \in D(\mathcal{A}) \) for each \( g \in D(\mathcal{A}) \). Define new probability measure \( \tilde{\mathbb{P}} \) by
\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = N(t).
\]

Then on probability space \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \tilde{\mathbb{P}})\), process \( \{ X(t), t \geq 0 \} \) is a Markov process with extended generator \( \hat{\mathcal{A}} \) given by (3.2.5) and the domain \( D(\hat{\mathcal{A}}) = D(\mathcal{A}) \).

**Proof.** Note that by assumption that function \( h \) is positive, we have that martingale \( \{ N(t), t \geq 0 \} \) is also positive. Moreover, exponential martingale \( \{ N(t), t \geq 0 \} \) is a multiplicative functional and hence from Theorem 3.1.7 process \( \{ X(t), t \geq 0 \} \) on the space \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \tilde{\mathbb{P}})\) is also a Markov process.

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We now show that operator $\hat{A}$ is an extended generator of process \( \{ X(t), t \geq 0 \} \) on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \hat{\mathbb{P})} \). Keeping condition (A.3) in mind, according to the definition of the extended generator it suffices to show that process

\[
\hat{M}^g(t) = g(X(t)) - \int_0^t (\hat{A}g)(X(s)) \, ds
\]

is a \( \hat{\mathbb{P}} \) - local martingale for \( g \in D(\mathcal{A}) \). Note that by Remark 3.2.1, \( \{ \hat{M}^g(t), t \geq 0 \} \) is a càdlàg process. Following Liptser and Shiryaev [84], Lemma 4.5.3, page 188, it suffices to show that the process \( \{ N(t) \hat{M}^g(t), t \geq 0 \} \) is a \( \mathbb{P} \) - local martingale. In fact, then for fundamental sequence of stopping times \( \{ \tau_n \} \) and \( s \leq t \leq \tau_n \) we have

\[
\hat{E} \left( \hat{M}^{\tau_n}(t) | \mathcal{F}_s \right) = \frac{1}{N^{\tau_n}(s)} \hat{E} \left( \hat{M}^{\tau_n}(t) N^{\tau_n}(t) | \mathcal{F}_s \right) = \frac{\hat{M}^{\tau_n}(s) N^{\tau_n}(s)}{N^{\tau_n}(s)} = \hat{M}^{\tau_n}(s) .
\]

Note that in that case we prove also inclusion \( D(\mathcal{A}) \subset D(\hat{A}) \). To prove the reverse inclusion it suffices to change measure by

\[
\frac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t} = (N(t))^{-1} ,
\]

which can be done by 3.1.1 and assumption of positivity of function \( h \), because then exponential martingale fulfills \( \mathbb{P}(N(t) > 0) = 1 \) for all \( t \geq 0 \).

By the integration-by-parts formula for semimartingales we get

\[
\hat{M}^g(t) N(t) = \int_0^t \hat{M}^g(s-) \, dN(s) + \int_0^t N(s-) \, d\hat{M}^g(s) + \left[ N, \hat{M}^g \right]_t .
\]

However, from definition of the process \( \hat{M}^g(t) \) and operator \( \hat{A} \) we have

\[
\hat{M}^g(t) = M^g(t) - \int_0^t \frac{<g, h >_A(X(s))}{h(X(s))} \, ds .
\]

Hence

\[
\hat{M}^g(t) N(t) = \int_0^t \hat{M}^g(s-) \, dN(s) + \int_0^t N(s-) \, dM^g(s) + \left[ N, \hat{M}^g \right]_t - \int_0^t \frac{<g, h >_A(X(s))}{h(X(s))} \, ds .
\]

The first two components are local martingales in view of 2.2.15 and 2.1.19. We now consider the third component. Let \( X^c, X^d \) be respectively continuous and pure-jump components of process \( \{ X(t), t \geq 0 \} \) and \( \Delta f(X(t)) = f(X(t)) - f(X(t-)) \). Then we have

\[
d \left[ N, \hat{M}^g \right]_t = d \left[ g(X), N \right]_t - d \left[ \int_0^t \hat{A}g(X(s)) \, ds, N(t) \right]_t .
\]

Note that by (A.3), process \( \left\{ \int_0^t \hat{A}g(X(s)) \, ds, t \geq 0 \right\} \) is continuous of finite variation and by 2.2.14

\[
\left[ \int_0^t \hat{A}g(X(s)) \, ds, N(t) \right]_t = 0 .
\]

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Thus
\[
d \left[ N, \hat{M}^d \right]_t = [g(X), N]_t
\]
which is by (3.2.3) equal to
\[
d \left[ g(X(t)), \int_0^t \frac{1}{h(X(s-))} N(s-) \, dh(X(s)) \right]_t
\]
\[-d \left[ g(X(t)), \int_0^t \frac{1}{h(X(s-))} \mathcal{A}h(X(s)) \, ds \right]_t.
\]
Note that
\[
\int_0^t \frac{1}{h(X(s-))} N(s-) \mathcal{A}h(X(s)) \, ds \leq \frac{1}{h(X(0))} \exp \left\{ \int_0^t \left| \frac{\mathcal{A}h(X(s))}{h(X(s))} \right| \, ds \right\} \int_0^t |\mathcal{A}h(X(s))| \, ds
\]
which is finite for all \( t \geq 0 \) by (A.1) and (A.2). Thus process
\[
\{ \int_0^t \frac{1}{h(X(s-))} N(s-) \mathcal{A}h(X(s)) \, ds, t \geq 0 \}
\]
is continuous of finite variation. Hence by 2.2.14 the second component is equal to 0. Then we have
\[
d[N, \hat{M}^d]_t = d \left[ g(X^c(t)) + g(X^d(t)), \int_0^t \frac{1}{h(X(s-))} N(s-) \, d(h(X^c(s)) + h(X^d(s))) \right]_t
\]
\[= \frac{1}{h(X(t-))} N(t-) \Delta g(X(t)) \Delta h(X(t)) = \frac{1}{h(X(t-))} N(t-) \, d \left[ g(X^d(t)), h(X^d(t)) \right]_t
\]
\[= \frac{1}{h(X(t-))} N(t-) \, d[g(X(t)), h(X(t))]_t.
\]
Using again integration-by-parts formula for semimartingales, one gets
\[
d \left[ N, \hat{M}^d \right]_t = \frac{1}{h(X(t-))} N(t-) \, d \left( g(X(t))h(X(t)) \right.
\]
\[-\int_0^t g(X(s-)) \, dh(X(s)) - \int_0^t h(X(s-)) \, dg(X(s)) \right)
\]
\[= \frac{1}{h(X(t-))} N(t-) \, dM^h(t) \quad (3.2.7)
\]
\[-\frac{1}{h(X(t-))} N(t-) g(X(t-)) \, dM^h(t) \quad (3.2.8)
\]
\[-\frac{1}{h(X(t-))} N(t-) h(X(t-)) \, dM^k(t) \quad (3.2.9)
\]
\[+ \frac{1}{h(X(t-))} N(t-) \left( (\mathcal{A}g)h(X(t)) - g(X(t-))(\mathcal{A}g)(X(t)) \right) \, dt.
\]
\[33\]
The first three components (3.2.7), (3.2.8) and (3.2.9) are local martingales by 2.2.15 and
the fourth (3.2.10), as a Riemann differential, equals to

$$< g, h >_A (X(t)) = h(X(t)) N(t -) dt,$$

which completes the proof in a view of (3.2.6).

\[ \square \]

To check assumptions of Theorem 3.2.2 we can use Proposition 3.2.3. In this
proposition \( \Omega \) is a space of càdlàg functions \( D_E[0, +\infty) \) or a space of continuous
functions \( C_E[0, +\infty) \) with values in a Polish space \( E \). Filtration \( \{ \mathcal{F}_t \} \) is the right-
continuous filtration \( \{ \mathcal{F}^X_t \} \).

**Proposition 3.2.3** Suppose that \( \{ X(t), t \geq 0 \} \) is the canonical process
\( (X(\omega, t) = \omega(t)) \) on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \mathbb{P}) \). Let also restrict the domain of the extended
generator \( D(\mathcal{A}) \) to the continuous functions \( g \in \mathcal{C}(E) \) such that \( \mathcal{A}g \in \mathcal{C}(E) \). If
additionally process \( \{ N(t), t \geq 0 \} \) given in (3.2.2) is a martingale for positive
function \( h \in D(\mathcal{A}) \) such that \( gh \in D(\mathcal{A}) \) for each \( g \in D(\mathcal{A}) \), then all assumptions of
Theorem 3.2.2 are fulfilled.

**Proof.** The assertion of this proposition follows from Remarks 3.1.2 and 3.2.1. In
fact, by assumptions of corollary functions \( g, \mathcal{A}g, \mathcal{A}(gh), \frac{\mathcal{A}(gh)}{h}, \frac{\mathcal{A}h}{h} \) are continuous
functions on \( E \). Thus conditions (A.1) and (A.2) hold. Condition (A.3) holds by
the following inequality

$$|\mathcal{A}g| \leq \frac{|\mathcal{A}(gh)|}{h} + \frac{|\mathcal{A}h|}{h}.$$

\[ \square \]

### 3.3 Piecewise deterministic process

In this section we apply the results of Theorem 3.2.2 to the so called **piecewise
deterministic Markov process** (PDMP) \( \{ X(t), t \geq 0 \} \). This is an important class
of Markov processes introduced by Davis [30]. In this section we show that such
a process is again a PDMP after the exponential change of measure and find its
parameters.

The evolution of a PDMP is a combination of deterministic motions and random
jumps. That is, the jump epochs and also the jump sizes are random in general but
the trajectories between the jumps are governed by a deterministic rule. We now
give a formal definition.

*State space:

We construct PDMP process \( \{ X(t), t \geq 0 \} \) on state space \( E \) which can be identified
with a subset of some Euclidean space \( \mathbb{R}^d \). More specific, let \( \mathcal{I} \) be an arbitrary
finite set which is not empty, and \( \{ d_\nu, \nu \in \mathcal{I} \} \) a family of natural numbers. For each
\( \nu \in \mathcal{I} \), let \( \mathcal{O}_\nu \) be an open subset of \( \mathbb{R}^{d_\nu} \). Put

$$E = \{(\nu, z) : \nu \in \mathcal{I}, z \in \mathcal{O}_\nu \}$$

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and by $\mathcal{B}(E)$ we denote the $\sigma$-algebra of Borel sets of $E$. Thus $\mathcal{I}$ is the set of possible different external states of the process and $\mathcal{O}_\nu$ is the state space if the external state is $\nu$. Moreover, $E$ is a Borel space with a metric $\rho$ such that $\rho$ restricted to any component $\mathcal{O}_\nu$ is equivalent to usual Euclidean metric (see Davis [30], page 58). Note also that $E$ can be identified with $\mathbb{R}^d$, where $d = \sum_{\nu \in I} d_\nu$.

- **Behavior between jumps:**

  Between jumps, the process $\{X(t), t \geq 0\}$ follows a deterministic path, while the external state is $\nu$, say. Starting at some point $z \in \mathcal{O}_\nu$, the development of the deterministic path is completely determined by its velocities at all points of $\mathcal{O}_\nu$, i.e. by a function $d_\nu(z) = (d_{\nu,1}(z), \ldots, d_{\nu,d_\nu}(z)) : \mathcal{O}_\nu \to \mathbb{R}^{d_\nu}$, provided some regularity assumptions are fulfilled. Such a function is called a *vector field*. Let

  $$X_\nu g(x) = \sum_{i=1}^{d_\nu} d_{\nu,i}(x) \frac{\partial g}{\partial x_i}(x)$$

be a *differential operator* describing *vector field*, which determine a deterministic path between jumps. That is, deterministic path $\phi_\nu(t,z)$, called the *integral curve*, is the solution to the differential equation

$$\frac{d}{dt} g(\phi_\nu(t,z)) = (X_\nu g)(\phi_\nu(t,z)),$$  \hspace{1cm} \phi_\nu(0,z) = z \hspace{1cm} (3.3.1)$$

for all differentiable functions $g : \mathcal{O}_\nu \to \mathbb{R}$. Notation $\phi_\nu(t,z)$ emphasize the dependence on initial point $(\nu,z) \in E$. We assume that there exists an unique solution of the above equation. To get this assumption fulfilled we may assume that functions $d_{\nu,i}(z)$ are *locally Lipschitz continuous functions* on $\mathbb{R}^{d_\nu}$, that is for each compact set $D \subset \mathbb{R}^{d_\nu}$ there exists constant $K_D$ such that

$$|d_{\nu,i}(x) - d_{\nu,i}(y)| \leq K_D \|x - y\|.$$  

We also assume that there is no explosions of the flow $\phi_\nu(\cdot,z)$, that is $|\phi_\nu(t,z)| < +\infty$ for all $t \geq 0$. We will write $X_\nu g(x)$ in place of the more cumbersome $X_\nu g_\nu(z)$ for $x = (\nu,z)$.

Denote by $\partial \mathcal{O}_\nu$ the boundary of $\mathcal{O}_\nu$ and let

$$\partial^* \mathcal{O}_\nu = \{z \in \partial \mathcal{O}_\nu : z = \phi_\nu(t,z') \text{ for some } (t,z') \in \mathbb{R}_+ \times \mathcal{O}_\nu\},$$

$$\Gamma = \{(\nu,z) \in \partial E : \nu \in \mathcal{I}, z \in \partial^* \mathcal{O}_\nu\},$$

$$t^*(\nu,z) = \sup\{t > 0 : \phi_\nu(t,z) \text{ exists and } \phi(t,z) \in \mathcal{O}_\nu\}.$$

The set $\Gamma$ is called the *active boundary of $E$* and denotes the set of boundary points of $E$, which can be reached from $E$ via integral curves within finite time and $t^*(\nu,z)$ is the time needed to reach the boundary from the point $(\nu,z)$. We will assume that $\phi_\nu(t^*(\nu,z),z) \in \Gamma$ if $t^*(\nu,z) < +\infty$, which means that the integral curves cannot "disappear" inside $E$.

- **The jump mechanism:**

  To define piecewise deterministic process one needs also measurable function describing *jump intensity*, that is a function

$$\lambda : E \to \mathbb{R}_+$$

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and a transition kernel
\[ Q : E \cup \Gamma \times \mathcal{B}(E) \to [0, 1] , \]
where \( Q(x, \cdot) \) is a probability measure for all \( x \in E \cup \Gamma \) and \( Q(\cdot, B) \) is measurable for all \( B \in \mathcal{B}(E) \). We assume that for each \( x = (\nu, z) \) there exists \( \epsilon(x) > 0 \) such that function \( s \to \lambda(\nu, \phi_b(s, z)) \) is integrable on \([0, \epsilon(x)) \). Note that \( \lambda \) can be interpreted as a "force of transition", whereas \( Q(x, \cdot) \) is the "after jump" distribution of a jump from the state \( x \) if \( x \in E \) or from the boundary point \( x \in \Gamma \). We also assume that
\[
\lim_{k \to +\infty} \sigma_k = +\infty \quad \text{a.e.} \quad (3.3.2)
\]
where \( \sigma_k \) is a \( k \)th moment of jump. To get this condition we can assume that \( \lambda(x) \) is bounded and one of the following conditions is fulfilled: \( t^*(x) = +\infty \) for each \( x \in E \), that is there are no active boundary points (e.g. \( \Gamma = \emptyset \)), or for some \( \epsilon > 0 \) we have \( Q(x, B_\epsilon) = 1 \) for all \( x \in \Gamma \), where \( B_\epsilon = \{ x \in E : t^*(x) \geq \epsilon \} \). The last condition means that the minimal distance between consecutive boundary hitting times is not smaller than \( \epsilon \) (see Davis [30], Proposition 24.6, page 60). Let PDMP starts at given state \( X(0) = x_0 \).

From Davis [30], Theorem 26.14, page 69, we get the following theorem.

**Theorem 3.3.1** The extended generator of piecewise deterministic process \( \{X(t), t \geq 0\} \) defined on a space \( (D_E[0, +\infty), \mathcal{F}, \{\mathcal{F}_t^X\}, \mathbb{P}) \) is
\[
\mathcal{A}g(x) = \mathcal{X}g(x) + \lambda(x) \int_E (g(y) - g(x))Q(x, dy) ,
\]
where \( x \in E \) and the domain \( \mathcal{D}(\mathcal{A}) \) consists of the functions \( g \) being restriction measurable function \( g^* : E \cup \Gamma \to \mathbb{R} \), where functions \( g^* \) satisfy three conditions given in

(i) for each \( (\nu, z) \in E \) the function \( t \to g^*(\nu, \phi_b(t, z)) \) is absolutely continuous on \( (0, t^*(\nu, z)) \),

(ii) the following boundary condition holds for each \( x \in \Gamma \):
\[
g^*(x) = \int_E g^*(y)Q(x, dy) ,
\]

(iii) for each \( t \geq 0 \)
\[
\mathbb{E} \left( \sum_{i=1}^n |g^*(X(\sigma_i)) - g^*(X(\sigma_i^-))| \right) < +\infty \quad \text{for } n = 1, 2, \ldots \quad (3.3.3)
\]

**Remark 3.3.1** Davis [30] proves Theorem 3.3.1 under the assumption of usual conditions, but by considering fundamental sequence of \( \mathcal{F}_t^X \)-stopping times \( \sigma_n \) by 2.3.4 the extended generator and the domain do not change under the augmentation.
Remark 3.3.2 If we replace condition (iii) in Theorem 3.3.1 by the condition
\[
\mathbb{E} \left( \sum_{\sigma_i \leq t} |g^*(X(\sigma_i)) - g^*(X(\sigma_i^-))| \right) < +\infty \quad \text{for all } t \geq 0,
\] (3.3.4)
then \( D(\mathcal{A}) \) is the domain of the full generator, that is process \( \{M^y(t), t \geq 0\} \) given in (3.2.1) is a martingale.

Remark 3.3.3 It may happen that the local characteristics of PDMP \( \{X(t), t \geq 0\} \) contain explicit time variation, i.e. take the form \( d_r(t, x), \lambda(t, x) \) and \( Q(t, x, dy) \), where \( x \in E \) and \( t \geq 0 \). A simple expedient removes this. We define an augmented PDMP \( \{\overline{X}(t) = (t, X(t)), t \geq 0\} \) on a state space \( \overline{E} = \mathbb{R}_+ \times E \) with local characteristics:
\[
\overline{d}_r(\overline{x}) = (1, d_r(t, x)), \quad \overline{\lambda}(\overline{x}) = \lambda(t, x), \quad \overline{Q}(\{t\} \times A, \overline{x}) = Q(t, x, A),
\] (3.3.5)
where \( \overline{x} = (t, x) \). Then by Theorem 3.3.1 the extended generator of the process \( \{(t, X(t)), t \geq 0\} \) is equal to
\[
(\mathcal{A}g)(t, x) = \frac{\partial}{\partial t} g(t, x) + \frac{\partial}{\partial x} g(t, x) + \lambda(t, x) \int_E (g(t, y) - g(t, x))Q(t, x, dy),
\]
where the domain \( D(\mathcal{A}) \) consist of the functions \( g(t, x) = \overline{g}(\overline{x}) : \mathbb{R}_+ \times E \to \mathbb{R} \) fulfilling conditions (i), (ii) and (iii) of Theorem 3.3.1 for jump intensity \( \overline{\lambda}(\overline{x}) \) and transition kernel \( \overline{Q} \) given in (3.3.5).

We can now prove the following theorem.

Theorem 3.3.2 Assume that all assumptions of Theorem 3.2.2 hold and define new probability measure \( \mathbb{P}^\nu \) by \( d\mathbb{P}^\nu/d\mathbb{P}_t = N(t), \) where \( \{N(t), t \geq 0\} \) is an exponential martingale (3.2.2). Then on the new probability space \((D_E[0, +\infty), F, \{\mathcal{F}_t^X\}, \mathbb{P}^\nu)\), process \( \{X(t), t \geq 0\} \) is still a piecewise deterministic Markov process with unchanged differential operator \( \mathcal{X} \) and the following jump intensity and transition kernel
\[
\hat{\lambda}(x) = \frac{\lambda(x)A(x)}{h(x)}, \quad \hat{Q}(x, dy) = \frac{h(y)}{A(x)}Q(x, dy),
\]
where \( A(x) = \int_E h(y)Q(x, dy) \).

Proof. Let \( x = (\nu, z) \), where \( z \in \mathbb{R}^d_v \). Then from Theorem 3.2.2 after the exponential change of measure the domain of the generator does not change and the generator \( \hat{\mathcal{A}} \) of process \( \{X(t), t \geq 0\} \) on the new probability space is equal to
\[
(\hat{\mathcal{A}}g)(x) = (\mathcal{A}g)(x) + \frac{<g, h>_A(x)}{h(x)} =
\]

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\[
= \sum_{i=1}^{d_{\nu}} d_{\nu,i} \frac{\partial g}{\partial z_i} (\nu, z) + \lambda(x) \int_E (g(y) - g(x)) Q(x, dy) \\
+ \sum_{i=1}^{d_{\nu}} d_{\nu,i} \frac{1}{h(\nu, z)} \frac{\partial g h}{\partial z_i} (\nu, z) + \lambda(x) \int_E \left( \frac{h(y) g(y)}{h(x)} - g(x) \right) Q(x, dy) \\
- \sum_{i=1}^{d_{\nu}} d_{\nu,i} \frac{\partial g}{\partial z_i} (\nu, z) - \lambda(x) \int_E (g(y) - g(x)) Q(x, dy) \\
- \sum_{i=1}^{d_{\nu}} d_{\nu,i} g(\nu, z) \frac{1}{h(\nu, z)} \frac{\partial h}{\partial z_i} (\nu, z) - \lambda(x) \int_E \left( \frac{g(x) h(y)}{h(x)} - g(x) \right) Q(x, dy) \\
= \sum_{i=1}^{d_{\nu}} d_{\nu,i} \frac{\partial g}{\partial z_i} (\nu, z) + \lambda(x) \int_E (g(y) - g(x)) \frac{h(y)}{h(x)} Q(x, dy) \\
= (A g)(x) + \frac{\lambda(x) A(x)}{h(x)} \int_E (g(y) - g(x)) \frac{h(y)}{A(x)} Q(x, dy).
\]

\[\square\]

**Remark 3.3.4** Note that function \( A(x) \) is chosen in such a way that \( \hat{Q}(x, \cdot) \) is a probability measure. Let us also note that all assumptions made during describing the piecewise deterministic Markov process are fulfilled besides condition \( \lim_{k \to +\infty} \sigma_k = +\infty \). This condition must be checked again after the exponential change of measure.

**Remark 3.3.5** We now discuss a particular case of importance. We consider

(B.1) canonical PDMP \( \{X(t), t \geq 0\} \) on stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_{t+}\}, \mathbb{P})\), that is \( X(\omega, t) = \omega(t) \),

(B.2)

- \( \mathcal{O}_\nu = \mathbb{R} \)

or

- \( \mathcal{O}_\nu = \mathbb{R}_+ \)

and in that case \( \phi_\nu(t, z) = 0 \) if and only if \( t = z = 0 \) (thus the state space \( E \) is Polish and there is no active boundary points: \( \Gamma = \emptyset \) and \( t^*(\nu, z) = +\infty \) for each \( \nu \) and \( z \)),

(B.3)

- \( \lim_{k \to +\infty} \sigma_k = +\infty \) a.e.
(B.4) \( A g(\nu, z) = d(\nu) \frac{d}{dz} g(z) \),

where functions \( d(\nu) \) are locally Lipschitz continuous; in particular functions \( d(\nu) \) can be constants,

(B.5) \( \lambda(\nu, \cdot) \in C(\mathcal{O}_\nu) \)

for all \( \nu \in \mathcal{I} \),

(B.6) \( x \rightarrow \int_E g(y) Q(x, dy) \)

is continuous function; this condition is fulfilled if measure \( Q \) depends on \( x = (\nu, z) \) only through \( \nu \).

Note that from Theorem 3.3.1 if \( g(\nu, \cdot) \in C^1_0(\mathcal{O}_\nu) \), then \( g \in D(\mathcal{A}) \). Thus we can restrict the domain of the extended generator to the set of bounded and continuously differentiable functions for each \( \nu \in \mathcal{I} \):

(B.7) \( g(\nu, \cdot) \in C^1_0(\mathcal{O}_\nu) \).

Moreover, if we take positive function \( h(\nu, \cdot) \in C^1(E) \) such that

(B.8) \( \frac{Ah}{h} \in B(E) \),

then by 3.2.2 process \( \{N(t), t \geq 0\} \) given in (3.2.2) is an exponential martingale. Note also that in that case \( gh(\nu, \cdot) \in C^1(\mathcal{O}_\nu) \) for each \( g(\nu, \cdot) \in C^1_0(\mathcal{O}_\nu) \). Then by (B.2)-(B.7) \( g \in C(E) \) and \( \mathcal{A} g \in C(E) \) (or in other notation \( g(\nu, \cdot) \in C(\mathcal{O}_\nu) \) and \( \mathcal{A} g(\nu, \cdot) \in C(\mathcal{O}_\nu) \)) for all \( g \in D(\mathcal{A}) \).

Thus by Proposition 3.2.3 and 3.2.2 all assumptions of Theorem 3.2.2 and hence Theorem 3.3.2 are fulfilled. This is the most often used setup in this dissertation.

This result was used in a few papers, e.g. in Palmowski and Rolski [100], Palmowski and Rolski [101]. Similar considerations can be found in Asmussen [7], Asmussen [2].
3.4 Diffusion process

In this section we assume that \( \{X(t), t \geq 0\} \) a Markov diffusion process. Although we will not study the fluid models governed by a diffusion, but some special cases were considered in literature; see e.g. Kulkarni and Rolski [75].

Let \( \{X(t), t \geq 0\} \) be a Markov diffusion process on a state space \( E = \mathbb{R}^d \) with given initial state \( X(0) \). We consider here \( \Omega = C_{\mathbb{R}^d}(0, + \infty) \) and \( X(\omega, t) = \omega(t) \) with right-continuous filtration \( \{\mathcal{F}_t^X\} \). We show that Cameron-Martin-Girsanov Theorem (see Stroock [117], Theorem 4.6, page 109) can be obtained from Theorem 3.2.2.

In fact, let consider measurable functions \( a = \{a_{ij}\} \in \mathbb{R}^d \rightarrow S^+(\mathbb{R}^d) \) and \( b = \{b_i\} : \mathbb{R}^d \rightarrow \mathbb{R}^d \), where \( S^+(\mathbb{R}^d) \) is a space of strictly positive definite symmetric matrices. We assume that functions \( b_i, a_{ij} \in C(\mathbb{R}^d) \) fulfill following conditions

\[
|a_{ij}(\mathbf{x})| \leq L(1 + ||\mathbf{x}||^2), \tag{3.4.1}
\]

\[
|b_i(\mathbf{x})| \leq L(1 + ||\mathbf{x}||) \tag{3.4.2}
\]

for some constant \( L \). Then the extended generator \( \mathcal{A} \) of the diffusion process \( \{X(t), t \geq 0\} \) is

\[
(\mathcal{A}g)(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) \frac{\partial^2 g(\mathbf{x})}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(\mathbf{x}) \frac{\partial g(\mathbf{x})}{\partial x_i}, \tag{3.4.3}
\]

where \( \mathbf{x} \in \mathbb{R}^d \). The family of twice continuously differentiable functions \( g \in C^2(\mathbb{R}^d) \) are included in the domain of this generator \( D(\mathcal{A}) \); see Karatzas and Shreve [61], Proposition 4.2, page 312, and Stroock-Varadhan Theorem (Rogers and Williams [106], Theorem 24.1, page 170). We restrict the domain \( D(\mathcal{A}) \) to the family of functions \( C^2(\mathbb{R}^d) \). To get Cameron-Martin-Girsanov Theorem we must put stronger than (3.4.1) assumption on function \( a \), namely that \( a_{ij}(x) \in C_0(\mathbb{R}^d) \).

Let \( c(\mathbf{x}) = (c_1(x_1), \ldots, c_d(x_d)) : \mathbb{R}^d \rightarrow \mathbb{R}^d \), where \( c_i(x) \in C_1^1(\mathbb{R}) \). Then we define process

\[
R(t) = \exp \left( \int_0^t c(X(s))dX(s) - \int_0^t c(X(s)) \cdot b(X(s))ds \right.
\]

\[
\left. - \frac{1}{2} \int_0^t (c, a \cdot c)_{\mathbb{R}^d}(X(s))ds \right), \tag{3.4.4}
\]

where

\[
\int_0^t c(X(s))dX(s) = \sum_{j=1}^{d} \int_0^t c_j(X_j(s))dX_j(s)
\]

and \((\cdot, \cdot)_{\mathbb{R}^d}\) is a scalar product on \( \mathbb{R}^d \).

3.4.1 Process \( \{R(t), t \geq 0\} \) is a martingale; see e.g. Stroock [117], Theorem 4.6, page 109, and Rogers and Williams [106], Theorem 27.1, page 177. Note that the assumption that \( b_i \) are bounded functions is not necessary to get this result. In fact,
by Stroock [117], page 75, and Liptser and Shiryaev [84], Theorem 7 (4), page 94, we have
\[
< \int_0^t c(X(s)) \, dX(s) - \int_0^t c(X(s)) \cdot b(X(s)) \, ds >_1 = < \int_0^t c(X(s)) \, dX(s) - \int_0^t b(X(w)) \, dw >_1 = \int_0^t (c(a \cdot c))_t(X(s)) \, ds \leq K_t
\]
for some constant $K_t$. Thus from Rogers and Williams [106], Theorem 37.8, page 77, process $\{R(t), t \geq 0\}$ is the martingale.

Define function $h : \mathbb{R}^d \rightarrow \mathbb{R}_+$ by
\[
h(x) = \prod_{j=1}^d \exp \left( \int_{x_{0,j}}^{x_j} c_j(z) \, dz \right) = \exp \left( \sum_{j=1}^d \int_{x_{0,j}}^{x_j} c_j(z) \, dz \right).
\]
Let $x_0 = (x_{0,1}, \ldots, x_{0,d})$, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. Denote
\[
\int_{x_0}^x c(y) \, dy = \sum_{j=1}^d \int_{x_{0,j}}^{x_j} c_j(y_j) \, dy_j.
\]
Hence in short
\[
h(x) = \exp \left( \int_{x_0}^x c(y) \, dy \right). \tag{3.4.5}
\]
When $d = 1$, then simply
\[
h(x) = e^{\int_{x_0}^x c(y) \, dy}.
\]
Note that function $h(x)$ fulfills following equation:
\[
\frac{1}{h(x)} \frac{\partial h(x)}{\partial x_j} = c_j(x_j). \tag{3.4.6}
\]
Moreover, for function $h$ we have that $h \in D(A) = C^2(\mathbb{R}^d)$. Thus by Lemma 3.2.1 process $\{N(t), t \geq 0\}$ given in (3.2.2) is a local martingale. It can be proved a stronger result.

**Lemma 3.4.1** Process $\{R(t), t \geq 0\}$ given in (3.4.4) is a mean-one exponential martingale $\{N(t), t \geq 0\}$ given in (3.2.2) for a function $h$ defined by (3.4.5).

**Proof.** We prove that $N(t) = R(t)$ for all $t \geq 0$. Then by 3.4.1 local martingale $\{N(t), t \geq 0\}$ is a true martingale.
From (3.4.3)
\[
\frac{A h(x)}{h(x)} = \frac{1}{h(x)} \left( \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 h(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial h(x)}{\partial x_i} \right) = \frac{1}{h(x)} \left( \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial c_j(x_j) h(x)}{\partial x_i} + \sum_{i=1}^{d} b_i(x) c_i(x_i) h(x) \right) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial c_j(x_j)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) c_i(x_i) c_j(x_j) + \sum_{i=1}^{d} b_i(x) c_i(x_i) \right) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial c_j(x_j)}{\partial x_i} + \frac{1}{2} (c, a \cdot c)_{\mathbb{R}^d}(x) + b(x) \cdot c(x) .
\]

But
\[
\frac{\partial c_j(x_j)}{\partial x_i} = 0 \quad \text{for} \quad i \neq j .
\]

Hence
\[
\frac{A h(x)}{h(x)} = \frac{1}{2} \sum_{i=1}^{d} a_{ii}(x) c'_i(x_i) + \frac{1}{2} (c, a \cdot c)_{\mathbb{R}^d}(x) + b(x) \cdot c(x) , \quad (3.4.7)
\]
where \( c'_i(x) = \frac{d}{dx} c_i(z) \big|_{z=x} \). Then
\[
N(t) = \exp \left( \int_{X(0)}^{X(t)} c(x) \, dx \right) \exp \left( - \int_{0}^{t} \sum_{i=1}^{d} a_{ii}(X(s)) c'_i(X_i(s)) \, ds \right) \cdot \exp \left( - \frac{1}{2} \int_{0}^{t} (c, a \cdot c)_{\mathbb{R}^d}(X(s)) \, dX(s) - \int_{0}^{t} b(X(s)) \cdot c(X(s)) \, ds \right) .
\]

Thus it suffices to show that
\[
\int_{X(0)}^{X(t)} c(x) \, dx = \int_{0}^{t} c(X(s)) \, dX(s) + \frac{1}{2} \int_{0}^{t} \sum_{i=1}^{d} a_{ii}(X(s)) c'_i(X_i(s)) \, ds \quad (3.4.8)
\]
or in another notation that
\[
\sum_{i=1}^{d} \int_{X_i(0)}^{X_i(t)} c_i(z) \, dz = \sum_{i=1}^{d} \int_{0}^{t} c_i(X_i(s)) \, dX_i(s) + \frac{1}{2} \int_{0}^{t} \sum_{i=1}^{d} a_{ii}(X(s)) c'_i(X_i(s)) \, ds . \quad (3.4.9)
\]
We prove that
\[
\int_{X_i(0)}^{X_i(t)} c_i(z) \, dz = \int_{0}^{t} c_i(X_i(s)) \, dX_i(s) + \frac{1}{2} \int_{0}^{t} a_{ii}(X(s)) c'_i(X_i(s)) \, ds . \quad (3.4.10)
\]
From Stroock [117], page 75, we have
\[
\{ d < X(t) - \int_{0}^{t} b(X(s)) \, ds > \} = \{ a_{ij}(X(t)) dt \} .
\]
In particular
\[ d < X(t) - \int_0^t b_i(X(s)) \, ds >_i a_i(X(t)) \, dt. \]
Now, (3.4.10) follows directly from Ito’s formula (see 2.2.12) for the function
\[ f(x) = \int_{x_0}^x c_i(z) \, dz. \]
\[
\int_{X(t)}^{X_i(0)} c_i(z) \, dz = \\
\quad = \int_0^t c_i(X_i(s)) \, dX_i(s) + \frac{1}{2} \int_0^t c'_i(X_i(s)) \, d < X_i(s) > \\
\quad = \int_0^t c_i(X_i(s)) \, dX_i(s) \\
\quad + \frac{1}{2} \int_0^t c'_i(X_i(s)) \, d < X_i(s) > - \int_0^s b_i(X(v)) \, dv > \\
\quad = \int_0^t c_i(X_i(s)) \, dX_i(s) + \frac{1}{2} \int_0^t a_i(X(s)) c'_i(X_i(s)) \, ds,
\]
where (3.4.11) follows by the fact that process \( \{ \int_0^t b_i(X(s)) \, ds, t \geq 0 \} \) has finite variation \( (b_i \) is a continuous function). \( \square \)

3.4.2 Note that for diffusion process \( \{ X(t), t \geq 0 \} \) all assumptions of Theorem 3.2.2 hold. In fact, by Lemma 3.4.1 process \( \{ N(t), t \geq 0 \} \) is a martingale. Moreover, for each \( g \in D(A) = C^2(\mathbb{R}^d) \) we have \( \mathcal{A}g \in C(\mathbb{R}^d) \) and \( gh \in C^2(\mathbb{R}^d) \). Thus by Proposition 3.2.3 all assumptions of Theorem 3.2.2 are fulfilled.

As an illustration of the derived Theorem 3.2.2 we give another proof of Cameron-Martin-Girsanov Theorem. In this theorem we consider new probability measure \( \tilde{\mathbb{P}} \) defined by \( d\tilde{\mathbb{P}}_t/d\mathbb{P}_t = R(t) \), where martingale \( \{ R(t), t \geq 0 \} \) is given by (3.4.4).

**Theorem 3.4.2** On the new probability space \( (\mathcal{C}[0, +\infty), \mathcal{F}, \{ \mathcal{F}_t \}, \tilde{\mathbb{P}}) \) the process \( \{ X(t), t \geq 0 \} \) is a diffusion process with parameters
\[ a = a, \quad \tilde{b} = b + a \cdot c. \]

**Proof.** By 3.4.2 all assumptions of Theorem 3.2.2 are fulfilled. From Lemma 3.4.1 and Theorem 3.2.2 we have that \( D(A) = D(\mathcal{A}) = C^2(\mathbb{R}^d) \) and
\[
(\mathcal{A}g)(x) = \\
\quad = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial g(x)}{\partial x_i} \\
\quad + \frac{1}{2} \sum_{i,j=1}^d \frac{a_{ij}(x)}{h(x)} \frac{\partial^2 g(x)}{\partial x_i \partial x_j} h(x) + \sum_{i=1}^d \frac{b_i(x) \partial g(x)}{h(x)} \frac{h(x)}{\partial x_i}
\]

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\[-\frac{1}{h(x)} \frac{1}{2} \sum_{i,j=1}^{d} h(x)a_{ij}(x) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} - \frac{1}{h(x)} \sum_{i=1}^{d} h(x)b_i(x) \frac{\partial g(x)}{\partial x_i} \]

\[-\frac{g(x)}{h(x)} \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 h(x)}{\partial x_i \partial x_j} - \frac{g(x)}{h(x)} \sum_{i=1}^{d} b_i(x) \frac{\partial h(x)}{\partial x_i}.\]

But

\[\frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 h(x)g(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial h(x)g(x)}{\partial x_i} \]

\[= \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_i} \left( \frac{\partial h(x)g(x)}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial h(x)g(x)}{\partial x_i} \]

\[= \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_i} \left( h(x) \frac{\partial g(x)}{\partial x_j} + g(x) \frac{\partial h(x)}{\partial x_j} \right) \]

\[+ \sum_{i=1}^{d} b_i(x) \left( h(x) \frac{\partial g(x)}{\partial x_j} + g(x) \frac{\partial h(x)}{\partial x_i} \right) \]

\[= h(x) \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + g(x) \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \]

\[+ \frac{1}{2} \sum_{i=1}^{d} \frac{\partial g(x)}{\partial x_i} \sum_{j=1}^{d} a_{ij}(x) \frac{\partial h(x)}{\partial x_j} + \frac{1}{2} \sum_{j=1}^{d} \frac{\partial g(x)}{\partial x_j} \sum_{i=1}^{d} a_{ij}(x) \frac{\partial h(x)}{\partial x_i} \]

\[+ h(x) \sum_{i=1}^{d} b_i(x) \frac{\partial g(x)}{\partial x_i} + g(x) \sum_{i=1}^{d} b_i(x) \frac{\partial h(x)}{\partial x_i}.\]

Hence using the fact that \(a\) is symmetric matrix we have

\[(\mathcal{A}g)(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left( b_i(x) + \sum_{j=1}^{d} a_{ij}(x) \frac{1}{h(x)} \frac{\partial h(x)}{\partial x_j} \right) \frac{\partial g(x)}{\partial x_i}.\]

Thus after the exponential change of measure the diffusion process \(\{X(t), t \geq 0\}\) change its infinitesimal variance and drift functions in the following way

\[\tilde{a}_{ij}(x) = a_{ij}(x)\]

and

\[\tilde{b}_i(x) = b_i(x) + \sum_{j=1}^{d} a_{ij}(x) \frac{1}{h(x)} \frac{\partial h(x)}{\partial x_j},\]

or in the other notation by (3.4.6)

\[\tilde{a} = a\]

and

\[\tilde{b} = b + a \cdot c.\]

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Example 3.4.1 Take $d = 1$ and $c_i(x) = -\alpha$. In this case diffusion process with infinitesimal variance function $a(x)$ and infinitesimal drift function $b(x)$ after the exponential change of measure change its parameters on
\[ \hat{a}(x) = a(x), \quad \hat{b}(x) = b(x) - \alpha a. \]
In particular, if we consider Wiener process $\{B(t), t \geq 0\}$, that is
\[ a(x) = 1, \quad b(x) = 0, \]
then after exponential change of measure $\{B(t), t \geq 0\}$ become a Gaussian process with $a(x) = 1$ and $b(x) = -\alpha$.

Example 3.4.2 The second example concerns the so called Gauss-Markov process (see Karatzas and Shreve [61], page 355). The Gauss-Markov process $\{X(t), t \geq 0\}$ is the solution of stochastic differential equation
\[ dX(t) = BX(t)dt + AdB(t), \]
where $B$, $A$ are $d \times d$ matrices and $\{B(t), t \geq 0\}$ is a $d$-dimensional Brownian motion. We assume that all eigenvalues of $B$ have negative real parts (so is nonsingular). Then by Stroock [117], Theorem 2.6, page 91, $\{X(t), t \geq 0\}$ is a diffusion with
\[ a(x) = AA^T, \quad b(x) = Bx, \]
where $x = (x_1, \ldots, x_d)$. After exponential change of measure process $\{X(t), t \geq 0\}$ on new probability space is a diffusion process with parameters
\[ \hat{a}(x) = AA^T, \quad \hat{b}(x) = Bx + AA^Tc(x), \]
where $c(i, x) = c_i(x) \in C_b^1(\mathbb{R})$. In particular, when $d = 1$, $B = -\alpha \in \mathbb{R}$ and $A = \sqrt{\alpha} \in \mathbb{R}_+$, then Gauss-Markov $\{X(t), t \geq 0\}$ is a one-dimensional Ornstein-Uhlenbeck process. Taking $c(x) = c \in \mathbb{R}$ after exponential change of measure $\{X(t), t \geq 0\}$ is a Ornstein-Uhlenbeck process with changed drift from $-\alpha$ to $-\alpha + a \cdot c$; see Simonian [115], Kulkarni and Rolski [75].
Chapter 4

Single source fluid model

In this chapter we develop exponential bounds for the distribution of the buffer content process in the fluid model whose input traffic is modulated by a superposition of semi-Markov processes. We will use the technique of the \textit{exponential change of measure} described in previous chapters. The considered model generalizes the one generated by a superposition of \textit{on-off} processes (see Palmowski and Rolski [101]). Another special case are tandem fluid models described in Sections 6.3 and 6.4.

In Sections 4.1 and 4.2 we define a fluid model and present general facts. In Section 4.3 we prove some properties of semi-Markov processes useful in further considerations. In Sections 4.4 and 4.5 we present and prove exponential bounds for the probability of a buffer overflow in a single fluid model. In Section 4.6 we give further generalizations of Theorem 4.4.1.

4.1 The Model

A stochastic fluid model is a stochastic system, where the input is described as a continuous flow of fluid that enters into buffer according to a randomly varying rate and leave the system with a constant rate. Such models are motivated as approximations to discrete queueing models of manufacturing systems, high speed data networks, etc.

In this dissertation we consider fluid models in which the fluid enter the buffer from \( N \) independent sources. The traffic generated from \( k \)-th \((k = 1, \ldots, N)\) source is driven by a stationary \textit{environment process} \( \{Z^{*,k}(t), t \in \mathbb{R}\} \). That is, when the environment process is in state \( m \), fluid is generated into the buffer at a rate \( r^{k}_m \).

Fluid is removed from the buffer by a channel with constant capacity \( c \); see Figure 1, page 2.

Although fluid models are the special case of models considered in dam theory (see Moran [92]), there are no general theorems in dam theory which can be applied to get results presented in this dissertation. More recently fluid models have been used in modeling and analysis of \textit{high-speed telecommunication networks} (mainly \textit{asynchronous transfer mode} (ATM)). In ATM network information flows in the network in the form of 53-byte packets or cells. The high speed (e.g. 155-622 Mbits/sec) of the ATM network implies that it can transmit millions of cells per second. This makes fluid models useful in describing the flow of cells. Following
the large literature using fluid models for communication systems, we analyze the packetized traffic by approximating it by fluids. Anick et al [1], Elwalid and Mitra [44], Chen and Yao [26] and [27], Ott and Shanthikumar [98], Harrison [55], Chen and Mandelbaum [25], etc., demonstrate how to convert any discrete arrival system into a fluid system and apply the fluid model results. Even in a network where burst are divided into larger packets than in the ATM network, the fluid approximation reveals some essential aspects of the way quality of service depends on the nature of offered traffic. Therefore our results can be applied to a wide variety of networks, not just high-speed ATM networks.

Typically, there is considered a single buffer which receives data from several independent sources, each source switching between on and off states according to an alternating renewal Markov process, see Anick et al [1], Kosten [72]. That is, \( r_0^k = 0, r_1^k = r \) and each \( \{ Z^{*,k}(t), t \in \mathbb{R} \} \) \( k = 1, \ldots, N \) is an alternating on-off process, where on and off times are exponential distributed; see Section 6.2. Such a model is called the AMS model. Palmowski and Rolski [101] consider fluid model for more general on-off input, in particular when on and off times are phase-type; see Section 6.1. Such models are called on-off fluid models. In this dissertation we consider a more general case, when \( \{ Z^{*,k}(t), t \in \mathbb{R} \} \) is a semi-Markov process.

The high-speed networks are expected to handle a wide variety of traffic, that is a cell may carry one of different types of information: voice, video, data, etc. We assume that each source produces a single class of traffic but different sources may produce traffic belonging to distinct classes. In other words, environment processes \( \{ Z^{*,k}(t), t \in \mathbb{R} \} \) \( k = 1, \ldots, N \) may be different. Such a case we call a heterogeneous one. If all sources generates the fluid according to the same law, then we call it a homogeneous case. Recall that sources are independent.

The optimal design and admission control problems frequently require computing a certain Quality of Service (QoS) or Grade of Service that the network users need to be assured. This QoS is based mainly on cell-loss probability and delay. The granularity of real arrival process introduces a supplementary delay but this is relatively small, especially with the short ATM cells. Therefore we mainly focus on the cell-loss probability aspect where cell loss occurs whenever a buffer overflows in a steady-state. In the ATM network, the buffer has finite capacity, say \( B \), very large in comparison to cell-size. If \( X_B(t) \) is an amount of fluid in the buffer at time \( t \), then buffer overflows happens, when \( X_B(t) = B \). Thus the stationary cell-loss probability is equal to

\[
P(X_B^* = B) = \lim_{t \to \infty} \P(X_B(t) = B)
\]

In a great majority of papers concerning fluid models, instead of finite buffer fluid models, there are considered infinite capacity buffer fluid models. Let \( X(t) \) be the amount of fluid in the buffer at time \( t \) in the infinite capacity buffer fluid model with the same input process like in the infinite buffer case, and let \( X^* \) be the steady-state buffer content in this model. Then \( X_B(t) \leq X(t) \) for each realization of the buffer content process; see Figure 2.
Hence
\[ \mathbb{P}(X_B^* = B) \leq \mathbb{P}(X^* \geq B) . \]
In this dissertation we consider only the case, when \( \mathbb{P}(X^* \geq B) = \mathbb{P}(X^* > B) \). Thus an upper bound for \( \mathbb{P}(X^* > B) \) is also an upper bound for the cell-loss probability. Moreover, in some cases we also have
\[
\lim_{B \to +\infty} \frac{\mathbb{P}(X_B^* = B)}{\mathbb{P}(X^* > B)} = 1 .
\]
(4.1.1)

For example, following Mitra [90], [91] in the AMS fluid model we have
\[
\mathbb{P}(X_B^* \geq x) = \sum_{i=1}^{m} a_i(B)e^{zi}, \quad 0 \leq x \leq B ,
\]
(4.1.2)
\[
\mathbb{P}(X^* > x) = \sum_{i=1}^{m} a_i e^{zi}, \quad x \geq 0
\]
(4.1.3)
for certain quantities \( a_i(B), a_i \) and \( z_0, \ldots, z_{m-1}, \ldots, z_m \) fulfilling
\[
\Re(z_m) \leq \Re(z_{m-1}) \leq \ldots \leq \Re(z_1) \leq z_0 = 0 \leq \Re(z_m) \leq \Re(z_{m-1}) \leq \ldots \leq \Re(z_{m+1}) ,
\]
which can be found by solving given system of equations. We can observe, by (4.1.2) and (4.1.3), that \( a_i(B) \to a_i \) for \( i = 1, \ldots, m \) and \( a_i(B) \exp\{z_i B\} \to 0 \) for \( i = m + 1, \ldots, m \) when \( B \to +\infty \). Hence

\[
\lim_{B \to +\infty} \frac{\mathbb{P}(X^*_B = B)}{a_1 e^{z_1 B}} = \lim_{B \to +\infty} \frac{\mathbb{P}(X^* > B)}{a_1 e^{z_1 B}} = 1 ,
\]

which yields equation (4.1.1). Thus knowledge of the asymptotic tail of distribution function of \( X^* \) in an infinite capacity buffer fluid model seems to be useful in estimating of the cell-loss probability. In this dissertation we will study the following function:

\[
\Psi(x) = \mathbb{P}(X^* > x) = \lim_{t \to +\infty} \mathbb{P}(X(t) > x) ,
\]

which is called the buffer overflow probability.

The function \( \Psi(x) \) is known only for few fluid models, in particular for the AMS fluid model. However in practice, even for these models, computations is frequently cumbersome. Therefore most of results concern different asymptotics of \( \Psi(x) \). We consider here only the case, when function \( \Psi(x) \) decreases exponentially fast to 0. That is, there exist constants \( C^* \) and \( C_* \) and quantity \( \eta > 0 \) called the adjustment coefficient, such that

\[
C_* e^{-\eta x} \leq \Psi(x) \leq C^* e^{-\eta x} .
\]  

(4.1.4)

Notice that it may happen that inequalities (4.1.4) do not hold. For instance, if on time in the single on-off fluid model has a subexponential distribution function, then bounds of type like in (4.1.4) are impossible; see Rolski et al [107], Boxma [21]. For exponential type fluid models the following types of results are studied:

- **The logarithmic asymptotics:**

\[
\lim_{x \to +\infty} \frac{1}{x} \log \Psi(x) = -\eta \quad \text{for some } 0 < \eta < +\infty ;
\]

see Botvich and Duffield [20], Duffield [40], Palmowski and Rolski [101] and Duffield and O’Connel [39] for results for AMS and on-off fluid model.

- In a few models can be obtained following *exact asymptotics* of function \( \Psi(x) \) for \( x \to +\infty \)

\[
\Psi(x) = C e^{-\eta x} + o(e^{-\eta x}) ,
\]

where prefactor \( C \) is known. However, for general fluid model such a results are difficult to prove, in particular computing constant \( C \); see Anick et al [1], Mitra [90] and [91].

- For majority of fluid models we can only find the *exponential upper bound* of \( \Psi(x) \)

\[
\Psi(x) \leq C^* e^{-\eta x} ,
\]

where constant \( C^* \) and adjustment coefficient \( \eta \) are given explicitly, or *two-sided exponential bounds* given in (4.1.4):

\[
C_* e^{-\eta x} \leq \Psi(x) \leq C^* e^{-\eta x} .
\]

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One of the methods to analyze the buffer content process and getting bounds (4.1.4) is using the technique of the exponential change of measure described in previous chapter. In this dissertation using this method we get two-sided exponential inequalities (4.1.4) for general fluid models driven by superposition of semi-Markov processes. Another way of getting inequalities (4.1.4) is effective bandwidth approximation; see Elwalid et al [46] and Elwalid and Mitra [47]. The effective-bandwidth methodology, can be used only when the buffer sizes are very large, and the tail probabilities are small. In this dissertation we redress these shortcomings to all buffer sizes and the tail probabilities.

At the beginning we get exponential bounds for probability of the buffer overflow $\Psi(x) = \mathbb{P}(X^* > x)$ in a single source fluid model.

### 4.2 Single source model

In this chapter we consider a single fluid source that input traffic into an infinite capacity buffer. The traffic from the source is driven by stationary process $\{Z^*(t), t \in \mathbb{R}\}$, that is if $Z^*(t) = m$, then fluid is generated at rate $r_m$. Fluid is removed from the buffer by a channel with constant capacity $c$. Let $X(t)$ be the amount of fluid in the buffer at time $t$; see Figure 3.

**Figure 3.**
Note that the buffer content process \( \{X(t), t \geq 0\} \) is governed by the equation
\[
\frac{dX(t)}{dt} = \begin{cases} 
    rZ^*(t) - c, & \text{for } X(t) > 0 \\
    (rZ^*(t) - c)_+, & \text{for } X(t) = 0,
\end{cases}
\]
where \((x)_+ = \max\{x, 0\}\). The solution is given by (see Kulkarni and Rolski [75])
\[
X(t) = \sup_{0 \leq s \leq t} \left( L(t), \int_0^t (rZ^*(s) - c) \, ds \right),
\]
where
\[
L(t) = X(0) + \int_0^t (rZ^*(s) - c) \, ds.
\]
Denote by \(X^*\) the steady-state buffer content, that is
\[
\mathbb{P}(X^* > x) = \lim_{t \to +\infty} \mathbb{P}(X(t) > x).
\]
By Theorem 2.1 from Kulkarni and Rolski [75] we get the following proposition.

**Proposition 4.2.1** If \(\{Z^*(t), t \in \mathbb{R}\}\) is a stationary, ergodic such that
\[
rE Z^*(t) < c,
\]
then the stationary distribution \(X^*\) of the content process has the following representation:
\[
X^* \overset{\mathbb{P}}{=} \sup_{t \leq 0} \int_0^t (rZ^*(s) - c) \, ds.
\]

Consider now the reversed càdlàg version \(\{Z(t), t \in \mathbb{R}\}\) of the environment process \(\{Z^*(t), t \in \mathbb{R}\}\), that is the càdlàg version of process \(\{Z^*(-t), t \in \mathbb{R}\}\).

We then get following corollary of Proposition 4.2.1.

**Corollary 4.2.2**
\[
X^* \overset{\mathbb{P}}{=} \sup_{t \geq 0} \int_0^t (rZ^*(s) - c) \, ds.
\]

In this dissertation we model \(\{Z^*(t), t \in \mathbb{R}\}\) as a canonical stationary, ergodic semi-Markov process (SMP) on the state space \(\{1, 2, \ldots, \ell\}\) fulfilling condition (4.2.1).

In the proof of the main theorem concerning fluid models we use the reversed SMP. Therefore in the next section we find conditions under which process \(\{Z(t), t \in \mathbb{R}\}\) is also a SMP. In further sections we find two-sided exponential bounds for
\[
\Psi(x) = \mathbb{P}(\sup_{t \geq 0} \int_0^t (rZ^*(s) - c) \, ds > x),
\]
where \(\{Z(t), t \geq 0\}\) is a SMP.
4.3 Semi-Markov process

4.3.1 General

Consider \( \{(Z_n^*, S_n^*), n \geq 0\} \) be a Markov renewal sequence, that is:

- \( S_{n+1}^* \geq S_n^*, Z_n^* \in \{1, \ldots, l\} \) and
- for all \( n \geq 0 \)

\[
\begin{align*}
\mathbb{P}(Z_{n+1}^* = j, S_{n+1}^* - S_n^* \leq x | Z_n^* = i, S_n^*, Z_{n-1}^*, \ldots, Z_0^*, S_0^*) \\
= \mathbb{P}(Z_{n+1}^* = j, S_{n+1}^* - S_n^* \leq x | Z_n^* = i) \\
= \mathbb{P}(Z_1^* = j, S_1^* \leq x | Z_0^* = i) = F_{ij}(x).
\end{align*}
\]

(4.3.1)

In order to define stationary SMP it is necessary to have the process 'in progress' at time \( t = 0 \). That is, \( S_0^* \) is a nonnegative random variable, whose distribution will be a part of initial measure for SMP and will be specified in 4.3.2 for stationary SMP. Define counting process

\[
N(t) = \sup\{n \in \mathbb{Z} : S_n^* \leq t\}
\]

and then process

\[
Z^*(t) = \begin{cases} Z^*(0), & 0 \leq t < S_0^*, \\
Z_{N(t)}, & t \geq S_0^*.
\end{cases}
\]

\( \{Z^*(t), t \geq 0\} \) is called a semi-Markov (SMP) process. Matrix

\[
F^*(x) = \{F_{ij}^*(x)\}_{i,j=1,\ldots,l} = \{\mathbb{P}(S_1^* \leq x; Z_1^* = j | Z_0^* = i)\}_{i,j=1,\ldots,l}.
\]

(4.3.3)

is called the kernel of semi-Markov process \( \{Z^*(t), t \geq 0\} \). Let \( \mathbb{P}(Z_0^* = j | Z^*(0) = i) = F_{ij}^*(+\infty) \). We assume that all \( F_{ij}^*(x) \) are absolutely continuous. Then following Kulkarni [76], page 511, each SMP is specified by kernel distributions and initial distribution \( (Z^*(0), S_0^*) \). Moreover, realizations of such defined SMP are càdlàg. Note that \( S_0^* \) is the time of the \( n \)th jump epoch in the SMP \( \{Z^*(t), t \geq 0\} \). \( Z_n^* \) can be interpret as the state of the SMP immediately after \( n \)th jump:

\[
Z_n^* = Z^*(S_n^*+).
\]

\( \{Z_n^*, n \in \mathbb{Z}_+\} \) is a discrete time Markov chain (DTMC), where \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). We assume that this DTMC is irreducible and recurrent (then SMP is called irreducible and recurrent). Denote transition probability matrix of underlying DTMC \( \{Z_n^*, n \in \mathbb{Z}_+\} \) by

\[
P^* = F^*(+\infty) = \{p_{ij}^*\}_{i,j=1,\ldots,l}
\]

(4.3.4)

and its stationary distribution by

\[
\pi_i = \lim_{n \to \infty} \mathbb{P}(Z_n^* = i),
\]

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which is given by the unique non-negative solution to
\[ \pi = \pi P^* \quad \text{and} \quad \sum_{i=1}^\ell \pi_i = 1 . \]  
(4.3.5)

We denote by
\[ \tau_i^* = \mathbb{E}(S_t^* | Z_0^* = i) < +\infty \]
the expected time that SMP spends at state \( i \). Denote
\[ F_i^*(x) = \sum_{j=1}^\ell F_{ij}^*(x) . \]

4.3.1 Let \( T_j \) be the first time after \( t = S_0^* \) at which the SMP \( \{ Z^*(t), t \geq 0 \} \) enters the state \( j \) and
\[ G_{ij}(t) = \mathbb{P}(T_j \leq t | Z_0^* = i) . \]
If \( \{ G_{ii}(x) \} \) is nonlattice, then SMP is said to be aperiodic. If \( G_{ii}(+\infty) = 1 \) and \( \int_{0}^{+\infty} x dG_{ii}(x) = 1 \), then we say that SMP is positive recurrent.

4.3.2 We will consider only the SMP \( \{ Z^*(t), t \geq 0 \} \) which is irreducible, aperiodic and positive recurrent. In that case we have
\[ p_i = \lim_{t \to +\infty} \mathbb{P}(Z^*(t) = i) = \frac{\pi_i \tau_i^*}{\sum_{m=1}^\ell \pi_m \tau_m^*} . \]
(4.3.6)
Moreover, if
\[ \mathbb{P}(S_0^* \leq x, Z^*(0) = i) = \int_0^x \pi_i(x) \, dx , \]
where
\[ \pi_i(x) = p_i \frac{1 - F_i^*(x)}{\tau_i^*} , \]
then \( \{ Z^*(t), t \geq 0 \} \) is a stationary SMP; see Koroljuk and Turbin [73], Theorem 4.6, page 64. Thus using Kolmogorov Theorem we can construct it on whole real line.

4.3.3 With process \( \{ Z^*(t), t \in \mathbb{R} \} \) we associate following quantity
\[ V^*(t) = \sup \{ u \geq t : Z^*(s) = Z^*(t) \} \quad \text{for all} \quad s \in [t, u] - t , \]
which is simply remaining time to the moment of jump of SMP \( \{ Z^*(t), t \in \mathbb{R} \} \). Note that \( V^*(0) = S_0^* \). Another quantity of interest is a supplementary age component \( S^*(t) \). We then define
\[ S^*(t) = \begin{cases} S^*(0) + t & \text{if } t < V^*(0) \\ t - \inf \{ u \leq t : Z^*(s) = Z^*(t) \} & \text{for all} \quad s \in [u, t] \end{cases} \quad \text{if } t \geq V^*(0) . \]

By Koroljuk and Turbin [73], Theorem 1.13, page 25, processes \( \{(Z^*(t), S^*(t)), t \in \mathbb{R} \} \) and \( \{(Z^*(t), V^*(t)), t \in \mathbb{R} \} \) are Markov processes. Both processes has the same unique stationary distribution given by density function \( \pi \). In other words, 
\[ \mathbb{P}(S^*(0) \leq x, Z^*(0) = i) = \mathbb{P}(V^*(0) \leq x, Z^*(0) = i) = \int_0^x \pi_i(x) \, dx , \]
where \( \pi_i(x) \) is given in (4.3.7).
4.3.2 Reversed SMP

As mentioned in Section 4.2, we will use reversed SMP in the proof of Theorem 4.4.1. Hence, we collect the relevant results about reversed SMP here.

Consider the reversed càdlàg version \( \{Z(t), t \in \mathbb{R}\} \) of the process \( \{Z^*(t), t \in \mathbb{R}\} \).

4.3.4 In that case, Markov process \( \{(Z(t), S(t)), t \in \mathbb{R}\} \) is a reversed process of \( \{(Z^*(t), V^*(t)), t \in \mathbb{R}\} \), where \( S(t) \) is a supplementary age component of \( \{Z(t), t \in \mathbb{R}\} \).

Similarly, \( \{(Z(t), V(t)), t \in \mathbb{R}\} \) is a reversed process of \( \{(Z^*(t), S^*(t)), t \in \mathbb{R}\} \), where \( V(t) \) is a remaining time to jump of SMP \( \{Z(t), t \in \mathbb{R}\} \). In particular, all processes has the same stationary distribution with density given in (4.3.7).

Note also that both processes \( \{Z^*(t), t \in \mathbb{R}\} \) and \( \{Z(t), t \in \mathbb{R}\} \) have the same stationary distribution \( p = (p_1, \ldots, p_\ell) \). Let \( S_n = -S^*_n \) and \( Z_n = Z(S^*_n+) \).

4.3.5 Process \( \{Z_n, n \in \mathbb{Z}_+\} \) is a Markov chain being a reversed version of Markov chain \( \{Z^*_n, n \in \mathbb{Z}_+\} \) with transition matrix

\[
P = \{p_{ij}\}
\]

given in (4.3.11). Moreover, Markov chains \( \{Z^*_n, n \in \mathbb{Z}_+\} \) and \( \{Z_n, n \in \mathbb{Z}_+\} \) have also the same stationary distribution \( \pi = (\pi_1, \ldots, \pi_\ell) \). That is,

\[
\pi = \pi P \quad \text{and} \quad \sum_{i=1}^\ell \pi_i = 1. \tag{4.3.8}
\]

Definition 4.3.1 An SMP with kernel \( F^*(\cdot) \) is called a non-anticipative SMP if

\[
\{F^*_i(x)\} = \{F^*_i(x)p^*_i\},
\]

where \( F^*_i(x) = \sum_{j=1}^\ell F^*_i(x) \).

We call such an SMP "non-anticipative" since the sojourn time in the current state does not depend upon the following state. In other words, given the current state, the sojourn time and the next state, are independent of each other (a property that is exhibited by continuous time Markov chain). In case of a general SMP, the sojourn time depends on both the current state and the following state.

We now define

\[
Z^-(t) = Z((t - S(t))^-), \quad Z^+(t) = Z((t + V(t))^+)
\]

and

\[
Z^{*-}(t) = Z^*((t - S^*(t))^-), \quad Z^{*+}(t) = Z^*((t + V^*(t))^+). \]

Notice that \( Z^-(t) \) and \( Z^{*-}(t) \) are previously visited states by processes \( \{Z(t), t \in \mathbb{R}\} \) and \( \{Z^*(t), t \in \mathbb{R}\} \) respectively. Furthermore, \( Z^+(t) \) and \( Z^{*+}(t) \) are the states to
which processes \( \{Z(t), t \in \mathbb{R}\} \) and \( \{Z^*(t), t \in \mathbb{R}\} \) are going to jump first time after time \( t \). Let

\[
N_i(t) = \text{card}\{0 < u \leq t : Z((u-) = i) = Z((u) = i)\},
\]

We now state the following theorem giving sufficient and necessary conditions for SMP \( \{Z^*(t), t \in \mathbb{R}\} \) to have the reversed process also an SMP. This result is due to Chari [24] but we give here a new proof.

**Theorem 4.3.2** Let \( \{Z(t), t \in \mathbb{R}\} \) be a stationary semi-Markov process with kernel \( F^*(x) = \{F^*_i(x)\} \). Then the time-reversed stationary process \( \{Z(t), t \in \mathbb{R}\} \) is an SMP if and only if \( \{Z^*(t), t \in \mathbb{R}\} \) is a non-anticipative SMP. If \( \{Z^*(t), t \in \mathbb{R}\} \) is a non-anticipative SMP, then the reversed SMP is also a non-anticipative SMP with kernel

\[
F(x) = \{F_{ij}(x)\} = \{F_i(x)\pi_j\},
\]

where

\[
F_i(x) = F^*_i(x),
\]

\[
P = \{p_{ij}\} = \left\{\frac{p^*_i \pi_j}{\pi_i}\right\},
\]

and \( \pi \) is given by equation (4.3.5) and (4.3.8).

**Proof.** Assume that process \( \{Z(t), t \in \mathbb{R}\} \) is an SMP. We prove that \( \{Z^*(t), t \in \mathbb{R}\} \) is then non-anticipative. Since process \( \{Z(t), t \in \mathbb{R}\} \) is a stationary SMP, then by 4.3.3 and 4.3.4 we have that

\[
\mathbb{P}(Z^+(t) = j, Z(-t) = i, S(-t) \leq x, V(-t) \leq z)
\]

cannot depend on \( t \). In that case we have also by 4.3.3

\[
\mathbb{P}(Z^+(t) = j, Z(-t) = i, S(-t) \leq x, V(-t) \leq z)
\]

\[
= \frac{p_i}{\tau_i} \int_0^z \mathbb{P}(Z^+(t) = j, S(-t) \leq x | V(-t) = w, Z(t) = i)
\]

\[
(1 - F^*_i(w)) dw
\]

(4.3.13)

and moreover for \( t \geq z \) by 4.3.4

\[
= \mathbb{P}(Z^{*}\!(t) = j, Z^*(t) = i, S^*(t) \leq z, V^*(t) \leq x)
\]

\[
= p^*_j \frac{p_i}{\tau_j} \int_0^z (F^*_i(x + w) - F^*_i(w)) dw.
\]

(4.3.14)

We now prove equality (4.3.14).

**4.3.6** From Pyke and Schaufeule [103], page 1450 we get following equation

\[
\mathbb{P}(Z^{*}\!(t) = j, Z^*(t) = i, S^*(t) \leq z, V^*(t) \leq x)
\]

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\[
\begin{align*}
&= \frac{p_j}{\tau_j} \int_{t-u}^{t} (F^*_i(t + x - u) - F^*_i(t - u)) (1 - F^*_j(u)) \, du \\
&\quad + \int_{0}^{t} \int_{t-u}^{t-v} (F^*_i(t + x - u - v) - F^*_i(t - u - v)) dF^*_j(v) d\mathbb{E}(N_j(u)) .
\end{align*}
\]

However, by the stationarity \( \mathbb{E}(N_j(t)) = \mathbb{E}(N_j(t + s) - N_j(s)) \), hence \( \mathbb{E}N_j(t) = \frac{p_j}{\tau_j} t \).
Then after the change of variable \( s := t - u \) and \( w := s - v \) the equation given in 4.3.6 is equivalent to the following

\[
\frac{p_j}{\tau_j} \int_{0}^{z} \left[ p^*_j(1 - F^*_j(t - w)) + F^*_j(t - w) \right] [F^*_i(x + w) - F^*_i(w)] \, dw . \tag{4.3.15}
\]

As we said, quantity (4.3.12) and hence also (4.3.15) cannot depend on \( t \). Hence \( p^*_j(1 - F^*_j(x)) + F^*_j(x) = \text{const} \). Because \( p^*_j(1 - F^*_j(\infty)) + F^*_j(\infty) = F^*_j(\infty) = p^*_j \), hence constant is equal to \( p^*_j \) and \( F^*_j(x) = p^*_j - p^*_j(1 - F^*_j(x)) = p^*_j F^*_j(x) \). Thus SMP process \( \{Z^*(t), t \in \mathbb{R}\} \) must be non-anticipative. Then also

\[
p^*_j(1 - F^*_j(t - w)) + F^*_j(t - w) = p^*_j,
\]

which completes the proof of equation (4.3.14). Now, if process \( \{Z^*(t), t \in \mathbb{R}\} \) is non-anticipative, then distribution of \( S_n \) depends only on the current state. Thus using 4.3.5 we get that \( \{(Z_n, S_n), n \in \mathbb{Z}_+\} \) is a Markov renewal sequence and hence process \( \{Z(t), t \in \mathbb{R}\} \) is SMP. We prove now equalities (4.3.9), (4.3.10) and (4.3.11).

From (4.3.13) and (4.3.14) we get

\[
\frac{p_i}{\tau_i} \mathbb{P}(Z^+(t) = j, S(t) \leq x | V(t) = w, Z(t) = i) (1 - F^*_i(w)) = p^*_j \frac{p_j}{\tau_j} (F^*_j(x + w) - F^*_j(w)) .
\]

Taking \( w = 0 \) we have that

\[
F_{ij}(x) = p^*_j \frac{p_j}{\tau_j} \pi^*_j F^*_i(x) .
\]

In particular, a SMP process \( \{Z(t), t \in \mathbb{R}\} \) is non-anticipative and \( F_i(x) = F^*_i(x) \).
Moreover, from definition of stationary distribution \( p \) in (4.3.6) we have

\[
\frac{p_j \pi^*_j}{p_i \pi^*_i} = \frac{\pi_j}{\pi_i} .
\]

Hence we get the assertion of the theorem. \( \square \)

Note that by (4.3.10) we have

\[
\tau_i = \mathbb{E}[S_1 | Z_0 = i] = \tau^*_i .
\]

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4.3.3 Converting general SMP into non-anticipative case

There are several applications where the non-anticipative SMP model fails. In this subsection we state and prove a theorem that explains how to convert a general SMP into a non-anticipative SMP.

Let \( \{Z^* (t), t \in \mathbb{R}\} \) be an \( \ell \)-state (not necessarily non-anticipative) SMP with kernel \( \{F_{ij}^* (x)\} \), transition probabilities \( p_{ij}^* = F_{ij}^*(+\infty) \), and stationary probabilities \( \pi_j \) (from \( \pi = \pi F(+\infty) \)). Also, \( S^n_0 \) and \( Z^n_0 \) denote respectively the \( n \)th transition epoch and the state of the SMP immediately after the \( n \)th transition, i.e. \( Z^n_0 = Z^*(S^n_0) \) (\( n \in \mathbb{Z} \)). Let \( N(t) = \sup\{n \in \mathbb{Z} : S_n^* \leq t\} \). Define
\[
\overline{Z}(t) = (Z^*(S_{N(t)}^*), Z^*(S_{N(t)+1}^*)) .
\] (4.3.16)

Let \( \overline{S}_n \) be the \( n \)th transition epoch of the \( \{\overline{Z}(t), t \in \mathbb{R}\} \) process and \( \overline{Z}_n \) be the state of the \( \{\overline{Z}(t), t \in \mathbb{R}\} \) process immediately after the \( n \)th transition, i.e.,
\[
\overline{Z}_n = \overline{Z}(\overline{S}_n) .
\]

Observe that \( \overline{S}_n = S_n^* \) and \( \overline{Z}_n = (Z^n_0, Z^n_{n+1}) \).

**Theorem 4.3.3** The \( \ell^2 \)-state process \( \{\overline{Z}(t), t \in \mathbb{R}\} \) is a non-anticipative SMP with state space \( \{(i, j) : 1 \leq i, j \leq \ell\} \) and kernel
\[
\overline{F}_{(i,j),(k,l)} (x) = \begin{cases} \frac{p_{ki}^* F_{ij}^*(x)}{F_{ij}^*(+\infty)} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} .
\] (4.3.17)

Transition and stationary probabilities are respectively
\[
\overline{p}_{(i,j),(k,l)} = \begin{cases} p_{ki}^* & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} .
\] (4.3.18)

and
\[
\overline{\pi}_{(i,j)} = \pi_j p_{ij}^* .
\] (4.3.19)

**Proof.** Let
\[
\overline{F}_{(i,j),(k,l)} (x) = \mathbb{P}(\overline{Z}_{n+1} = (k, l), \overline{S}_{n+1} \leq x | \overline{Z}_n = (i, j), \overline{S}_n, \overline{Z}_{n-1}, \ldots )
\]
\[
= \mathbb{P}(Z_{n+2} = l, Z_{n+1} = k, S_{n+1}^* \leq x | Z_{n+1}^* = j, Z_{n}^* = i, S_n^*, Z_{n-1}^*, \ldots )
\]
\[
= \mathbb{P}(Z_2 = l, Z_1^* = k, S_1^* \leq x | Z_1^* = j, Z_0^* = i) ,
\]

since \( \{Z^*(t), t \in \mathbb{R}\} \) is an SMP. Also, given \( Z_1^* \), random variables \( Z_2^* \) and \( S_1^* \) are conditionally independent. Hence we can write
\[
\overline{F}_{(i,j),(k,l)} (x) = \mathbb{P}(Z_2^* = l, Z_1^* = k | Z_1^* = j, Z_0^* = i) \mathbb{P}(S_1^* \leq x | Z_1^* = j, Z_0^* = i)
\]
\[
= \begin{cases} p_{ki}^* \mathbb{P}(S_1^* \leq x | Z_1^* = j, Z_0^* = i) & \text{if } j = k \\ 0 \cdot \mathbb{P}(S_1^* \leq x | Z_1^* = j, Z_0^* = i) & \text{if } j \neq k \end{cases}
\]
\[
= \overline{p}_{(i,j),(k,l)} \mathbb{P}(S_1^* \leq x | Z_1^* = j, Z_0^* = i)
\]
\[
= \overline{p}_{(i,j),(k,l)} \frac{F_{ij}^*(x)}{F_{ij}^*(+\infty)}
\]
\[
= \overline{p}_{(i,j),(k,l)} \overline{F}_{ij}^*(x) .
\]
which is of the form (like the non-anticipative SMP)

$$\{F_{ab}^*(x)\} = \{F_a^*(x)p_{ab}^*\} .$$

Hence we get equations (4.3.17) and (4.3.18). It is easy to verify that \( \overline{\pi}_{i,j} = \pi_i p_{ij} \) satisfies

$$\overline{\pi} F(\infty) = \overline{\pi} \text{ and } \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \overline{\pi}_{i,j} = 1 .$$

Hence we get equation (4.3.19).

\[\square\]

4.4 Main Results

In this section we present exponential bounds for a probability of the buffer overflow given by Corollary 4.2.2 in the following way

$$\Psi(x) = \mathbb{P}(X^* > x) = \mathbb{P}(\sup_{t \geq 0} \int_0^t (r_{Z(s)} - c) \, ds > x) ,$$

where \( \{Z(t), t \geq 0\} \) is a non-anticipative semi-Markov process.

Consider canonical non-anticipative semi-Markov process \( \{Z(t), t \geq 0\} \) on a state space \( \{1, 2, \ldots, \ell\} \) with the kernel of the SMP

$$F(x) = \{F_{ij}(x)\}_{i,j=1,\ldots,\ell} = \{\mathbb{P}(S_1 \leq x; Z_1 = j | Z_0 = i)\}_{i,j=1,\ldots,\ell} \quad (4.4.1)$$

fulfilling

$$F_{ij}(x) = p_{ij} F_i(x) ,$$

where

$$F_i(x) = \mathbb{P}(S_1 \leq x | Z_0 = i) = \sum_{j=1}^{\ell} F_{ij}(x) ;$$

see Definition 4.3.1. For further notations and assumptions we refer to Subsections 4.3.1 and 4.3.2.

4.4.1 We also assume that distribution function \( F_i(x) \) is absolutely continuous with continuous density function \( f_i(x) = \frac{dF_i(x)}{dx} \) on the \( \mathbb{R}_+ \) as a support. Define hazard rate function by

$$\rho_i(x) = \frac{f_i(x)}{F_i(x)} ,$$

where \( \overline{F}_i(x) = 1 - F_i(x) \).

**Remark 4.4.1** Assumption that density function is continuous is equivalent to condition (B.5) in Remark 3.3.5 and is needed to fulfill assumptions (A.1) - (A.3) in perturbation Theorem 3.2.2. By Remark 3.2.1 it can be relaxed to the condition
that hazard rate function is dominated by some continuous function on $\mathbb{R}_+$. Following Remark 4.4.2 note that we need Theorem 3.2.2 only to prove lower exponential bound for the buffer overflow probability. Hence, upper exponential bound holds without assumption 4.4.1.

For stability we assume that

$$\sum_{m=1}^{\ell} p_m r_m < c \quad (4.4.2)$$

This condition is equivalent to (4.2.1) in Proposition 4.2.1. We also assume that

$$c < \max_{m \in \{1, \ldots, \ell\}} r_m \quad (4.4.3)$$

otherwise the buffer will be always empty in the steady-state.

Define

$$\Phi(\delta) = \{ \Phi_{ij}(\delta) \}_{i,j=1, \ldots, \ell} = \{ \mathbb{E}(e^{\delta(r_{ij} - \alpha)} ; Z_1 = j | Z_0 = i) \}_{i,j=1, \ldots, \ell} \quad (4.4.4)$$

and

$$\delta^* = \sup \{ \delta \geq 0 : \Phi_{ij}(\delta) < +\infty ; i, j = 1, \ldots, \ell \} \quad (4.4.5)$$

Throughout this dissertation we assume that

$$\delta^* > 0 \quad (4.4.6)$$

This assumption fails e.g. if kernel distributions $F_{ij}(x)$ are heavy-tailed. In this case asymptotics of probability of the buffer overflow is nonexponential; see Rolski et al [107], Boxma [21]. This case will not be considered in this dissertation.

Note that $\Phi(\delta)$ is a matrix with non-negative elements equal to $p_{ij} F_i(\delta r_i - c)$, where

$$\hat{F}(\alpha) = \int_0^{+\infty} e^{\alpha x} dF(x)$$

is the moment generating function of distribution function $F$. Let $\kappa(\Phi(\delta), c)$ be the Perron-Frobenius eigenvalue of positive matrix $\Phi(\delta)$. That is, $\kappa(\Phi(\delta), c)$ is a the biggest real eigenvalue of $\Phi(\delta)$. Let assume that there exists $\eta$ being the biggest positive value satisfying

$$\kappa(\Phi(\eta), c) = 1 \quad (4.4.7)$$

that is, there exists from Perron-Frobenius Theorem, positive vector $u$ (all components are strictly positive) such that

$$\Phi(\eta) u = u \quad (4.4.8)$$

This system of equation we call similarly like in Palmowski and Rolski [101], $(B)$ase $(S)$ystem of $(N)$onlinear $(E)$quations. We will abbreviate it by BSNE. The existence of solution of BSNE equation is the most important condition needed to get exponential bounds for probability of the buffer overflow. The $\eta$ is a constant in the exponent in the lower and upper exponential bound.

The BSNE equation not always has a unique solution $\eta$. If there is no solution of BSNE, then can happen two scenarios:
1. \( \eta = \delta^* = 0 \) and in this case (called heavy-tailed case; see Rolski et al. [107] and Boxma [21]) asymptotics of probability of the buffer overflow \( \Psi(x) \) is not exponential;

2. \( \kappa(\Phi(\eta), c) < 1 \) and \( \kappa(\Phi(\eta^+), c) = +\infty \) for some \( \eta > 0 \); then \( \eta = \delta^* \) and we can get exponential upper bound

\[
\Psi(x) \leq C^* e^{-\eta x};
\]

see Section 5.6.

We will discuss conditions for the solution of BSNE in Section 5.5.

Define also

\[
\Gamma(x) = \{ \Gamma(x) \}_{i,j=1,\ldots,\ell} = \{ E(e^{(r_i - c)(S_i - x)}; Z_1 = j | Z_0 = i, S_1 > x) \}_{i,j=1,\ldots,\ell} \quad (4.4.9)
\]

and

\[
s_m = \frac{p_m}{\tau_m \eta (r_m - c)} . \quad (4.4.10)
\]

The main theorem of this chapter is the following.

**Theorem 4.4.1** Let BSNE (4.4.8) has a unique solution \( \eta \). Then the steady-state distribution of the buffer content process is bounded as

\[
C^* e^{-\eta x} \leq P(X^* > x) \leq C^* e^{-\eta x} , \quad (4.4.11)
\]

provided that \( C^* > 0 \) and \( C^* < +\infty \), where

\[
C^* = \frac{s(I - P)u}{\min_{m: r_m > \epsilon} \inf_{x \geq 0} h_m(x)}
\]

\[
C^* = \frac{s(I - P)u}{\max_{m: r_m > \epsilon} \sup_{x \geq 0} h_m(x)}
\]

and

\[
h(x) = \Gamma(x) u . \quad (4.4.12)
\]

### 4.4.1 Outline of the proof

We now describe the main steps in the proof of this theorem below. Each step is discussed in detail in the subsequent subsections.

1. **Markov process and its generator**
   
   SMP \( \{Z(t), t \geq 0\} \) is not a Markov process in general. However, process \( \{w(t), t \geq 0\} \) defined by \( w(t) = (Z(t), S(t)) \) is a PDMP, where \( S(t) \) is the supplementary age process of \( \{Z(t), t \geq 0\} \). We consider canonical process \( \{w(t), t \geq 0\} (w(\omega, t) = \omega(t)) \) on a space \( D_E[0, +\infty) \), where \( E = \{1, \ldots, \ell\} \times \)
$\mathbb{R}_+$ is a Polish space. We adopt here a convention that $g(i, x) = g_i(x)$. Let $g(x) = (g_1(x), \ldots, g_k(x))$ and $\mathcal{A}$ be extended generator. From Theorem 3.3.1

$$(\mathcal{Ag})(i, x) = \frac{\partial}{\partial x} g_i(x) - \rho_i(x) g_i(x) + \rho_i(x) \sum_{j=1}^k \rho_{ij} g_j(0).$$

We restrict the domain $D(\mathcal{A})$ of the extended generator to the family of functions $g(i, x)$ ($i = 0, 1, x \in \mathbb{R}_+$) such that

$$g(i, \cdot) \in C^1_b(\mathbb{R}_+). \quad (4.4.13)$$

From Lemma 3.2.1 the process $\{N(t), t \geq 0\}$ defined by

$$N(t) = \frac{h(w(t))}{h(w(0))} e^{-\int_0^t \frac{(\mathcal{Ah})(w(s))}{h(w(s))} ds}$$

is a mean-one local martingale with respect to right-continuous filtration $\{\mathcal{F}_t\} = \{\mathcal{F}^w_t\}$ for any positive function $h(\cdot, \cdot)$ from the domain $D(\mathcal{A})$ of the extended generator $\mathcal{A}$. We assume that positive function $h(\cdot, \cdot)$ fulfills condition (4.5.3) and additionally condition (4.5.5). Notice that these both conditions turn out to be equivalent to requirement that $C^* < +\infty$ and $C_* > 0$ for constants given in Theorem 4.4.1.

2. Choice of $h(\cdot, \cdot)$

We next show (in Subsection 4.5.2) that it is possible to choose a function $h(\cdot, \cdot) \in D(\mathcal{A})$ such that

$$N(t) = \frac{h(w(t))}{h(w(0))} e^{-\int_0^t \frac{(\mathcal{Ah})(w(s))}{h(w(s))} ds} = \frac{h(w(t))}{h(w(0))} e^{-\int_0^t (r_Z(s) - c) ds} \delta_0 \int_0^t (r_Z(s) - c) ds,$$

for the biggest possible real number $\delta_0$. It turns out that $\delta_0 = \eta$ if $\eta$ is a solution of BSNE (4.4.8). Then local martingale $\{N(t), t \geq 0\}$ is a true martingale. Function $h(\cdot, \cdot)$ fulfilling above equation is given in (4.4.12).

3. Exponential change of measure

Let

$$\tau(x) = \inf\{t > 0 : \int_0^t (r_Z(s) - c) ds > x\}. \quad (4.4.15)$$

From the above definition, we have

$$\left\{ \sup_{t \geq 0} \int_0^t (r_Z(s) - c) ds > x \right\} \equiv \{\tau(x) < +\infty\},$$

hence,

$$\mathbb{P}(\sup_{t \geq 0} \int_0^t (r_Z(s) - c) ds > x) = \mathbb{P}(\tau(x) < +\infty).$$
By Remark 3.3.5 all assumptions of Theorem 3.2.2 hold. Therefore we can define new probability measure $\hat{\mathbb{P}}^{(i,x)}_t$ by

$$\frac{d\hat{\mathbb{P}}^{(i,x)}_t}{d\mathbb{P}^{(i,x)}_t} = N(t),$$

where $\mathbb{P}^{(i,x)}$ is the measure under which $w(0) = (i, x)$ and $\hat{\mathbb{P}}^{(i,x)}_t = \mathbb{P}^{(i,x)}_{\tau_x_i} \cdot \mathbb{P}^{(i,x)}_0$. Then from Lemma 3.1.8 we get

$$\mathbb{P}^{(i,x)}(\tau(x) < +\infty) = \mathbb{E}^{(i,x)}\left[N(\tau(x))^{-1}; \tau(x) < +\infty\right]. \quad (4.4.16)$$

Now from Lemma 4.5.4

$$\hat{\mathbb{P}}^{(i,x)}(\tau(x) < +\infty) = 1. \quad (4.4.17)$$

That is, after the exponential change of measure, the drift of the buffer content process changes from negative to positive.

4. Bounds

By (4.4.17) we have

$$\mathbb{P}(\tau(x) < +\infty) = \sum_{j=1}^\ell \int_0^{+\infty} \mathbb{E}^{(i,y)}\left[N(\tau(x))^{-1}; \tau(x) < +\infty\right] \pi_j(y) dy$$

$$= \sum_{j=1}^\ell \int_0^{+\infty} \mathbb{E}^{(i,y)}\left[N(\tau(x))^{-1}\right] \pi_j(y) dy,$$

where

$$\pi_i(x) dx = p_i \frac{1 - F_i(x)}{\tau_i} dx$$

is a stationary distribution of process $\{w(t), t \geq 0\}$ (see 4.3.2 and 4.3.3). By (4.4.14) we have

$$\mathbb{P}(\tau(x) < +\infty) = \sum_{j=1}^\ell \int_0^{+\infty} \mathbb{E}^{(i,y)}\left[\frac{h(j,y)}{h(w(\tau(x)))} \exp\{-\eta \int_0^{\tau(x)} (r_Z(s) - c) ds\}\right] \pi_j(y) dy.$$

Thus by definition of $\tau(x)$ in (4.4.15)

$$\mathbb{P}(\tau(x) < +\infty) = e^{-\eta x} \sum_{j=1}^\ell \int_0^{+\infty} \mathbb{E}^{(i,y)}\left[\frac{h(j,y)}{h(w(\tau(x)))}\right] \pi_j(y) dy. \quad (4.4.18)$$

Clearly, at time $\tau(x)$, $w(\tau(x))$ can only be in states $(i,s)$ such that $r_i > c$. Hence the lower bound on $\mathbb{E}^{(i,y)}\left\{\frac{1}{h(w(\tau(x)))}\right\}$ is $1/\{\max_{i;r_i > c} \sup_x h(i,x)\}$ and the upper bound on $\mathbb{E}^{(i,y)}\left\{\frac{1}{h(w(\tau(x)))}\right\}$ is $1/\{\min_{i;r_i > c} \inf_x h(i,x)\}$. These yields bounds for $\mathbb{P}(X^* > x).$
Remark 4.4.2 Note that we need perturbation Theorem 3.2.2 only to find parameters of SMP \( \{Z(t), t \geq 0\} \) after exponential change of measure (Lemma 4.5.3) and prove that drift of the buffer content process changes from negative to positive (Lemma 4.5.4). That is, we need Theorem 3.2.2 only to prove lower exponential bound for the buffer overflow probability. In fact, regardless of the sign of the drift by (4.4.16) and (4.4.18) we have

\[
\text{IP}(\tau(x) < +\infty) = e^{-nx} \sum_{j=1}^{\ell} \int_0^{+\infty} \mathbb{E}^{(j,y)} \left[ \frac{h(j,y)}{h(w(\tau(x)))} ; \tau(x) < +\infty \right] \pi_j(y) dy
\]

\[
\leq e^{-nx} \min_{i \in C(i)} \inf_x h(i,x) \sum_{j=1}^{\ell} \int_0^{+\infty} \text{IP}^{(j,y)} (\tau(x) < +\infty) h(j,y) \pi_j(y) dy
\]

\[
\leq e^{-nx} \min_{i \in C(i)} \inf_x h(i,x) \sum_{j=1}^{\ell} \int_0^{+\infty} h(j,y) \pi_j(y) dy = C^\ast e^{-nx},
\]

where \( C^\ast \) is an upper constant in the Theorem 4.4.1.

4.5 Proof of Theorem 4.2.1

4.5.1 Generator

We now consider the canonical process \( \{w(t), t \geq 0\} \) defined by

\[ w(t) = (Z(t), S(t)). \tag{4.5.1} \]

By 4.3.3, process \( \{w(t), t \geq 0\} \) is Markovian and moreover it is a piecewise deterministic Markov process (PDMP) on a space \( (DE[0, +\infty), \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \text{IP}) \), where \( E = \{1, \ldots, \ell\} \times \mathbb{R}_+ \) (see Section 3.3). The domain \( D(A) \) of the extended generator \( A \) of the process \( \{w(t), t \geq 0\} \) is restricted to the family functions \( g(x) = (g_1(x), \ldots, g_{\ell}(x)) \), such that \( g_i(x) (i = 1, \ldots, \ell) \) are bounded continuously differentiable functions:

\[ g_i(\cdot) \in C^1_b(\mathbb{R}_+) \).

Following Theorem 3.3.1, the extended generator of \( \{w(t), t \geq 0\} \) is as follows.

Lemma 4.5.1 The extended generator of the process \( \{w(t), t \geq 0\} \) is

\[
(Ag)(i, x) = \frac{\partial}{\partial x} g_i(x) - \rho_i(x) g_i(x) + \rho_i(x) \sum_{j=1}^{\ell} p_{ij} g_j(0). \tag{4.5.2}
\]

4.5.1 Using notations from Section 3.3 we have \( \lambda g_i(x) = \frac{\partial}{\partial x} g_i(x) \), \( \lambda(i, x) = \rho_i(x) \) and \( Q(d(j,y), (i,x)) = p_{ij} \delta(y) \), where \( \delta(y) \) is a Dirac measure, \( E = \{1, \ldots, \ell\} \times \mathbb{R}_+ \) is a Polish space. We assume that conditions of 4.4.1 hold. If we assume also that positive function \( h(\cdot, \cdot) \) fulfills the following conditions

\[ h(i, \cdot) \in C^1_b(\mathbb{R}_+) \tag{4.5.3} \]
and

\[ \frac{(\mathcal{A} h)_i}{h_i} \in B(\mathbb{R}_+) , \] (4.5.4)

where \( i = 1, \ldots, \ell \), then all conditions given in Remark 3.3.5 are fulfilled. In particular, from assumption that \( 0 < \tau_i < +\infty \), we have condition (B.3) fulfilled. By 4.4.1 (B.5) holds. Note that condition (B.6) is also fulfilled, because \( Q \) does not depend on \( x \). Thus by Remark 3.3.5 all assumptions of Theorems 3.2.2 and 3.3.2 are fulfilled. In particular, process \( \{N(t), t \geq 0\} \) is a martingale.

4.5.2 To get constant \( C^* \) for Theorem 4.4.1 finite we must assume the following additional condition on function \( h \):

\[ \inf_x h_i(x) > 0 \quad \text{for } i: r_i > c. \] (4.5.5)

4.5.2 The \( h \) function

We now fill up details of Step 2 and show how to choose function \( h(x) = (h_1(x), \ldots, h_\ell(x)) \in \mathcal{D}(\mathcal{A}) \) fulfilling conditions (4.5.4) and (4.4.14). Consider the following system of equations

\[ (\mathcal{A} h)(i, x) = \beta(r_i - c) h_i(i, x) \quad i = 1, \ldots, \ell . \] (4.5.6)

Then (4.4.14) and (4.5.4) hold. Thus we look for the smallest \( \beta \) fulfilling (4.5.6).

The \( i \)-th row of (4.5.6) is the first order nonhomogeneous differential equation

\[ \frac{\partial}{\partial x} h_i(x) + \rho_i(x) \left( \sum_{j=1}^{\ell} p_{ij} h_j(0) - h_i(x) \right) = \beta(r_i - c) h_i(x) \] (4.5.7)

whose the general solution is:

\[ h_i(x) = (F_i(x))^{-1} e^{\beta(r_i - c) x} \left[ h_i(0) - \int_0^x e^{-\beta(r_i - c) z} f_i(z) \, dz \sum_{j=1}^{\ell} p_{ij} h_j(0) \right] . \]

A necessary and sufficient condition for functions \( h_i \) to be positive is following

\[ h_i(0) - \int_0^{+\infty} e^{-\beta(r_i - c) z} f_i(z) \, dz \sum_{j=1}^{\ell} p_{ij} h_j(0) \geq 0 \]

for all \( i = 1, \ldots, \ell \). This is equivalent to \( u \geq \Phi(-\beta) u \), where \( u = (h_1(0), \ldots, h_\ell(0))^T \) and \( h_i(0) > 0 \). That is we look for the smallest \( \beta \) and vector \( u > 0 \) such that \( \Phi(-\beta) u \leq u \).
Lemma 4.5.2 If BSNE has a solution, then the smallest possible $\beta < 0$ fulfilling (4.5.6) is $-\eta$. Moreover,
\[ h(x) = \Gamma(x)u \] 
(4.5.8)
where $\Phi(\eta)u = u$, and $h$ is a positive function.

Proof. If there exists $-\beta_0 > \eta$ and $u > 0$ fulfilling inequality $\Phi(-\beta_0)u \leq u$, then multiplying it by the left Perron-Frobenius vector $\nu_{-\beta_0} > 0$, we would have $\kappa(\Phi(-\beta_0), \nu_{-\beta_0}u) \leq \nu_{-\beta_0}u$, so $\kappa(\Phi(-\beta_0), c) \leq 1$. But this is a contradiction because $\kappa(\Omega, \nu_{-\beta_0})$ is a convex function of $\alpha$ and $\kappa(\Phi(\eta), c) = 1$. Thus $-\beta_0 = \eta$ and $(h_1(0), \ldots, h_\ell(0)) = (u_1, \ldots, u_\ell)$, where $\Phi(\eta)u = u$. In this case:

\[ h_i(x) = (\tilde{F}_i(x))^{-1} e^{-\eta(r_i-c)x} \sum_{j=1}^\ell p_{ij} u_j \int_x^{+\infty} e^{\eta(r_i-c)z} f_i(z) \, dz \] 
(4.5.9)
\[ = \sum_{j=1}^\ell \Gamma_{ij}(x) u_j \] 
(4.5.10)
that is $h(x) = \Gamma(x)u$. Note that $h$ is a positive function.

\[ \square \]

4.5.3 Exponential change of measure

We assume that conditions (4.5.3) and (4.5.5) hold. Then by 4.5.1 all assumptions of Theorem 3.3.2 are fulfilled. Thus by Theorem 3.3.2 we can define uniquely a new probability measure $\tilde{P}^{(i,x)}$ on $(D[0, +\infty), \mathcal{F}, \{\mathcal{F}_t^n\})$ by

\[ \frac{d\tilde{P}^{(i,x)}}{dP^{(i,x)}} = N(t) \]

where $\{N(t), t \geq 0\}$ is exponential martingale given in (3.2.2).

Lemma 4.5.3 If the BSNE has a solution, then on the new probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}, \tilde{P}^{(i,x)})$ the process $\{Z(t), t \geq 0\}$ is again a semi-Markov process specified by $(\tilde{p}_{ij}, [\tilde{\tau}_i])$, where

\[ \tilde{p}_{ij} = \frac{p_{ij} u_j}{\sum_{k=1}^\ell p_{ik} u_k} = \Phi_{ij}(\eta)u_j, \] 
(4.5.11)
\[ \tilde{p}_i(x) = \frac{p_i(x) \sum_{j=1}^\ell p_{ij} u_j}{h_i(x)} = \frac{f_i(x)e^{\eta(r_i-c)x} \sum_{j=1}^\ell p_{ij} u_j}{\sum_{j=1}^\ell \int_x^{+\infty} e^{\eta(r_i-c)z} p_{ij} f_i(z) u_j \, dz}, \] 
(4.5.12)
\[ \tilde{f}_i(x) = \frac{f_i(x)e^{\eta(r_i-c)x} \sum_{j=1}^\ell p_{ij} u_j}{u_i}, \] 
(4.5.13)
and

\[ \tilde{\tau}_i = \frac{(\Phi_{ij}(\eta)u_j)}{u_i(r_i - c)}. \] 
(4.5.14)
Proof. By the unique existence of probability measure \( \hat{\mathbf{P}} \) (see Theorem 3.1.4), from Theorem 3.3.2, process \( \{\mathbf{u}(t), t \geq 0\} \) on the new probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \hat{\mathbf{P}}^{(x)})\) is still a PDMP with the extended generator:

\[
\hat{A}g \big( i, x \big) = \frac{\partial}{\partial x} g_i(x) + \sum_{j=1}^{\ell} \frac{p_{ij}(x)}{h_i(x)} \Phi_{ij}(\eta) u_j \left[ \sum_{k=1}^{\ell} \phi_{i,k}(\eta) u_k \right] \left( g_j(0) - g_i(x) \right). \tag{4.5.15}
\]

The first part of equation (4.5.11) and equation (4.5.12) are simple consequences of the definition of the generator of PDMP and definition of function \( h_i(x) \). Moreover, the second part of equation (4.5.11) follows by

\[
\frac{p_{ij} u_j}{\sum_{k=1}^{\ell} p_{ik} u_k} = \frac{p_{ij} \int_0^{\infty} e^{\eta (r_i - c) x} f_i(x) \, dx \, u_j}{\sum_{k=1}^{\ell} p_{ik} \int_0^{\infty} e^{\eta (r_i - c) x} f_i(x) \, dx \, u_k} = \frac{\Phi_{ij}(\eta) u_j}{\Phi_{i,k}(\eta) u_k}.
\]

From equation (4.5.12) we get

\[
\hat{f}_i(x) = \frac{f_i(x) e^{\eta (r_i - c) x}}{d_i} \sum_{j=1}^{\ell} \Phi_{ij}(\eta) u_j = \frac{\Phi_{i,k}(\eta) u_k}{u_i},
\]

where \( d_i \) is the norming constant such that \( \int_0^{\infty} f_i(x) \, dx = 1 \). Hence

\[
d_i = \int_0^{\infty} f_i(z) e^{\eta (r_i - c) z} \sum_{j=1}^{\ell} p_{ij} u_j \, dz = \sum_{j=1}^{\ell} \Phi_{ij}(\eta) u_j = u_i, \tag{4.5.17}
\]

and we get equation (4.5.13). Using (4.5.13) equation (4.5.14) is fulfilled by

\[
(r_i - c) \hat{\tau}_i = (r_i - c) \int_0^{\infty} x \hat{f}_i(x) \, dx = \sum_{k=1}^{\ell} \phi_{i,k}(\eta) u_k = \frac{\Phi^{(\eta)}(\eta)}{u_i}.
\]

This completes the proof. \( \square \)

Let \( \hat{\mathbf{\pi}} \hat{\mathbf{P}} = \hat{\mathbf{\pi}} \) be the stationary distribution of the Markov chain \( \{Z_n\} \) under the new probability measure. In matrix notation (4.5.11) reads

\[
\hat{\mathbf{P}} = \{ \hat{p}_{ij} \} = \text{diag}(\mathbf{u}^{-1}) \Phi(\eta) \text{diag}(\mathbf{u}). \tag{4.5.18}
\]

Therefore, by (4.5.18)

\[
\hat{\mathbf{\pi}} = \hat{\mathbf{\pi}} \text{diag}(\mathbf{u}^{-1}) \Phi(\eta) \text{diag}(\mathbf{u})
\]

and hence

\[
\hat{\mathbf{\pi}} = \nu \text{diag}(\mathbf{u}), \tag{4.5.19}
\]

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where $\nu$ is the left (row) eigenvector of $\Phi(\eta)$ corresponding to eigenvalue 1 such that $\nu \mathbf{u} = 1$, that is

$$\nu = \nu \Phi(\eta) .$$

The drift $\hat{d} = \mathbb{E}r_Z(+) - c$ is

$$\hat{d} = \sum_{i=1}^{\ell} \frac{r_i \bar{\tau}_i}{\sum_{j=1}^{\ell} \bar{\tau}_j} - c .$$

Hence $\hat{d} > 0$ if and only if

$$\sum_{i=1}^{\ell} \bar{\tau}_i (r_i - c) > 0$$

which is, by (4.5.14) and (4.5.19), equivalent to

$$\nu \Phi'(\eta) \mathbf{u} > 0 .$$

**Lemma 4.5.4** Following inequality holds

$$\hat{d} > 0 .$$

**Proof.** Consider the eigenvalue problem

$$\begin{align*}
\Phi(\alpha) \mathbf{h}_\alpha &= \kappa(\alpha) \mathbf{h}_\alpha , \\
\nu_\alpha \Phi(\alpha) &= \kappa(\alpha) \nu_\alpha , \\
\nu_\alpha \mathbf{h}_\alpha &= 1 , \\
\alpha &\geq 0 .
\end{align*}$$

(4.5.21)

Function $\kappa(\alpha)$ is convex by Kingman-Miller Theorem; see Miller [89] and the beautiful proof of this theorem by Kingman [65]. Differentiating the first line in (4.5.21) we obtain

$$\Phi'(\alpha) \mathbf{h}_\alpha + \Phi(\alpha) \mathbf{h}_\alpha' = \kappa'(\alpha) \mathbf{h}_\alpha + \kappa(\alpha) \mathbf{h}_\alpha' .$$

Multiplying both the sides from the right by $\nu_\alpha$ and rearranging we arrive at

$$\nu_\alpha \Phi'(\alpha) \mathbf{h}_\alpha = \kappa'(\alpha) + (\kappa(\alpha) - 1) \nu_\alpha \mathbf{h}_\alpha' .$$

(4.5.22)

Since $\Phi(0) = P$ we have

$$\mathbf{h}_0 = P \mathbf{e} = \mathbf{e}, \quad \nu_0 = \pi P = \pi, \quad \kappa(0) = 1 .$$

(4.5.23)

where $\mathbf{e} = [1, \ldots, 1]^T$ is the (column) vector of one’s. The stability condition $d = \sum_i p_i r_i - c < 0$ and (4.5.22) yields

$$\kappa'(0) = \pi \Phi'(0) \mathbf{e}$$

$$= \sum_{i=1}^{\ell} \pi_i \sum_{j=1}^{\ell} \frac{\partial}{\partial \alpha} \Phi_{ij}(\alpha(r_i - c)) \big|_{\alpha = 0}$$

$$= \sum_{i=1}^{\ell} \pi_i \bar{\tau}_i (r_i - c)$$

(4.5.24)

$$= \left( \sum_{j=1}^{\ell} \pi_j \bar{\tau}_j \right) \left( \sum_{i=1}^{\ell} p_i r_i - c \right) < 0 .$$

(4.5.25)
Since $\kappa(0) = 1$, $\kappa'(0) < 0$ and $\kappa(\alpha)$ is a convex function, there is $\kappa'(\eta) > 0$ for $\kappa(\eta) = 1$. Substituting $\alpha = \eta$ in (4.5.22) and bearing in mind (4.5.20), inequality $\tilde{d} > 0$ is equivalent to

$$\nu \Phi'(\eta) u = \kappa'(\eta) > 0.$$ 

The proof is completed. □

### 4.5.4 Bounds

Let $\tau(x)$ be as in equation (4.4.15). Starting from equation (4.4.18) we get

$$\mathbb{P}(\tau(x) < +\infty) = e^{-\eta x} \sum_{j=1}^{\ell} \int_{0}^{+\infty} \mathbb{E}_{\nu}^{(j,y)} \left\{ \frac{h_j(y)}{h(w(\tau(x)))} \right\} \pi_j(y) dy,$$

which can be bounded above by

$$\leq e^{-\eta x} \frac{1}{\min_{m,m',m''} \inf_{x \geq 0} h_m(x)} \sum_{j=1}^{\ell} \int_{0}^{+\infty} h_j(y) \pi_j(y) dy.$$  \hspace{1cm} (4.5.26)

However,

$$\sum_{j=1}^{\ell} \int_{0}^{+\infty} h_j(y) \pi_j(y) dy = \sum_{j=1}^{\ell} \frac{p_j}{\tau_j} \int_{0}^{+\infty} h_j(y) \tilde{F}_j(y) dy \hspace{1cm} (4.5.27)$$

$$= \sum_{j=1}^{\ell} \frac{p_j}{\tau_j} \int_{0}^{+\infty} \sum_{k=1}^{\ell} p_{jk} u_k \int_{0}^{+\infty} e^{\eta (r_j - c)} f_j(z) dz e^{-\eta (r_j - c) y} dy$$

and integrating by parts,

$$= \sum_{j=1}^{\ell} \left[ \frac{p_j}{\tau_j \eta (r_j - c)} \sum_{k=1}^{\ell} p_{jk} u_k \left( \tilde{F}_j(\eta (r_j - c)) - 1 \right) \right]$$

$$= \sum_{j=1}^{\ell} s_j \sum_{k=1}^{\ell} p_{jk} u_k \tilde{F}_j(\eta (r_j - c)) - \sum_{j=1}^{\ell} s_j \sum_{k=1}^{\ell} p_{jk} u_k$$

$$= \sum_{j=1}^{\ell} s_j (\Phi(\eta) u)_j - \sum_{j=1}^{\ell} s_j (P u)_j = s (I - P) u.$$ 

Thus by equation (4.5.26) the proof of the upper bound in equation (4.4.11) is completed. In the similar way we obtain the lower bounds in equation (4.4.11). Note that conditions (4.5.3) and (4.5.5), which we assume in the proof, are equivalent to assumptions that

$$C_* > 0 \quad \text{and} \quad C^* < +\infty.$$  \hspace{1cm} (4.5.28)

This completes the proof of the theorem. □
4.6 Parameters of reversed SMP in terms of original process

For numerical results we should convert the constants $C_\ast$ and $C^\ast$ and $\eta$ from Theorem 4.4.1 in terms of original SMP (non-anticipative) process $\{Z^\ast(t), t \in \mathbb{R}\}$.

4.6.1 Non-anticipative case

Let

$$\chi(\delta) = \{\chi_{ij}(\delta)\}_{i,j \in 1,\ldots,\ell} = \{\mathbb{E}(e^{\delta(r_i - c)}S^\ast; Z_1^\ast = j|Z_0^\ast = i)\}_{i,j \in 1,\ldots,\ell} \quad (4.6.1)$$

and by $\zeta$ we denote the Perron-Frobenius eigenvalue of $\chi(\delta)$, that is $\zeta$ is the biggest real-positive value satisfying

$$\mathbf{v} = \mathbf{v} \chi(\zeta) \quad (4.6.2)$$

where $\mathbf{v} = [v_1, v_2, \ldots, v_\ell]$ is the corresponding positive left eigenvector.

From the definition of matrix $\Phi(\delta)$ in equation (4.4.4) and equation (4.6.1), we obtain by Theorem 4.3.2

$$\Phi_{ij}(\delta) = \hat{F}_{ij}(\delta(r_i - c)) = p_{ij}\hat{F}_i(\delta(r_i - c)) = \hat{F}_i^\ast(\delta(r_i - c))p_{ji}\frac{\pi_j}{\pi_i}. \quad (4.6.3)$$

Moreover,

$$\chi_{ij}(\delta) = \hat{F}_{ij}^\ast(\delta(r_i - c)) = \hat{F}_i^\ast(\delta(r_i - c))p_{ji}^\ast. \quad (4.6.4)$$

**Lemma 4.6.1** For a given positive number $0 < \delta < \delta^\ast$, the matrices $\Phi(\delta)$ and $\chi(\delta)$ have identical eigenvalues. Therefore

$$\kappa(\chi(\delta), c) = \kappa(\Phi(\delta), c)$$

and $\zeta = \eta$.

**Proof.** Let $\Lambda$ and $\Xi$ be diagonal matrices defined by

$$\Lambda = \text{diag}\{\hat{F}_1^\ast(\delta(r_1 - c)), \hat{F}_2^\ast(\delta(r_2 - c)), \ldots, \hat{F}_\ell^\ast(\delta(r_\ell - c))\}$$

and

$$\Xi = \text{diag}\{\pi_1, \pi_2, \ldots, \pi_\ell\}.$$

Clearly,

$$\chi(\delta) = \Lambda \Pi^\ast \quad \text{and}$$

$$\Phi(\delta) = \Lambda \Xi^{-1}\Pi^\ast^T \Xi.$$
Let $\lambda$ be an eigenvalue of $\chi(\delta)$. Therefore if $|D|$ denotes the determinant of the square matrix $D$ and if $D^T$ denotes the transpose of a matrix $D$, we have

\begin{align*}
|\lambda P^* - \lambda I| &= 0 \\
\Rightarrow |A| |P^* - \lambda A^{-1}| &= 0 \\
\Rightarrow |A| |P_{\ast}^T - \lambda A^{-1}| &= 0 \\
\Rightarrow |A| |P_{\ast}^T \Xi - \lambda \lambda A^{-1}| &= 0 \\
\Rightarrow |\lambda \lambda A^{-1}| &= 0,
\end{align*}

Thus $\lambda$ is also an eigenvalue of $\Phi(\delta)$. For a given positive number $\delta$, the matrices $\Phi(\delta)$ and $\chi(\delta)$ have identical eigenvalues. Specifically, the largest positive eigenvalues of $\Phi(\delta)$ and $\chi(\delta)$ are identical. Also, the smallest $\delta$ that satisfies the largest positive eigenvalues of $\Phi(\delta)$ and $\chi(\delta)$ to be identical and equal to one, is unique. Therefore

$$\kappa(\chi(\delta), c) = \kappa(\Phi(\delta), c)$$

and $\zeta = \eta$.

Since $\zeta = \eta$, the eigenvectors $v$ and $u$ are given by

$$v \chi(\zeta) = v \quad \text{and} \quad \Phi(\zeta) u = u.$$

**Lemma 4.6.2** The eigenvector $u$, the function $h(x)$ and the matrix $\Gamma(x)$ are related to the corresponding terms of the original non-anticipative SMP $\{Z^*(t), t \in \mathbb{R}\}$ as given below:

$$u_j = \frac{\hat{F}^*_j(\zeta(r_j - c))}{\pi_j} v_j,$$  \hspace{1cm} (4.6.6)

$$\Gamma_{ij}(x) = \frac{\int_x^\infty e^{\zeta(r_j - c)} y dF^*_i(y) p_{ji} \pi_i}{\int_x^\infty e^{\zeta(r_j - c)} y dF^*_i(y) \pi_i},$$  \hspace{1cm} (4.6.7)

and

$$h_i(x) = \frac{\int_x^\infty e^{\zeta(r_i - c)} y dF^*_i(y) v_i}{\int_x^\infty e^{\zeta(r_i - c)} y dF^*_i(y) \pi_i}.$$  \hspace{1cm} (4.6.8)

**Proof.** Since $\Phi(\zeta) u = u$, from equation (4.6.3) for all $i$, we get

$$u_i = \sum_{j=1}^\ell \Phi_{ij}(\zeta) u_j$$

$$= \sum_{j=1}^\ell \hat{F}^*_i(\zeta(r_i - c)) p_{ji} \frac{\pi_j}{\pi_i} u_j.$$  \hspace{1cm} (4.6.9)

Note that

$$u_j = \frac{\hat{F}^*_j(\zeta(r_j - c))}{\pi_j} v_j$$

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satisfies equation (4.6.9) since

\[
\sum_{j=1}^{\ell} \hat{F}^*_i(\zeta(r_i - c)) p^*_j \frac{\pi_j}{\pi_i} u_j = \hat{F}^*_i(\zeta(r_i - c)) \sum_{j=1}^{\ell} \frac{p^*_j \pi_j}{\pi_i} u_j
\]

\[
= \hat{F}^*_i(\zeta(r_i - c)) \sum_{j=1}^{\ell} \hat{F}^*_j(\zeta(r_j - c)) p^*_j v_j
\]

\[
= \hat{F}^*_i(\zeta(r_i - c)) \sum_{j=1}^{\ell} \chi_{ji}(\zeta) v_j
\]

\[
= \hat{F}^*_i(\zeta(r_i - c)) v_i,
\]

using equation (4.6.5) and the relation \( \mathbf{u} \chi(\zeta) = \mathbf{v} \). From equation (4.4.9)

\[
\Gamma_{ij}(x) = \frac{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i(y) p_{ij}}{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i(y) p_{ij}}
\]

\[
= \frac{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i^*(y) \frac{p_{ij}^* \pi_j}{\pi_i}}{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i^*(y) \frac{p_{ij}^* \pi_j}{\pi_i}}.
\] (4.6.10)

Note that from equation (4.5.8)

\[
\mathbf{h}(x) = \mathbf{\Gamma}(x) \mathbf{u}.
\] (4.6.11)

Using equation (4.6.10) and

\[
v_i = \sum_{j=1}^{\ell} \hat{F}^*_i(\zeta(r_j - c)) p^*_j v_j,
\]

we have

\[
\mathbf{h}_i(x) = \sum_{j=1}^{\ell} \frac{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i^*(y) \frac{p_{ij}^* \hat{F}^*_j(\zeta(r_j - c))}{\pi_i} \pi_j}{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i^*(y) \pi_i}
\]

\[
= \frac{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i^*(y) v_i}{\int_{x}^{+\infty} e^{\zeta(r_i - c)y} dF_i^*(y) \pi_i}.
\]

\[
\square
\]

From the definition of \( s_i \) in equation (4.4.10), it follows that

\[
s_i = \frac{p_i}{\pi_i \zeta(r_i - c)}.
\] (4.6.12)

The following theorem describes the bounds of Theorem 4.4.1 in terms of the original non-anticipative SMP \( \{Z^*(t), t \in \mathbb{R}\} \).
Theorem 4.6.3 The steady-state distribution of the buffer content process is bounded as

\[ C_* e^{-C x} \leq \mathbb{P}(X^* > x) \leq C* e^{-C x}, \]

where

\[ C^* = \frac{\sum_{i=1}^\ell \frac{\nu_i}{\zeta(r_i - c) + \zeta(r_i - c) - 1}}{\min_{i: r_i > c} \inf_x \left\{ \frac{\sum_{i=1}^\ell \nu_i}{\zeta(r_i - c) + \zeta(r_i - c) - 1} \right\} \int^y_x e^{(r_i - c) y} dF_i(x)} \] (4.6.13)

and

\[ C_* = \frac{\sum_{i=1}^\ell \frac{\nu_i}{\zeta(r_i - c) + \zeta(r_i - c) - 1}}{\max_{i: r_i > c} \sup_x \left\{ \frac{\sum_{i=1}^\ell \nu_i}{\zeta(r_i - c) + \zeta(r_i - c) - 1} \right\} \int^y_x e^{(r_i - c) y} dF_i(x)} \] (4.6.14)

provided that (4.6.2) has a unique solution and

\[ C_* > 0 \text{ and } C^* < +\infty. \] (4.6.15)

Proof. Using the time reversed process \{Z(t), t \in \mathbb{R}\}, we get from Theorem 4.4.1

\[ C^* = \frac{s(I - P)u}{\min_{i: r_i > c} \inf_x h_i(x)} \] (4.6.16)

\[ C_* = \frac{s(I - P)u}{\max_{i: r_i > c} \sup_x h_i(x)}. \] (4.6.17)

Now we rewrite \( s, P, u \) and \( h_i(x) \) in equations (4.6.16) and (4.6.17) in terms of the parameters of the \{Z^*(t), t \in \mathbb{R}\} process using equations (4.6.12), (4.3.11), (4.6.6) and (4.6.8). After some algebra, we obtain equations (4.6.13) and (4.6.14). Hence the proof is complete.

\[ \square \]

4.6.2 General SMP Proof

Theorem 4.6.3 of the previous subsection holds only for the special case when the SMP driving the input to a buffer can be modeled as a non-anticipative SMP. There are several applications where the non-anticipative SMP model fails. In this subsection, we prove Theorem 4.6.7 (which is stated for a general SMP) using Theorem 4.6.3 for the non-anticipative SMP. We will use Theorem 4.3.3 to convert a general SMP into a non-anticipative SMP.

Let \{Z^*(t), t \in \mathbb{R}\} be a general (not necessarily non-anticipative) SMP. Construct a non-anticipative SMP \{\overline{Z}(t), t \in \mathbb{R}\} as described in equation (4.3.16). Now we use Theorem 4.6.3 for the \ell-state non-anticipative SMP \{\overline{Z}(t), t \geq 0\}, which is non-anticipative. Similar to the equations (4.6.4), and (4.3.3) in Section 4, we now define \( \overline{X}(v), \overline{\tau}(i, j), \) and \( \overline{F}_{(i, j)}(v) \).

From the definition of the non-anticipative SMP \{\overline{Z}(t), t \in \mathbb{R}\} and equations (4.3.18), (4.3.17) and (4.3.19) it is straightforward to show that

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\[
\mathbf{r}_{(i,j)} = \begin{cases} 
 r_i & \text{if } p_{ij}^* > 0 \\
 0 & \text{if } p_{ij}^* = 0 ,
\end{cases} 
\]

(4.6.18)

\[
\hat{F}_{(i,j)}(v\mathbf{r}_{(i,j)} - c)) = \frac{\hat{F}_i^*(v(r_i - c))}{\hat{F}_i^*(+\infty)} = \frac{\chi_{ij}(v)}{p_{ij}^*} .
\]

(4.6.19)

We have,

\[
\chi_{(i,j),(k,l)}(v) = \mathbb{E}(e^{v(\mathbf{r}_{(i,j)} - c)\mathbf{r}_{(k,l)}} | \mathbf{Z}_1 = (k, l), \mathbf{Z}_0 = (i, j))
\]

\[
= \begin{cases} 
 0 & \text{if } k \neq j \\
 \hat{F}_{(i,j)}(v(\mathbf{r}_{(i,j)} - c)\mathbf{r}_{(k,l)}) & \text{if } k = j 
\end{cases}
\]

\[
= \begin{cases} 
 0 & \text{if } k \neq j \\
 \frac{\chi_{ij}(v)}{p_{ij}^*} p_{ji}^* & \text{if } k = j .
\end{cases}
\]

(4.6.20)

**Lemma 4.6.4** The smallest real positive value of \( \delta \) that satisfies

\[
\kappa(\chi(\delta), c) = 1
\]

is \( \zeta \) if and only if \( \zeta \) is the smallest real positive value that satisfies

\[
\kappa(\chi(\delta), c) = 1 .
\]

**Proof.** For a given \( v \), let \( \lambda_i \) be such that

\[
w^i \chi(v) = \lambda_i w^i ,
\]

where \( \lambda_i \) is the \( i \)th eigenvalue of \( \chi(v) \) and \( w^i \) is the corresponding eigenvector. Therefore \( \lambda_1, \lambda_2, \ldots, \lambda_\ell \) are the \( \ell \) eigenvalues of \( \chi(v) \) and \( w^1, w^2, \ldots, w^\ell \) the corresponding eigenvectors. Without loss of generality we can assume that

\[
\Re(\lambda_1) \geq \Re(\lambda_2) \geq \ldots \geq \Re(\lambda_\ell) ,
\]

where \( \Re(x) \) denotes the real part of the complex number \( x \).

From the expression of \( \chi(v) \) in equation (4.6.20), it follows that there are at most \( \ell \) linearly independent rows for the matrix \( \chi(v) \). Hence at least \( \ell^2 - \ell \) eigenvalues are zero. Let \( \lambda_1, \lambda_2, \ldots, \lambda_\ell \) be the \( \ell \) possibly non-zero eigenvalues of \( \chi(v) \) and \( \overline{w^1}, \overline{w^2}, \ldots, \overline{w^\ell} \) the corresponding eigenvectors. Without loss of generality we can assume that

\[
\Re(\overline{\lambda_1}) \geq \Re(\overline{\lambda_2}) \geq \ldots \geq \Re(\overline{\lambda_\ell}) .
\]

From Perron-Frobenius Theorem, \( \overline{\lambda_1} \) is real and positive and hence \( \Re(\overline{\lambda_1}) \geq 0 \). Note that we can choose each eigenvector \( \overline{w^i} \) such that

\[
\overline{w^i}_{(i,j)} = p_{ij}^* w_i^* .
\]

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Then also for a given \( v \), \( \overline{\lambda}_n = \lambda_n \) for all \( n \in 1,2,\ldots,\ell \). In fact,

\[
\{ \overline{w}_n \overline{\chi}(v) \}_{(k,l)} = \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \overline{w}_{(i,j)} \overline{\chi}_{(i,j),(k,l)}(v)
\]

\[
= \sum_{i=1}^{\ell} \overline{w}_{(i,k)} \overline{\chi}_{(i,k),(k,l)}(v)
\]

\[
= \sum_{i=1}^{\ell} \overline{w}_{(i,k)} \frac{\chi_{ik}(v)}{p_{kl}^*} p_{kl}^*
\]

\[
= \sum_{i=1}^{\ell} p_{ik}^* w_k \frac{\chi_{ik}(v)}{p_{kl}^*} p_{kl}^*
\]

\[
= \lambda_n p_{kl}^* w_k
\]

\[
= \lambda_n \overline{w}_{(k,l)}
\]

\[
= \overline{\lambda}_n \overline{w}_{(k,l)}.
\]

When \( v = \zeta \), the largest real-positive eigenvalue for both \( \overline{\chi}(\zeta) \) and \( \chi(\zeta) \) are equal to one. Hence the proof.

\[
\square
\]

In the next Lemma we derive the relationship between the left eigenvectors \( \mathbf{v} \) and \( \overline{\mathbf{v}} \).

**Lemma 4.6.5** Let \( \overline{\mathbf{v}} \overline{\chi}(\zeta) = \overline{\mathbf{v}} \) and \( \mathbf{v} \chi(\zeta) = \mathbf{v} \), then

\[
\overline{v}_{(i,j)} = v_i p_{ij}^*. \quad (4.6.21)
\]

**Proof.** From the definition of \( \overline{\mathbf{v}} \) we have

\[
\overline{v}_{(k,l)} = \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \overline{v}_{(i,j)} \overline{\chi}_{(i,j),(k,l)}(\zeta)
\]

\[
= \sum_{i=1}^{\ell} \overline{v}_{(i,k)} \overline{\chi}_{(i,k),(k,l)}(\zeta)
\]

\[
= \sum_{i=1}^{\ell} \overline{v}_{(i,k)} \frac{\chi_{ik}(\zeta)}{p_{kl}^*} p_{kl}^*,
\]

Since \( v_j = \sum_{i=1}^{\ell} \chi_{ij}(\zeta) v_i \), verify that

\[
\overline{v}_{(i,j)} = v_i p_{ij}^* \quad (4.6.22)
\]

solves \( \overline{\mathbf{v}} \overline{\chi}(\zeta) = \overline{\mathbf{v}} \).

\[
\square
\]

Define \( \overline{\tau}_{(i,j)} \) as the expected time the non-anticipative SMP \( \{ \overline{Z}(t), t \in \mathbb{R} \} \) spends in state \( (i,j) \). Therefore,

\[
\overline{\tau}_{(i,j)} = \mathbb{E}\{ \overline{S}_1 | \overline{Z}_0 = (i,j) \}
\]

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\[
\begin{align*}
&= \int_0^{+\infty} \left( 1 - F_{(i,j)}(x) \right) dx \\
&= \int_0^{+\infty} \left( 1 - F_{ij}^*(x) / F_{ij}^*(+\infty) \right) dx.
\end{align*}
\]

Let \( \overline{p}_{(i,j)} \) be the probability that in the steady-state the non-anticipative SMP \( \{Z(t), t \in \mathbb{R}\} \) is in state \((i, j)\), and this probability is given by
\[
\overline{p}_{(i,j)} = \frac{\overline{p}_{(i,j)} \overline{\pi}_{(i,j)}}{\sum_{k=1}^{\ell} \sum_{l=1}^{\ell} \overline{p}_{(k,l)} \overline{\pi}_{(k,l)}}.
\]

Subsequently we need an expression for \( \frac{\overline{p}_{(i,j)} p_{ij}^*}{\overline{p}_{(i,j)}} \). This is shown in the following lemma.

**Lemma 4.6.6** For all \( i, j = 1, \ldots, \ell \)
\[
\frac{\overline{p}_{(i,j)} p_{ij}^*}{\overline{p}_{(i,j)}} = \frac{\tau_i}{p_i}.
\] (4.6.23)

**Proof.** The following standard transformations yields the result:
\[
\begin{align*}
\frac{\overline{p}_{(i,j)} p_{ij}^*}{\overline{p}_{(i,j)}} &= \frac{\overline{p}_{(i,j)} \overline{\pi}_{ij} p_{ij}^*}{\overline{p}_{(i,j)} \overline{\pi}_i} \\
&= \frac{\overline{p}_{(i,j)} \overline{\pi}_{ij} p_{ij}^*}{\overline{p}_{(i,j)} \overline{\pi}_i} \\
&= \frac{\overline{p}_{(i,j)} \overline{\pi}_{ij} p_{ij}^*}{\overline{p}_{(i,j)} \overline{\pi}_i} \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \sum_{l=1}^{\ell} \overline{p}_{(k,l)} \overline{\pi}_{(k,l)} \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \sum_{l=1}^{\ell} \pi_k p_{ij}^* \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \int_0^{+\infty} (1 - F_{ki}^*(x) / F_{kj}^*(+\infty)) dx \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \int_0^{+\infty} (1 - F_{kj}^*(x)) dx \\
&= \frac{1}{\pi_i} \sum_{k=1}^{\ell} \pi_k \tau_k \\
&= \frac{\tau_i}{p_i}.
\end{align*}
\]

\( \square \)

Define
\[
H = \sum_{i=1}^{\ell} \frac{v_i}{\zeta(r_i - c)} \left( \sum_{j=1}^{\ell} (\lambda_{ij}(\zeta)) - 1 \right),
\] (4.6.24)
\[
\Psi_{\text{max}}(i, j) = \sup_x \left\{ \frac{v_i e^{-\zeta(r_i - c)x} \int_{r_i}^{+\infty} e^{\zeta(r_i - c)y} dF_{ij}^*(y)}{p_i \int_{r_i}^{+\infty} dF_{ij}^*(y)} \right\}
\] (4.6.25)
and
\[
\Psi_{\text{min}}(i, j) = \inf_x \left\{ \frac{v_i e^{-\zeta(r_i - c)x} \int_x^{+\infty} e^{\zeta(r_i - c)y} dF_{ij}^*(y)}{\mu_i \int_x^{+\infty} dF_{ij}^*(y)} \right\} ,
\]
(4.6.26)

Hence, we can now state the following theorem giving exponential bounds for \( \Psi(x) \) in terms of original general SMP environment process \( \{Z^*(t), t \in \mathbb{R}\} \).

**Theorem 4.6.7** Suppose that there exists a unique solution of equation (4.6.2) and
\[
\Psi_{\text{min}}(i, j) > 0 \quad \text{and} \quad \Psi_{\text{max}}(i, j) < +\infty \quad \text{for all} \ i, j \in \{1, \ldots, \ell\} .
\]
(4.6.27)
Then
\[
C_0 e^{-\zeta x} \leq \mathbb{P}(X^* > x) \leq C^* e^{-\zeta x} , \quad \text{for all} \ x \geq 0 ,
\]
(4.6.28)
where \( \zeta \) is from equation (4.6.2) and
\[
C^* = \frac{H}{\min_{i \neq r_i < c, j} \{ \Psi_{\text{min}}(i, j) \}} ,
\]
\[
C_0 = \frac{H}{\max_{i \neq r_i < c, j} \{ \Psi_{\text{max}}(i, j) \}} .
\]

**Proof.** From equation (4.6.13) for the non-anticipative SMP \( \{Z(t), t \in \mathbb{R}\} \), we have
\[
C^* = \frac{\sum_{i=1}^\ell \sum_{j=1}^\ell \frac{\tau_{(i,j)}}{\zeta(r_i - c)} \left( \frac{\hat{F}_{ij}(\zeta(r_i - c))}{\hat{F}_{ij}(\zeta)} - 1 \right)}{\min_{i \neq r_i < c} \inf_x \left\{ \frac{\tau_{(i,j)}}{\hat{F}_{ij}(y)} \int_x^{+\infty} e^{\zeta(r_i - c)y} dF_{ij}^*(y) \right\}} ,
\]
(4.6.29)
We substitute the expressions \( \tau_{(i,j)}, \hat{F}_{ij}(\zeta), \hat{F}_{ij}(\zeta), \hat{F}_{ij}(y) \) and \( \tau_{(i,j)} \hat{F}_{ij}/\zeta(r_i - c) \) from equations (4.6.22), (4.6.18), (4.6.19), (4.3.17) and (4.6.23) respectively into equation (4.6.29) to get
\[
C^* = \sum_{i=1}^\ell \sum_{j=1}^\ell \frac{\tau_{(i,j)}}{\zeta(r_i - c)} \left( \frac{\hat{F}_{ij}^*(\zeta(r_i - c))}{\hat{F}_{ij}^*(\zeta)} - 1 \right)
\min_{i \neq r_i < c} \inf_x \left\{ \frac{v_i e^{\zeta(r_i - c)x} \int_x^{+\infty} e^{\zeta(r_i - c)y} dF_{ij}^*(y)}{\mu_i \int_x^{+\infty} dF_{ij}^*(y)} \right\} ,
\]
\[
= \sum_{i=1}^\ell \frac{\tau_{(i,j)}}{\zeta(r_i - c)} \left( \sum_{i=1}^\ell \chi_{ij}(\zeta) - 1 \right)
\min_{i \neq r_i < c} \inf_x \left\{ \frac{v_i e^{-\zeta(r_i - c)x} \int_x^{+\infty} e^{\zeta(r_i - c)y} dF_{ij}^*(y)}{\mu_i \int_x^{+\infty} dF_{ij}^*(y)} \right\} ,
\]
Note that assumptions (4.6.27) yield (4.6.15). Using a similar analysis for \( C_0 \), Theorem 4.6.7 is proved. \(\square\)
Chapter 5

Multiplexing of SMP sources

5.1 Model

In this chapter we consider a fluid model that admits traffic from \(N\) independent sources, each driven by a random environment SMP process \(\{Z_*^{*,k}(t), t \in \mathbb{R}\}\) with the state space \(\{1, \ldots, \ell^k\}\) (see Figure 1, page 2). Note that \(Z_*^{*,k}(t)\) can be thought of as the state of the \(k\)th input source \((k = 1, \ldots, N)\) at time \(t\). When source \(k\) is in state \(Z_*^{*,k}(t)\), it generates fluid at rate \(r_*^k\) into buffer. Let \(X^*\) be the steady-state buffer content. In this chapter we generalize the results of Chapter 4 to the case of \(N\) sources fluid model and find the exponential bounds for a function

\[
\Psi(x) = \mathbb{P}(X^* > x).
\]

Similarly like in Chapter 4 we can prove that

\[
\Psi(x) = \mathbb{P}(X^* > x) = \mathbb{P} \left( \sup_{t \le 0} \int_0^t \left( \sum_{k=1}^N r_*^k - c \right) ds > x \right).
\]

Let assume that each environment SMP’s \(\{Z_*^{*,k}(t), t \in \mathbb{R}\}\) is non-anticipative. Then from Theorem 4.3.2 its reversed process \(\{Z_*^{k}(t), t \in \mathbb{R}\}\) is non-anticipative SMP’s and we have that

\[
\Psi(x) = \mathbb{P} \left( \sup_{t \ge 0} \int_0^t \left( \sum_{k=1}^N r_*^k - c \right) ds > x \right).
\]

Note that assumption of non-anticipativity is not restrictive because we can treat a general SMP process as SMP non-anticipative process on the vector state space as it was presented in section 4.3.3.

5.2 Notations

Let \(\{Z_*^k(t), t \ge 0\}\) be a non-anticipative SMP. It modulates input from the \(k\)th source. Denote by

\[
F_*^k(x) = \{F_*^k(x)\} = \{y\} \cdot F_*^k(x)
\]
the kernel of \( \{Z^k(t), t \geq 0\} \). We assume that distribution function \( F^k_i(x) \) is absolutely continuous with continuous density function \( f^k_i(x) = \frac{dF^k_i(x)}{dx} \) and \( \mathbb{R}_+ \) as a support. By \( \rho^k_i(x) = \frac{f^k_i(x)}{F^k_i(x)} \) we denote the hazard rate function. The expected time the \( k \)th SMP spends in state \( i \) is \( r^k_i \). Let \( \{Z^k(t), t \geq 0\} \) fulfills condition 4.3.2. Then there exists unique stationary distributions of the \( k \)th SMP \( \{Z^k(t), t \geq 0\} \) and its underlying DTMC \( Z^k_n = Z^k(S^k_n+) \), where \( S^k_n \) are the \( n \)th jump epoch of \( Z^k(t) \), denoted by \( p^k \) and \( \pi^k \) respectively. We assume for stability and non-triviality that

\[
\sum_{k=1}^{N} \sum_{m=1}^{\ell^k} p^k_{m} r^k_{m} < c < \sup_{k,m} r^k_{m} .
\]

Moreover, we set

\[
P^k = \{p^k_{ij} \}_{i,j=1,\ldots,\ell^k} = \{\mathbb{P}(Z^k_i = j | \mathbb{P}(Z^k_0 = i) \}_{i,j=1,\ldots,\ell^k}, \quad (5.2.1)
\]

\[
\Phi^k(\delta) = \{\Phi^k_{ij}(\delta) \}_{i,j=1,\ldots,\ell^k} = \{\mathbb{E}(e^{\delta(r^k_i - c^k)}S^k_i; Z^k_i = j | Z^k_0 = i) \}_{i,j=1,\ldots,\ell^k} \quad (5.2.2)
\]

for auxiliary constant \( c^k \). Define also

\[
\delta^*= \min_k \{\delta^k,*\} ,
\]

where

\[
\delta^k,* = \sup \{\delta \geq 0 : \Phi^k(\delta) < +\infty \} .
\]

We assume that

\[
\delta^* > 0 .
\]

Following the analysis for single source model for each source \( k \), we now compute \( \eta, u^k \) and auxiliary constants \( c^k \) from the following system of equations, called (B)asic (S)ystem of (N)onlinear (E)quations:

\[
\Phi^k(\eta) \ u^k = u^k \quad (k = 1, \ldots, N) \quad (5.2.4)
\]

\[
\sum_{k=1}^{N} c^k = c .
\]

All these quantities are needed to determine bounds for probability of the buffer overflow \( \mathbb{P}(X^* > x) \) given in Theorem 5.3.1. Under some additional assumptions put on the kernel distribution there exists unique solution of BSNE (see Section 5.5). The case when BSNE has not solution is discussed in Section 5.6.

Finally, we let

\[
s^k_i = \frac{p^k_i}{\pi^k_i \eta(r^k_i - c^k)} ,
\]

\[
\Gamma^k(x) = \{\Gamma^k_{i,j}(x)\} = \{\mathbb{E}(e^{\eta(r^k_i - c^k)}S^k_i; Z^k_i = j | Z^k_0 = i, S^k_i > x) \} \quad (5.2.6)
\]

and \( h^k(i, x) \) satisfy

\[
h^k(i, x) = (\Gamma^k(x) u^k)_i . \quad (5.2.7)
\]

We can now state main theorems concerning the fluid model driven by multiplexing of SMP sources.
5.3 Results

**Theorem 5.3.1** Assume that the BSNE given in (5.2.4) has a solution (which is unique). Then the distribution of the steady-state buffer content process is bounded as

\[ C_* e^{-\zeta x} \leq \mathbb{P}(X^* > x) \leq C^* e^{-\zeta x}, \tag{5.3.1} \]

provided that \( C_*>0 \) and \( C^* < +\infty \), where

\[
C^* = \frac{\prod_{k=1}^N s^k(I - P^k) u^k}{\min_{i^1, i^2, \ldots, i^N} \sum_{k=1}^N r_{ik} \geq c \prod_{k=1}^N \{ \inf_x h^k(i^k, x) \}},
\]

\[
C_* = \frac{\prod_{k=1}^N s^k(I - P^k) u^k}{\max_{i^1, i^2, \ldots, i^N} \sum_{k=1}^N r_{ik} \geq c \prod_{k=1}^N \{ \sup_x h^k(i^k, x) \}}.
\]

Following Section 4.6.1, in this case we can also convert constant \( C_* \) and \( C^* \) in terms of original input environment processes \( \{Z^*,k(t), t \in \mathbb{R}\} \). Let \( F^*,k(x) = \{F^*,k(x) \} \) be the kernel of SMP \( \{Z^*,k(t), t \in \mathbb{R}\} \) and

\[ \chi^k(\delta) = \{\chi^k_0(\delta)\} = \{\mathbb{P}(e^{\delta(r^k - \delta)}s^k_i, Z^*,k = j|Z^*,k = i)\}. \]

Let \( \zeta \) satisfies following system of equations

\[
\begin{align*}
\nu^k \chi^k(\zeta) &= \nu^k \quad (k = 1, \ldots, N) \\
\sum_{k=1}^N \zeta^k &= c.
\end{align*}
\tag{5.3.2}
\]

Define also

\[
H^k = \sum_{i=1}^{\ell^k} \frac{v_i^k}{\zeta(r_i^k - c^k)} \left( \sum_{j=1}^{\ell^k} (\chi^k_{ij}(\zeta)) - 1 \right),
\tag{5.3.3}
\]

\[
\Psi^k_{\min}(i,j) = \inf_x \left\{ \frac{v_i^k e^{-\zeta(r_i^k - c^k)x} \int_x^{+\infty} e^{\zeta(r_i^k - c^k)y} dF^*,k_{ij}(y)}{\Psi^k_{ij}(x)} \right\},
\tag{5.3.4}
\]

and

\[
\Psi^k_{\max}(i,j) = \sup_x \left\{ \frac{v_i^k e^{-\zeta(r_i^k - c^k)x} \int_x^{+\infty} e^{\zeta(r_i^k - c^k)y} dF^*,k_{ij}(y)}{\Psi^k_{ij}(x)} \right\}.
\tag{5.3.5}
\]

Then we have following theorem.

**Theorem 5.3.2** If system of equations (5.3.2) has a solution \( \zeta \) and \( \Psi^k_{\min}(i,j) > 0 \) and \( \Psi^k_{\max}(i,j) < +\infty \) for all \( i,j \in \{1, \ldots, \ell^k \} \), then the bounds on the limiting distribution of the buffer content process \( \{X(t), t \geq 0\} \) driven by \( N \) independent SMP sources are given by

\[ C_* e^{-\zeta x} \leq \mathbb{P}(X^* > x) \leq C^* e^{-\zeta x}, \]

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where

\[
C^* = \frac{\prod_{k=1}^{N} H^k}{\min_k \prod_{k=1}^{N} \Psi_{\min}^k (i^k, j^k)},
\]

and

\[
K = \left\{ (i^1, j^1), (i^2, j^2), \ldots, (i^N, j^N) : \sum_{k=1}^{N} x^k > c \text{ and } \forall k, P^k(i^k, j^k) > 0 \right\}. \quad (5.3.6)
\]

5.4 Proof of the main theorem

We describe only a brief outline of the proof of the Theorem 5.3.1 by illustrating the main steps of the proof since most of the analysis is identical to the proof of theorem presented in Section 4.4.

Proof.

1. Markov process and its generator

Let \( E^k = \{1, \ldots, \ell^k\} \times \mathbb{R}_+ \). Consider canonical PDMP \( \{w^k(t), t \geq 0\} = \{(Z^k(t), S^k(t)), t \geq 0\} \) on stochastic basis \((D_{E^k}[0, +\infty), \mathcal{F}^k, \{\mathcal{F}^{w^k}\}, \mathbb{P}^k)\), where \( S^k(t) \) is the supplementary age process of \( \{Z^k(t), t \geq 0\} \). Then by 2.1.25 we can define probability space \((D_{E}[0, +\infty), \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\), where

\[
E = \times_{k=1}^{N} E^k
\]

\[
\mathcal{F}_t = \bigotimes_{k=1}^{N} \mathcal{F}^{w^k}_{t^+}, \quad \mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t
\]

and

\[
\mathbb{P} = \mathbb{P}^1 \otimes \mathbb{P}^2 \otimes \ldots \otimes \mathbb{P}^N.
\]

Note that on \((D_{E}[0, +\infty), \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\), the processes \( \{w^k(t), t \geq 0\} \) are independent. On stochastic basis \((D_{E}[0, +\infty), \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) we construct a Markov process \( \{W(t), t \geq 0\} \) defined by

\[
W(t) = (Y^1(t), S^1(t), Y^2(t), S^2(t), \ldots, Y^N(t), S^N(t)).
\]

2. Exponential change of measure

5.4.1 By Lemma 4.5.2 functions \( h^k(i, x) (i = 1, \ldots, \ell^k, x \in \mathbb{R}_+) \) defined in (5.2.7) are positive.
We assume that
\[ h^k(i, \cdot) \in C^1_b(\mathbb{R}_+) \quad \text{for all } k = 1, \ldots, N \text{ and } i = 1, \ldots, \ell^k. \]  
(5.4.1)
To get constant \( C^* \) finite we also assume the following condition:
\[ \min_{i_k, i_{k+1}, \ldots, i_N} \sum_{k=1}^N r_k \geq c \begin{array}{l} \prod_{k=1}^N \inf_x h(i^k, x) > 0. \end{array} \]  
(5.4.2)
Then, similarly like in a single source case (see Lemma 4.5.2 and the second point of the sketch of the proof of Theorem 4.4.1, in particular (4.4.14)), processes
\[ N^k(t) = \frac{h^k(w^k(t))}{h^k(i, x)} e^{-\int_0^t (A^k h^k)(w^k(s)) ds} \]
\[ = \frac{h^k(w^k(t))}{h^k(i, x)} e^{-\int_0^t \sum_{k=1}^N (r_k h_{i^k} - c) ds} \]
are exponential martingales. Since, on stochastic basis \((D_E[0, +\infty), \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) the environment processes \(\{Z^k(t), t \geq 0\}\) are independent, then also processes \(\{N^k(t), t \geq 0\}\) are independent. Thus
\[ N(t) = \prod_{k=1}^N N^k(t) = \frac{h^1(w^1(t)) \ldots h^N(w^N(t))}{h^1(i_1, x_1) \ldots h(i_N, x_N)} e^{-\int_0^t \sum_{k=1}^N (r_k h_{i^k} - c) ds} \]
\[ = \frac{h(W(t))}{h(i, x)} e^{-\int_0^t \sum_{k=1}^N (r_k h_{i^k} - c) ds} \]  
(5.4.3)
is a \(\mathbb{P}\)-martingale, where \(h(i, x) = h(i_1, x_1, \ldots, i_N, x_N) = \prod_{k=1}^N h^k(i^k, x^k)\) and \(i = (i_1, \ldots, i_N), i^k \in \{1, \ldots, \ell^k\}\) and \(x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N\).

Similarly like in a single source case, under conditions (5.4.1) and (5.4.2) by Remark 3.3.5 all assumptions of Theorem 3.3.2 are fulfilled. Hence from Theorem 3.3.2 we can construct uniquely a new probability space \((D_E[0, +\infty), \mathcal{F}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})\) by
\[ \frac{d\tilde{\mathbb{P}}(i, x)}{d\mathbb{P}(i, x)} = N(t). \]

Note that
\[ \tilde{\mathbb{P}}(i, x) = \tilde{\mathbb{P}}(i, x_1) \circ \tilde{\mathbb{P}}(i, x_2) \circ \cdots \circ \tilde{\mathbb{P}}(i, x_N), \]
where \(\tilde{\mathbb{P}}(i^k, x^k)\) is given by
\[ \frac{d\tilde{\mathbb{P}}(i^k, x^k)}{d\mathbb{P}(i^k, x^k)} = N^k(t) \]
and \(\mathbb{P}(i^k, x^k)\) is the probability measure on \((D_E[0, +\infty), \mathcal{F}^k, \{\mathcal{F}^k_{t+}\})\) under which \(w^k(0) = (i^k, x^k)\).
Thus from Lemma 4.5.4
\[ \hat{d}^k = \mathbb{E}(r^k_{\mathbf{Z}(t)} - c^k) > 0 \]
and then
\[ \hat{d} = \mathbb{E} \left( \sum_{k=1}^{N} r^k_{\mathbf{Z}(t)} - c \right) > 0 . \] (5.4.4)
Let
\[ \tau(x) = \inf \{ t > 0 : \int_{0}^{t} \left( \sum_{k=1}^{N} r^k_{\mathbf{Z}(s)} - c \right) ds > x \} . \] (5.4.5)
From (5.4.4)
\[ \mathbb{P}(\tau(x) < +\infty) = 1 . \] (5.4.6)
Moreover,
\[ \left\{ \sup_{t \geq 0} \int_{0}^{t} \left( \sum_{k=1}^{N} r^k_{\mathbf{Z}(s)} - c \right) ds > x \right\} \equiv \{ \tau(x) < +\infty \} . \]
Hence
\[ \mathbb{P}(\sup_{t \geq 0} \int_{0}^{t} \left( \sum_{k=1}^{N} r^k_{\mathbf{Z}(s)} - c \right) ds > x) = \mathbb{P}(\tau(x) < +\infty) . \]
Now from Lemma 3.1.8
\[ \mathbb{P}(\tau(x) < +\infty) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \pi^k_y(y^k) dy^k \mathbb{E}(\mathbf{j} \mid \mathbf{y}) \left[ N(\tau(x))^{-1}; \tau(x) < +\infty \right] , \]
where \( \mathbf{j} = (j^1, \ldots, j^N) \), \( \mathbf{y} = (y^1, \ldots, y^N) \) and
\[ \pi^k_y(y) dy = p^k \frac{1 - F^k(y)}{\tau^k_y} dy \]
is a stationary distribution of \( \{ \mathbf{w}^k(t), t \geq 0 \} \).

3. **Bounds**

Then by (5.4.6) we have
\[ \mathbb{P}(\tau(x) < +\infty) = \prod_{k=1}^{N} \sum_{j^k=1}^{t^k} \int_{\mathbb{R}^N} \prod_{k=1}^{N} \sum_{j^k=1}^{t^k} h^k(j^k, y^k) \pi^k_{j^k}(y^k) dy^k \mathbb{E}(\mathbf{j} \mid \mathbf{y}) \left[ \frac{1}{h(\mathbf{W}(\tau(x)))} \right] . \] (5.4.7)
(5.4.8)
(5.4.9)

At time \( \tau(x) \), the \( \mathbf{W}(t) \) process can only be in states \( (i^1, x^1, i^2, x^2, \ldots, i^N, x^N) \) that satisfy \( \sum_{k=1}^{N} r^k_{i^k} > c \). Hence we have
\[ \mathbb{E} \left\{ \frac{1}{h(\mathbf{W}(\tau(x)))} \right\} \leq \min_{i^1, i^2, \ldots, i^N; \sum_{k=1}^{N} r^k_{i^k} > c} \frac{1}{\prod_{k=1}^{N} \{ \inf_{x} h^k(i^k, x) \} .} \]

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and
\[
\mathbb{E}\left\{ \frac{1}{h(\mathbf{W}(\tau(x)))} \right\} \geq \frac{1}{\max_{\beta, \ldots, \beta^N} \sum_{k=1}^{N} \epsilon_k \max_{\beta, \ldots, \beta^N} \sup_{x} h^k(i_k, x)}
\]

Thus we have assertion of Theorem 5.3.1 by equality
\[
\prod_{k=1}^{N} \sum_{j=1}^{C^k} \int_{0}^{+\infty} h^k(y^k) \pi_{ij}^k(y^k) dy^k = \prod_{k=1}^{N} s^k(I - P_k)u^k.
\]

Note that similarly like in a single source case conditions (5.4.1) and (5.4.2) are equivalent to assumptions \( C_* > 0 \) and \( C_* < +\infty \) respectively.

\[ \square \]

5.5 Discussion of the BSNE and theorem conditions

In this section we give conditions sufficient for assumptions made in Theorem 5.3.1 to be fulfilled in terms of kernel distributions \( F_{ij}(x) \). Note that in the proof of Theorem 5.3.1 we use only two assumptions:

(i) conditions (5.4.1) and (5.4.2) hold,

(ii) there exists unique solution of BSNE.

Conditions (5.4.1) and (5.4.2) are equivalent to assumption that \( C_* > 0 \) and \( C_* < +\infty \) respectively. In next lemma we find distributions \( F_{ij}(x) \) fulfilling condition (i).

**Lemma 5.5.1** If for all \( i \) there exists \( j \) such that
\[
\inf_{x \geq 0} \Gamma_{ij}^k(x) > 0 \quad k = 1, \ldots, N \tag{5.5.1}
\]
and for all \( i \) such that \( r_i^k = e^k > 0 \)
\[
\sup_{x \geq 0} \Gamma_{ij}^k(x) < +\infty \quad k = 1, \ldots, N \tag{5.5.2}
\]
then functions \( h^k(x) \) fulfill conditions (5.4.1) and (5.4.2).

**Proof.** Note that
\[
h_i^k(x) = \sum_{j=1}^{r_i^k} (F_i^k(x))^{-1} \int_{x}^{+\infty} e^{\eta(r_i^k - e^k)(z-x)} p_{ij}^k f_i^k(z) \, dz. \tag{5.5.3}
\]
Thus if \( r_i^k - c^k < 0 \), then \( h_i^k(x) \leq \sum_{j=1}^{\ell^k} p_{ij}^k = 1 \). Hence condition (5.4.1) is equivalent to the assumption that

\[
\sup_{x \geq 0} (\tilde{F}_i^k(x))^{-1} \int_x^\infty e^{\eta(r_i^k - c^k)(z-x)} p_{ij}^k f_i^k(z) \, dz < +\infty \quad (5.5.4)
\]

for \( i \) such that \( r_i^k - c^k > 0 \). Moreover, sufficient condition for (5.4.2) is

\[
\inf_x h_i^k(x) > 0
\]

for all \( k = 1, \ldots, N \) and \( i = 1, \ldots, \ell^k \). Thus for (5.4.2) it suffices to assume that for all \( i \) there exists \( j \) fulfilling

\[
\inf_{x \geq 0} (\tilde{F}_i^k(x))^{-1} \int_x^\infty e^{\eta(r_i^k - c^k)(z-x)} p_{ij}^k f_i^k(z) \, dz > 0 \quad , \quad (5.5.5)
\]

which completes the proof in a view of (5.2.6).

Note that a sufficient condition for (5.5.1) is: for all \( i \)

\[
\lim_{x \to +\infty} \inf_{x} \rho_i^k(x) > 0 \quad (5.5.6)
\]

and for (5.5.2) is: for all \( i \) such that \( r_i^k - c^k > 0 \)

\[
\lim_{x \to +\infty} \sup_{x} \rho_i^k(x) = \nu_i^k > 0 \quad \text{and} \quad \nu_i^k > \eta(r_i^k - c^k) \quad . \quad (5.5.7)
\]

We give now necessary and sufficient conditions for the existence of a unique solution of BSNE given in (5.2.4). Let \( \kappa(\Phi^k(\delta), c^k) \) be the Perron-Frobenius eigenvalue of matrix

\[
\Phi^k(\delta) = \{ p_{ij} \tilde{F}_i(\delta(r_i^k - c^k)) \} \quad .
\]

A necessary conditions for the solution of BSNE are

\[
\sum_{i=1}^{\ell^k} p_{ij}^k r_i^k < c^k , \quad k = 1, \ldots, N ,
\]

because otherwise \( d/d\delta (\kappa(\Phi^k(\delta), c^k)) |_{\delta=0} > 0 \) and hence there is no \( \eta \) by the convexity of function \( \kappa(\Phi^k(\alpha), c^k) \) with respect to variable \( \alpha \).

**Lemma 5.5.2** Let \( \lim_{\delta \to \eta_i^k} \Phi_{ij}^k(\delta) = +\infty \) for some \( \eta_i^k \) and \( i, j = 1, \ldots, \ell^k \) \( k = 1, \ldots, N \). \( \kappa(\Phi^k(\delta), c^k) \) is a continuous and strictly convex function of \( \delta \). Moreover, \( \kappa(\Phi(\delta), c^k) \) as a function of \( c^k \) is decreasing and

\[
\lim_{\delta \to \eta_i^k} \kappa(\Phi^k(\delta), c^k) = +\infty \quad .
\]
Proof. From Theorem 2 of Miller [89] \( \kappa(\Phi^k(\delta), c^k) = \kappa^k(\delta, c^k) \) is a continuous and strictly convex function of \( \delta \). Let \( \kappa_A \) be Perron-Frobenius eigenvalue of matrix \( A \). From the fact that if \( A \preceq B \), then \( \kappa_A \preceq \kappa_B \), we get that inequality \( c^k_1 < c^k_2 \) implies that \( \kappa(\Phi^k(\delta), c^k_1) > \kappa(\Phi^k(\delta), c^k_2) \). Hence \( \kappa(\Phi^k(\delta), c^k) \) as a function of \( c^k \) is decreasing. Moreover, from the convexity of \( \delta \) function \( \kappa(\Phi^k(\delta), c^k) \), assumption of lemma and Theorem 3.b of Miller [89] we get that \( \lim_{\delta \to \eta^k} \kappa(\Phi^k(\delta), c^k) = +\infty \).

Sufficient conditions for the unique existence of solution of BSNE are given in the following theorem.

**Theorem 5.5.3** Let \( \lim_{\delta \to \eta^k} \Phi^k_{ij}(\delta) = +\infty \) for some \( \eta^k \) and \( i, j = 1, \ldots, \ell^k \) \( (k = 1, \ldots, N) \). Then there exists uniquely \( \eta \) and \( c^1 > \sum_{n=1}^\ell p^1_n r^1_n, \ldots, c^N > \sum_{n=1}^\ell p^N_n r^N_n \) solving the BSNE given in (5.2.4).

Proof. Note that \( \kappa(\Phi^k(0), c^k) = 1 \). From Lemma 5.5.2 we get that for continuous and convex \( \delta \) function \( \kappa(\Phi^k(\delta), c^k) \) we have \( \lim_{\delta \to \eta^k} \kappa(\Phi^k(\delta), c^k) = +\infty \). Hence there exists solution \( \eta \) of equation \( \kappa(\Phi^k(\eta), c^k) = 1 \). Finally, note that if \( c^k \preceq \sum_{n=1}^\ell p^k_n r^k_n \), then \( \frac{\partial}{\partial \delta} \Phi^k_{ij}(\delta)|_{\delta = 0} \to 0 \). Thus \( \frac{\partial}{\partial \delta} \kappa(\Phi^k(\delta), c^k)|_{\delta = 0} \to 0 \). Moreover, \( \kappa(\Phi^k(\delta), c^k) \) as a function of \( c^k \) is by Lemma 5.5.2 decreasing. Therefore each graph of function of \( c^k \) \( \kappa(\Phi^k(\eta), c^k) \) moves from 0 to a certain point at level 1 as \( c^k \) moves from \( \sum_{i=1}^{\ell^k} r^k_i P^k_i \) to \( \max_i r^k_i \) to get solution of BSNE (see Figure 4).

---

**Figure 4.**

---

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Note that a large class of phase-type distributions $F^k_i(x)$ fulfills all assumption of Theorem 5.3.1; see also Palmowski and Rolski [101]. Distribution function $F^k_i(x)$ is a phase-type with representation $(\alpha^k, T^k)$, when

$$F^k_i(x) = \alpha^k \exp(-xT^k)e,$$  \hspace{1cm} (5.5.8)

where $\alpha^k$ is a probability vector, $T^k$ is a transient intensity matrix and $e$ is a column vector consisting of 1’s. Then

$$\hat{F}^k(s) = -\alpha^k(sI + T^k)^{-1}t^{\circ,k}$$  \hspace{1cm} (5.5.9)

$$f^k_i(x) = \alpha^k \exp(xT^k)t^{\circ,k},$$  \hspace{1cm} (5.5.10)

where $t^{\circ,k} = -T^k e$. For further properties of phase-type distributions we refer to Neuts [95], Rolski et al [108] and Asmussen [6].

**Corollary 5.5.4** If $F^k_i(x)$ are phase-type distributions, then there exists the unique solution of the BSNE and conditions (5.5.6) and (5.5.7) hold.

**Proof.** We assume that $T^k$ is a subintensity matrix (off-diagonal entries are non-negative and at least one row sums up to a strictly negative number) and hence the Perron-Frobenius eigenvalue $\lambda_1$ is negative and for all remaining eigenvalues $\Re(\lambda_j) < \Re(\lambda_1)$ ($j = 2, \ldots, n$). Therefore from (5.5.9) we get that $\lim_{s \to +\lambda_1} \hat{F}^k_i(s) = +\infty$. That is, the assumption of Theorem 5.5.3 are fulfilled and there exists solution of BSNE. Moreover, for phase-type distributions conditions (5.5.6) and (5.5.7) are also fulfilled. Suppose that $\phi_i$ and $\xi_i$ are the left and right eigenvector corresponding to $\lambda_i$ respectively. If moreover the eigenvectors $\{\phi_1, \ldots, \phi_n\}$ are independent (for instance in the case when eigenvalues of $T^k$ are distinct), then

$$\exp(xT^k) = \sum_{i=1}^n e^{\lambda_i x} \xi_i \phi_i.$$ 

Using this representation, equations (5.5.8), (5.5.10) and definition of vector $t^\circ$ we have

$$\lim_{x \to +\infty} \rho^k_i(x) = \frac{\alpha^k \xi_i \phi_i t^{\circ,k}}{\alpha^k \xi_i \phi_i e} = -\lambda_1 > 0.$$

Hence condition (5.5.6) holds. Moreover, from the fact that $\Phi^{k}_{ij}(\eta) < +\infty$ we have that for $i$:

$$r^k_i - c^k > 0$$

$$\eta (r^k_i - c^k) < -\lambda_1.$$  \hspace{1cm} (5.5.11)

Thus condition (5.5.7) also holds.

We show now that if distributions $F^k_i$ are IFR or DFR, that is they have increasing or decreasing hazard rate function respectively, then constants $C_1$ and $C_\ast$ in Theorem 5.3.1 can be computed directly. Note that to do this we need in fact to calculate directly the $\inf_x h^k(i, x)$ and $\sup_x h^k(i, x)$, where $k = 1, \ldots, N$ and $i = 1, \ldots, \ell^k$.  

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**Theorem 5.5.5** If $F_i^k$ is IFR or DFR, then $\sup_x h^k(i,x)$ and $\inf_x h^k(i,x)$ respectively occur at $x^*$ taking values given by the following tables

<table>
<thead>
<tr>
<th>$x^*$</th>
<th>$r_i^k &gt; c^k$</th>
<th>$r_i^k \leq c^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sup_x h^k(i,x)$</td>
<td>0</td>
<td>$\rho_i^k(\infty)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{k} \gamma_j u_j^k \frac{\rho_i^k(\infty)}{\rho_i^k(\infty) - \gamma_j(x^*)}$</td>
<td>$u_i^k$</td>
<td></td>
</tr>
<tr>
<td>$\inf_x h^k(i,x)$</td>
<td>$\sum_{j=1}^{k} \gamma_j u_j^k \frac{\rho_i^k(\infty)}{\rho_i^k(\infty) - \gamma_j(x^*)}$</td>
<td>$u_i^k$</td>
</tr>
</tbody>
</table>

where

$$\rho_i^k(\infty) = \lim_{x \to \infty} \rho_i^k(x).$$

**Proof.** Let $Y^t$ be the remaining life associated with the random variable $Y$ with distribution function $F_i^t(x)$. Then $Y^t$ will have distribution $F_i^t(x)$ given by

$$1 - F_i^t(x) = \mathbb{P}(Y^t > x) = \mathbb{P}(Y - t > x | Y > t).$$

It is known that (see Ross [110]) if $Y$ is IFR (DFR), then $Y^t$ is stochastically decreasing (increasing) in $t$ and hence $\mathbb{E}[f(Y^t)]$ is decreasing (increasing) in $t$ for all increasing functions $f$. Therefore if $Y$ is IFR (DFR) and $r_i^k > c^k$, then

$$h^k(i,x) = \sum_{j=1}^{k} \gamma_j u_j^k \mathbb{E}(e^{\gamma_j(r_i^k - c^k)} | Y > x) = \sum_{j=1}^{k} \gamma_j u_j^k \frac{e^{-\gamma_j(r_i^k - c^k)x} \int_{x}^{\infty} e^{\gamma_j(r_i^k - c^k)y} dF_i^k(y)}{\int_{x}^{\infty} dF_i^k(y)}$$

is decreasing (increasing) in $x$. Similarly, if $Y$ is IFR (DFR), then $Y^t$ is stochastically decreasing (increasing) in $t$ and hence $\mathbb{E}[f(Y^t)]$ is increasing (decreasing) in $t$ for all decreasing functions $f$. Therefore if $Y$ is IFR (DFR) and $r_i^k \leq c^k$, then

$$h^k(i,x) = \sum_{j=1}^{k} \gamma_j u_j^k \mathbb{E}(e^{\gamma_j(r_i^k - c^k)} | Y > x) = \sum_{j=1}^{k} \gamma_j u_j^k \frac{e^{-\gamma_j(r_i^k - c^k)x} \int_{x}^{\infty} e^{\gamma_j(r_i^k - c^k)y} dF_i^k(y)}{\int_{x}^{\infty} dF_i^k(y)}$$

is increasing (decreasing) in $x$. Hence using deL’Hospital rule and noting that

$$h^k(i,0) = \sum_{j=1}^{k} \gamma_j u_j^k \Phi_i^k(\eta(r_i^k - c^k)) = \sum_{j=1}^{k} u_j^k \Phi_i^k(\eta) = u_i$$

we get bounds given in the table.

If $Y$ is not an IFR or DFR random variable, then $C^*$ and $C_*$ can be obtained only numerically.

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5.6 When BSNE has no solution

It can happen that the BSNE has no solution. This is a case when e.g. kernel distributions \( F_{ij}(x) \) are heavy-tailed. Then we cannot get exponential bounds for the buffer overflow probability (see Rolski et al [107], Boxma [21]). This case will not be considered in this dissertation.

Throughout this dissertation we assume that matrices \( \Phi^k(\delta) \) are finite on some positive interval \([0, \delta^*]\). Thus in particular \( F_{ij}(x) \) cannot be heavy-tailed. In this section we find an upper exponential bound of the probability of the buffer overflow by steady-state buffer content \( X^* \). We prove also that lower bound does not exist in general.

Firstly, we consider the single source case and then we generalize it to the \( N \) sources case.

In general, function \( \kappa(\delta) = \kappa(\Phi(\delta), c) \) of \( \delta \) is convex and continuous and only three possibilities can hold (see Asmussen e.g. [6], page 256, and Asmussen [4], page 51):

1. \( \lim_{\delta \to +\infty} \kappa(\Phi(\delta), c) = +\infty \),
2. \( \lim_{\delta \to \eta^*} \kappa(\Phi(\delta), c) = +\infty \) for some \( \eta^* > 0 \),
3. \( \lim_{\delta \to \eta^*} \kappa(\Phi(\delta), c) = \kappa(\Phi(\eta^*), c) = \alpha \) and \( \kappa(\Phi(\eta^*\), c) = +\infty \) for some \( \eta^* > 0 \) and \( \alpha \).

If BSNE has no solution, then only the third case holds for \( \alpha < 1 \) (see Figure 5).

\[ \text{FIGURE 5.} \]

\[ \begin{align*}
\kappa(\Phi(\delta), c) & \quad \eta^* \\
1 & \quad \delta
\end{align*} \]
Thus following equation holds

$$ u = \frac{1}{\alpha} \Phi(\eta^*) u $$

(5.6.1)

for certain constant $\alpha < 1$ and $\eta^* = \delta^*$, where $\delta^*$ is given by (4.4.5). Define

$$ C = \frac{\dot{s} (I - \dot{P}) \dot{u}}{\min_{m, x > e} \inf_{h \geq 0} \frac{1}{h(x)}}, $$

(5.6.2)

where $\dot{u}$, $\dot{P}$, $\dot{s}$ and $\dot{h}(x)$ are given in (4.4.8), (4.3.4), (4.4.10) and (4.4.12) respectively calculated for model with SMP input with kernel distributions functions $\{\dot{H}_{ij}(x)\}_{i,j=1,\ldots,\ell}$ given in (5.6.4) and (5.6.5). We can now prove the following upper bound.

**Theorem 5.6.1** Let

$$ \eta^* = \sup \{ \delta > 0 : \Phi_{ij}(\delta) < +\infty ; \ i, j = 1, \ldots, \ell \}.$$  

Then

$$ \Psi(x) = \mathbb{P}(X^* > x) \leq Ce^{-\eta^* x}, $$

where constant $C$ is given in (5.6.2).

**Proof.** Let $\mathcal{W}$ be the family of diagonal matrix $B = \text{diag}(\alpha_i)$ such that $\alpha_i = 1$ for $i$ : $r_i - c < 0$ and $\alpha_i > 1$ for $i$ : $r_i - c > 0$. We prove that there exists matrix $B^0 \in \mathcal{W}$ fulfilling

$$ B^0 \Phi(\eta^*) \dot{u} = \dot{u}, $$

(5.6.3)

that is $\kappa(B^0 \Phi(\eta^*), c) = 1$. In fact, note that $\kappa(B^0 \Phi(\eta^*), c) > \kappa(\Phi(\eta^*), c) = \alpha$, because if $A > B$, then Perron-Frobenius eigenvalue of matrix $A$ is greater than Perron-Frobenius eigenvalue of matrix $B$ (see Miller [89]). Moreover, consider a family of diagonal matrices $B(\theta) \in \mathcal{W}$ such that $\alpha_i(\theta) = \theta$ for $i$ : $r_i - c > 0$, where $\theta > 1$. Then from Seneta [113], page 4, taking $\theta \to +\infty$ we get

$$ \kappa(B(\theta) \Phi(\eta^*), c) \geq \min_{i} \sum_{j} (B(\theta) \Phi(\eta^*))_{ij} \geq \theta \Phi_{ii, \theta} (\eta^*) \to +\infty, $$

where $r_{i_0} - c > 0$. Thus to prove the existence of matrix $B^0$ fulfilling (5.6.3) it suffices to prove that function $\vartheta(\theta) = \kappa(B(\theta) \Phi(\eta^*), c)$ is continuous, that is $\vartheta(\theta(1 + \epsilon)) \to \vartheta(\theta)$ and $\vartheta(\theta(1 - \epsilon)) \to \vartheta(\theta)$ as $\epsilon \to 0$, where $\epsilon > 0$. Note that

$$ (1-\epsilon)B(\theta) \Phi(\eta^*) < B(\theta(1-\epsilon)) \Phi(\eta^*) < B(\theta) \Phi(\eta^*) < B(\theta(1+\epsilon)) \Phi(\eta^*) < (1+\epsilon)B(\theta) \Phi(\eta^*). $$

Hence

$$ (1-\epsilon)\vartheta(\theta) \leq \vartheta(\theta(1-\epsilon)) < \vartheta(\theta) < \vartheta(\theta(1+\epsilon)) \leq (1+\epsilon)\vartheta(\theta), $$

which completes the proof of continuity of function $\vartheta(\theta)$. Thus there exists matrix $B^0 = \text{diag}(\alpha^0_i) \in \mathcal{W}$ fulfilling (5.6.3). Now, note that for $i$ such that $r_i - c > 0$
there exists distribution function $H_i(x)$ independent of kernel distribution functions $F_{ij}(x)$ ($i, j = 1, \ldots, \ell$) of SMP process $\{Z(t), t \geq 0\}$ such that $H_i(0) = 0$ and

$$
\hat{H}_i(\eta^*) = \alpha_i^0 > 1.
$$

Let

$$
\hat{F}_{ij}(x) = H_i * F_{ij}(x), \quad \text{for } i: r_i - c > 0
$$

and

$$
\check{F}_{ij}(x) = F_{ij}(x), \quad \text{for } i: r_i - c < 0. \quad (5.6.4)
$$

We consider now a fluid model driven by semi-Markov process with kernel $\{\check{F}_{ij}(x)\}$. Obviously $\hat{F}_{ij}(x) \leq \check{F}_{ij}(x)$ for $i$ such that $r_i - c > 0$. Hence probability of the buffer overflow in the fluid model with the SMP input with kernel $\{\check{F}_{ij}(x)\}_{i,j=1,\ldots,\ell}$ (which we denoted by $\Psi(x)$) is larger than probability of the buffer overflow in the previous model. That is,

$$
\Psi(x) \geq \check{\Psi}(x).
$$

However, we now have

$$
B^* \Phi(\eta^*) = \check{\Phi}(\eta^*).
$$

Hence $\kappa(\check{\Phi}(\eta^*), c) = 1$ and the BSNE has a solution. Moreover, by convexity of $\kappa(\check{\Phi}(\delta), c)$ as a function of $\delta$, we have $\frac{\partial}{\partial \delta} \kappa(\check{\Phi}(\delta), c)|_{\delta = 0} < 0$. Thus by (4.5.25) the stability condition holds in the new fluid model. Therefore by Theorem 4.4.1 we can find the upper bound for $\check{\Psi}(x)$:

$$
\check{\Psi}(x) \leq Ce^{-\gamma^*x},
$$

where $C$ is given in (5.6.2). This completes the proof.

We will write $f(x) \approx g(x)$ if and only if $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \infty$ for certain constant $\infty \in (0, +\infty)$ and say that functions $f$ and $g$ are asymptotically equivalent. It is easy to find an example proving that the lower bound does not exist in general. In particular, the probability of the overflow of the buffer can be asymptotically equivalent to $x^{-\sigma} e^{-\eta^*x}$. In fact, consider a single source on-off fluid model (see Chapter 6.1) with on time denoted by $T_{on}$ with distribution function $F_{on}$ and off time denoted by $T_{off}$ with exponential distribution function $F_{off} \overset{D}{=} \text{Exp}(\beta)$. Let $r$ be the input intensity and $c$ - output intensity. We denote by $p = \frac{r}{r + c}$ the probability that stationary on-off input process is in the state on. Assume also that stability condition $pr < c$ holds. Then we have following lemma.

**Lemma 5.6.2** Let in the single on-off fluid model off-time is exponential distributed and on-time has distribution function fulfilling:

$$
\check{F}_{on}(x) \approx x^{-\sigma} e^{-\eta^*x} \quad (5.6.6)
$$

for some $\sigma > 1$. Then for steady-state buffer content $X^*$

$$
\mathbb{P}(X^* > x) \approx x^{-\sigma} e^{-\eta^*x}.
$$

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Proof. Following Klüppelberg [69], the class of distribution functions fulfilling (5.6.6) is denoted by $S((r-c)\eta)$. Denote by $T_{on}^s$ the residual time on with distribution function:

$$F_{on}^*(x) = \mathbb{P}(T_{on}^s > x) = \int_x^{+\infty} F_{on}(y) \, dy \over \mathbb{E}T_{on}.$$

By Klüppelberg [66], page 139, we have

$$F_{on}^*(x) \approx x^{-\sigma} e^{-\eta (r-c)x}.$$ \hfill (5.6.7)

Hence

$$\mathbb{P}((r-c)T_{on}^s > x) \approx x^{-\sigma} e^{-\eta x}.$$ \hfill (5.6.7)

Let $W$ be the virtual waiting time in $G|G|1$ queue with service time $(r-c)T_{on}$ and interarrival time $cT_{off}$. If BSNE has no solution, then

$$\mathbb{E}e^{(r-c)T_{on} - cT_{off}} < 1.$$ 

In this case by Borovkov [18], Theorem 12, page 132:

$$\mathbb{P}(W > x) \approx \mathbb{P}(((r-c)T_{on} - cT_{off})^* > x) \approx x^{-\eta} e^{-\eta x}.$$ \hfill (5.6.8)

5.6.1 There exists the following relation between probability of the buffer overflow $\Psi(x)$ in a single on-off model and virtual waiting time $W$:

$$\Psi(x) = \mathbb{P}(X^* > x) = \frac{r}{c} p \mathbb{P}(W + (r-c)T_{on}^s > x)(1 + o(1))$$

as $x \to +\infty$; see Kella and Whitt [68], Heath et al [56] and Boxma and Dumas [22].

From (5.6.8) and Theorem 2.1 of Klüppelberg [70] (see also Klüppelberg [69]), we have that the tail of distribution function of $W + (r-c)T_{on}^s$ is asymptotically equivalent to the tail of distribution function of variable $(r-c)T_{on}^s$. This completes the proof in a view of 5.6.1 and (5.6.7).

We consider now the case of $N$ sources. Let $\eta^*$ be the biggest $\delta > 0$ fulfilling

$$\kappa(\Phi^k(\delta), \epsilon^k) \leq 1, \quad (5.6.9)$$

for some $c^1, \ldots, c^N$ such that

$$\sum_{k=1}^{N} c^k = c. \quad (5.6.10)$$

Remark 5.6.1 Note that following the argument from the proof of Lemma 5.5.3, there exists a unique solution of system (5.6.9) and (5.6.10) (see also Figure 6).
Then similarly like in a single source case, for each source $k = 1, \ldots, N$ for which $\kappa(\Phi^k(\eta^*), e^k) < 1$ we can define a new SMP environment process with kernel distributions $\tilde{E}^k$. For matrix $\tilde{\Phi}^k(\eta^*)$ there exists solution of BSNE and the input process is stochastically larger in the new fluid model. Hence from Theorem 5.3.1 we get the following theorem.

**Theorem 5.6.3** Let $\eta^*$ be the biggest $\delta$ fulfilling (5.6.9). Then

$$\mathbb{P}(X^* > x) \leq C e^{-\eta^* x},$$

where

$$C = \frac{\prod_{k=1}^{N} \tilde{s}^k (I - \hat{P}^k) \tilde{u}^k}{\min_{i_1, i_2, \ldots, i_N} \sum_{k=1}^{N} r_{ik} > c \prod_{k=1}^{N} \{ \inf_x h^k(i^k, x) \}}$$

and $\tilde{u}^k$, $\hat{P}^k$, $\tilde{s}^k$ and $h^k(i^k, x)$ are given in (5.2.4), (5.2.1), (5.2.5) and (5.2.7) respectively calculated for kernel distributions functions $\{\tilde{F}^k_{ij}(x)\}_{i,j=1,\ldots,k} (k = 1, \ldots, N)$. 

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5.7 Effective bandwidth

A fundamental problem in the design of multi-service communication networks is admission control. Networks providers would like to develop a good way in which to decide whether or not to satisfy each successive connection request. Due to the diverse traffic that may be carried by these networks, the different sources can be expected to make very different demands on the network. Consequently, it is natural to accept the notion of the effective bandwidth associated with each kind of source in the following way. \( N \) sources with ‘bandwidths’ \( \alpha_i, i = 1, \ldots, N \), can be regarded as feasible if and only if condition \( \sum_{i=1}^{N} \alpha_i < c \) implies that the probability of the buffer overflow is less than \( v \). Typically \( v \) is small (of the order \( 10^{-9} \)). Output intensity \( c \) can be thought as a total available ‘bandwidth’. The concept of effective bandwidth has been studied extensively in literature; see e.g. Kesidis et al. [63], Kelly [67], Gibbens and Hunt [52]. In this section we prove that quantity \( c^k \), which appears in BSNE, is simply the effective bandwidth of \( k \)th source. In fact, suppose that we are free to choose the number \( N_k \) of sources of \( k \)th type from the finite set \( D_k \) and let \( \mathcal{S} \in \mathcal{D} = \times_{k=1}^{n} D_k \), where \( n \) is the number of different kinds of sources. Let \( \sum_{i=1}^{n} N_i = N \). We wish to choose the number of sources in such a way that for the probability of the buffer overflow over level \( x \) we want to have following inequality \( \mathbb{P}_\mathcal{S}(X^* > x) \leq v \) for some \( v \). Without loss of generality we can assume that for large buffer content \( x \) we have \( v = e^{-\eta_0 x} \) for some constant \( \eta_0 \), which we call adjustment coefficient. The way in which we can do it gives the following theorem.

**Theorem 5.7.1** Assume that \( \mathbb{P}_\mathcal{S}(X^* > x) \) has an exponential lower and upper bound. Let \( \mathcal{T}_0 = \{ \mathcal{S} \in \mathcal{D} : \lim_{x \to +\infty} \frac{\mathbb{P}_\mathcal{S}(X^* > x)}{\exp(-\eta_0 x)} < 1 \} \) for a certain constant \( \eta_0 \). Furthermore, let

\[
\mathcal{T} = \{ \mathcal{S} \in \mathcal{D} : \sum_{k=1}^{n} N_k c^k(\eta_0) < c \} , \quad \mathcal{T} = \{ \mathcal{S} \in \mathcal{D} : \sum_{k=1}^{n} N_k c^k(\eta_0) \leq c \} ,
\]

where \( c^k(\eta_0) \) are chosen in such way that Perron-Frobenius eigenvalue \( \kappa(\Phi^k(\eta_0), c^k(\eta_0)) \) of matrix \( \Phi^k(\eta_0) \) is 1. Then

\[
\mathcal{T} \subset \mathcal{T}_0 \subset \mathcal{T} .
\]

**Proof.** Given \( \mathcal{S} \) we know that we have following upper and lower bounds for the probability of the buffer overflow

\[
C_{\mathcal{S}} e^{-\eta_0 x} \leq \mathbb{P}_\mathcal{S}(X^* > x) \leq C_{\mathcal{S}}^* e^{-\eta_0 x}
\]

where \( \eta_0 \) is the solution of the BSNE. Hence

\[
\sum_{k=1}^{n} N_k c^k(\eta_0) = c
\]

and

\[
\kappa(\Phi^k(\eta_0), c^k(\eta_0)) = 1 .
\]

Then

\[
C_{\mathcal{S}} e^{-(0,\xi)(\xi)} x \leq \mathbb{P}_\mathcal{S}(X^* > x) \leq C_{\mathcal{S}}^* e^{-(0,\xi)(\xi)} x .
\]

(5.7.3)
To get $\mathcal{S} \in \mathcal{T}_0$ we want the above to be less than 1 for large sizes of buffer $x$, that is $\eta_S > \eta_0$. From Lemma 5.5.2, $\kappa(\Phi^k(\delta), c^k)$ is continuous and decreasing function of $c^k$ and convex function of $\delta$. Thus from (5.7.2)

$$\kappa(\Phi^k(\eta_0), c^k(\eta_S)) < 1$$

and then $c^k(\eta_S) \geq c^k(\eta_0)$. In view of equation (5.7.1) this completes the proof of $\mathcal{T} \subseteq \mathcal{T}_0$. Now, if $\mathcal{S} \not\in \mathcal{T}$, then $c^k(\eta_S) < c^k(\eta_0)$ and hence $\eta_S < \eta_0$. Thus from (5.7.3) we get that $\mathcal{S} \not\in \mathcal{T}_0$.

$\Box$
Chapter 6

Miscellaneous fluid models

In this chapter we consider different special cases of SMP fluid models. The most important is the so-called on-off fluid model presented in Section 6.1. In this model environment processes $\{Z^k(t), t \geq 0\}$ are alternating renewal processes in which consecutive off-periods alternate with on-periods. This is one of the most often fluid model used to modeling traffic in high-speed data networks. The classical AMS fluid model is a particular case of it, when on and off times are exponential distributed; see Section 6.2.2. Moreover, following Palmowski and Dębicki [37] and Kulkarni and Rolski [75] in heavy traffic environment the steady-state buffer content in the $N$th on-off fluid model converges in distribution as $N \to +\infty$ to the steady state buffer content in limiting fluid model with Gaussian environment process. Such a Gaussian fluid model also plays a great role in the analysis of data traffic. In Section 6.1.3 we find the asymptotics of the buffer overflow probability in the single on-off fluid model. In Sections 6.2 and 6.3, 6.4 we consider further examples of SMP fluid models: Markov fluid model and tandem fluid models.

6.1 On-off fluid model

6.1.1 Bounds

The results presented in this section were developed in Palmowski and Rolski [101]. Consider $N$ sources modulated by alternating renewal processes $\{Z^k(t), t \geq 0\}$ that alternates between on ($Z^k(t) = 1$) and off ($Z^k(t) = 0$) states. The random amount of time the process $\{Z^k(t), t \geq 0\}$ spends in the on state (called on-times) has distribution function $F_{on}^k(\cdot)$ with mean $\tau_{on}^k$ and the corresponding off-time distribution function is $F_{off}^k(\cdot)$ with mean $\tau_{off}^k$. The on and off time sequences are independent, each consisting of i.i.d. random variables and assume that $\tau_{on}^k + \tau_{off}^k < +\infty$ (hence there exists stationary version of the process $\{Z^k(t), t \geq 0\}$).

Fluid is generated continuously from the $k$th source at rate $r_k^1 = r_k$ during the on state and at rate $r_k^0 = 0$ during the off state. A source modulated by such a 2-state (on-off) process is called a general on-off source with on-time distribution $F_{on}^k(\cdot)$, off-time distribution $F_{off}^k(\cdot)$ and (peak) rate $r_k$. 
Assume that
\[ \sum_{k=1}^{N} r^k p^k < c < \max_k \{ r^k \} , \]
where
\[ p^k = \frac{\tau_{on}^k}{\tau_{on}^k + \tau_{off}^k} . \] (6.1.1)

Then from (4.4.4) matrix \( \Phi^k(\delta) \) reduces to
\[ \Phi^k(\delta) = \left[ \begin{array}{cc}
0 & \hat{F}_{on}^k(-\delta c^k) \\
\hat{F}_{on}^k(\delta(r^k - c^k)) & 0
\end{array} \right] , \]
where \( \hat{F}_{on}^k(\cdot) \) and \( \hat{F}_{off}^k(\cdot) \) are the moment generating functions of \( F_{on}^k(t) \) and \( F_{off}^k(t) \) respectively. From (5.2.4), the BSNE is equivalent to
\[ \kappa(\Phi^k(\eta), c^k) = 1 \]
\[ \sum_{k=1}^{N} c^k = c . \]

Note that
\[ \kappa(\Phi^k(\eta), c^k) = \sqrt{\hat{F}_{on}^k(\eta(r^k - c^k)) \hat{F}_{off}^k(-\eta c^k)} . \] (6.1.2)

Thus \( \eta \) is the real-positive solution of the following system of equations:
\[ \begin{cases} 
\hat{F}_{on}^k(\eta(r^k - c^k)) \hat{F}_{off}^k(-\eta c^k) = 1 , \\
\sum_{k=1}^{N} c^k = c .
\end{cases} \] (6.1.3)

Also, equation (5.2.4) gives
\[ u^k = [1 , \hat{F}_{on}^k(\eta(r^k - c^k))]^T . \]

Then
\[ h_0^k(x) = \frac{e^{\eta c^k}}{F_{off}^k(x)} \int_x^{+\infty} e^{-\eta y} dF_{off}^k(y) \] (6.1.4)

and
\[ h_1^k(x) = \hat{F}_{on}^k(\eta(r^k - c^k)) \frac{e^{-\eta(r^k - c^k)x}}{F_{on}^k(x)} \int_x^{+\infty} e^{\eta(r^k - c^k)y} dF_{on}^k(y) , \] (6.1.5)

From Theorem 5.3.1 we get following result.

**Theorem 6.1.1** Let \( \eta \) be a solution of BSNE given in (6.1.3). Then the probability of the buffer overflow can be estimated in the following way:
\[ C_* e^{-\eta x} \leq \mathbb{P}(X^* > x) \leq C^* e^{-\eta x} , \]

provided that \( C_* > 0 \) and \( C^* < +\infty \), where
\[ C_* = \min_{i_1,i_2,\ldots,i_N \in \{0,1\}: \sum_{k=1}^{N} r^k_{i_k} > c} \prod_{k=1}^{N} \{ \inf_{x} h^k(i_k,x) \} \]

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\[ C^* = \max_{i_1, i_2, \ldots, i_N \in \{0, 1\}^N} \prod_{k=1}^N C_k^{\eta} \prod_{k=1}^N \{\sup_x h_k(i_k, x)\} \]

and

\[ C_k = \frac{1}{\tau_{on}^k + \tau_{off}^k} \frac{r^k \hat{F}_{on}^k(\eta(r^k - c^k)) - 1}{\eta(r^k - c^k)}. \] (6.1.6)

**Proof.** By Theorem 5.3.1 it suffices to verify that

\[ C_k = s_k(I - P^k)u^k, \]

where

\[ P^k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

\[ s_k = \begin{bmatrix} 1 \left( \frac{1}{\tau_{off}^k + \tau_{on}^k} \eta c^k - 1 \right) \\ 1 \left( \frac{1}{\tau_{off}^k + \tau_{on}^k} \eta (r^k - c^k) - 1 \right) \end{bmatrix}. \]

In fact,

\[ s_k(I - P^k)u^k = \frac{1}{(\tau_{off}^k + \tau_{on}^k)\eta} \left( \frac{1}{r^k - c^k} + \frac{1}{c^k} \right) (\hat{F}_{on}^k(\eta(r^k - c^k)) - 1) = C_k. \]

\[ \square \]

In homogeneous case (when \( \{Z^k(t), t \geq 0\} \) have the same law and \( r^k = r \)) the BSNE given in (6.1.3) is equivalent to

\[ \hat{F}_{on}^k(\eta(r - \frac{c}{N})) \hat{F}_{off}^k(-\eta \frac{c}{N}) = 1, \] (6.1.7)

where \( F_{on}^k(x) = F_{on}(x) \) and \( F_{off}^k(x) = F_{off}(x) \) \( (k = 1, \ldots, N) \). To compare different fluid models we made some numerical experiments for \( \eta \) in this case. Following Anick et al [1] we take \( r = 1 \) and \( F_{off}^k(\cdot) = \text{Exp}(0.4) \) and \( c = 16.666 \) for \( N = 30, 50, c = 33.333 \) for \( N = 85, 100, c = 66.666 \) for \( N = 150, 200 \). In the first case \( F_{on} \) is a exponential distribution: \( F_{on}(\cdot) = \text{Exp}(1) \), in the second case \( F_{on}(\cdot) \) is a Erlang distribution Erl(2,2) and in the third case \( F_{on}(\cdot) \) is a hyperexponential distribution \( p\text{Exp}(2) + (1 - p)\text{Exp}(\frac{2p}{1-p}) \), where \( p = 0.1 \). In all cases the mean of on-time is equal 1.

**Numerical Comparisons of \( \eta \).**

<table>
<thead>
<tr>
<th>( F_{on} )</th>
<th>( c = 16.666 )</th>
<th>( c = 33.333 )</th>
<th>( c = 66.666 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 30 )</td>
<td>1.5299</td>
<td>0.2909</td>
<td>0.6251</td>
</tr>
<tr>
<td>( N = 50 )</td>
<td>2.304</td>
<td>0.41</td>
<td>0.8786</td>
</tr>
<tr>
<td>( N = 85 )</td>
<td>1.4677</td>
<td>0.2909</td>
<td>0.6043</td>
</tr>
<tr>
<td>( N = 100 )</td>
<td>|</td>
<td>|</td>
<td>|</td>
</tr>
<tr>
<td>( N = 150 )</td>
<td>|</td>
<td>|</td>
<td>|</td>
</tr>
<tr>
<td>( N = 200 )</td>
<td>|</td>
<td>|</td>
<td>|</td>
</tr>
</tbody>
</table>
We can justify the order $\eta_{\text{Hyper}} < \eta_{\text{Exp}} < \eta_{\text{Erl}}$ as follows. Let $F^{*}_{\text{on}}(\cdot) = \text{Erl}(2,2)$, $F_{\text{on}}(\cdot) = \text{Exp}(1)$ and $F^{\text{off}}(\cdot) = \text{Exp}(0.4)$. Since $\text{Erl}(2,2)$ is IFR (see Szekli [119], page 17), so $\text{Erl}(2,2) \sim \text{Exp}(1)$. Thus

$$
\hat{F}^{*}_{\text{on}}(\eta_{\text{Exp}}(r - \frac{c}{N}))\hat{F}^{\text{off}}(-\eta_{\text{Exp}} \frac{c}{N}) \leq \hat{F}^{*}_{\text{on}}(\eta_{\text{Exp}}(r - \frac{c}{N}))\hat{F}^{\text{off}}(-\eta_{\text{Exp}} \frac{c}{N}) = 1
$$

and hence $\eta_{\text{Erl}} \geq \eta_{\text{Exp}}$. Similar argument together with Szekli [119], Theorem E, page 17, proves the first inequality.

### 6.1.2 Generator of $W$

The superposition of alternating renewal processes is very important not only in theory of fluid models but also in other stochastic systems. Therefore it seems useful to compute the extended generator $\mathcal{A}$ and the domain $D(\mathcal{A})$ of extended generator of process $\{W(t) = (Z^{1}(t), S^{1}(t), \ldots, Z^{N}(t), S^{N}(t)), t \geq 0\}$. Recall that $S^{k}(t)$ is a supplementary age component of process $\{Z^{k}(t), t \geq 0\}$. This result will be presented in this section. Moreover, we prove that process given in (5.3) is in fact a exponential martingale (3.22) computed for generator $\mathcal{A}$ (see Corollary 6.1.4).

Note that by Lemma 4.5.1, PDMP $\{\mathbf{w}^{k}(t) = (Z^{k}(t), S^{k}(t)), t \geq 0\}$ $(k = 1, \ldots, N)$ has extended generator $(\mathcal{A}^{k}\mathbf{g})(i, x) = (\mathcal{A}^{k}(x)\mathbf{g}(x))_{i}$, where

$$
\mathcal{A}^{k}(x) = \begin{pmatrix}
\frac{\partial}{\partial x} - \rho^{k}_{\text{on}}(x)D & \rho^{k}_{\text{off}}(x)D \\
\rho^{k}_{\text{on}}(x)D & \frac{\partial}{\partial x} - \rho^{k}_{\text{off}}(x)
\end{pmatrix}.
$$

Here $\rho^{k}_{\text{on}}(x)$ and $\rho^{k}_{\text{off}}(x)$ denote hazard rate functions of on and off time respectively on $k$th source and the operator $D$ is defined by $(D\mathbf{g})(x) = \mathbf{g}(0)$. We restrict the domain $D(\mathcal{A}^{k})$ to family functions $\mathbf{g}(x) = (g_{0}(x), g_{1}(x)) \in C_{b}^{1}(\mathbb{R}_{+}) \times C_{b}^{1}(\mathbb{R}_{+})$.

For the notational convenience we use the Kronecker (tensor) product $\mathbf{A} \otimes \mathbf{B}$ for two matrices

$$
\mathbf{A} = (a_{ij})_{i=1,\ldots,m, \ j=1,\ldots,m} \quad \text{and} \quad \mathbf{B} = (b_{ij})_{i=1,\ldots,m', \ j=1,\ldots,m'}
$$

defined by

$$
\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix}
a_{11}\mathbf{B} & \cdots & a_{1m}\mathbf{B} \\
a_{21}\mathbf{B} & \cdots & a_{2m}\mathbf{B} \\
\vdots & \ddots & \vdots \\
a_{m1}\mathbf{B} & \cdots & a_{nm}\mathbf{B}
\end{pmatrix}.
$$

For example if $(\mathbf{g}^{k}(x^{k}))^{T} = (g_{0}^{k}(x^{k}), g_{1}^{k}(x^{k}))$ then $\mathbf{g}^{1}(x^{1}) \otimes \cdots \otimes \mathbf{g}^{N}(x^{N})$ is a column vector consisting of $g_{0}(x^{1})g_{0}(x^{2}) \cdots g_{0}(x^{N})$ and this component we denote by $\mathbf{g}^{1} \otimes \cdots \otimes \mathbf{g}^{N}(\mathbf{i}, \mathbf{x})$, where $\mathbf{i} = (i^{1}, \ldots, i^{N}) \in \{0,1\}^{N}$ and $\mathbf{x} = (x^{1}, \ldots, x^{N})$. We now define the Kronecker sum by

$$
\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_{m'} + \mathbf{I}_{m} \otimes \mathbf{B},
$$

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where $I_k$ means the unit matrix of size $k \times k$. Standard considerations yield

$$
\left[ (\mathcal{A}^1 \oplus \ldots \oplus \mathcal{A}^N)(g^1 \otimes \ldots \otimes g^N) \right](i,x) = (A^1 g^1 \otimes (g^2) \otimes \ldots \otimes (g^N)(i,x) + \ldots + (g^1) \otimes (g^2) \otimes \ldots \otimes (A^N g^N)(i,x).
$$

By $D(\mathcal{A}^1) \otimes D(\mathcal{A}^2) \otimes \ldots \otimes D(\mathcal{A}^N)$ we denote the family of functions $g^1 \otimes \ldots \otimes g^N(i,x)$, where $g^k(i^k,x^k) \in D(\mathcal{A}^k)$.

**Proposition 6.1.2** The extended generator $\mathcal{A}$ of the process

$$
\{ W(t) = (Z^1(t), S^1(t), \ldots, Z^N(t), S^N(t)), t \geq 0 \}
$$

is equal to

$$
\mathcal{A} = \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \ldots \oplus \mathcal{A}^N
$$

and $D(\mathcal{A}^1) \otimes \ldots \otimes D(\mathcal{A}^N) \subset D(\mathcal{A})$.

**Proof.** It suffices to prove that process

$$
\mathcal{A} g^1 \otimes \ldots \otimes g^N(W(t)) = \int_0^t (\mathcal{A}^1 \oplus \ldots \oplus \mathcal{A}^N)(g^1 \otimes \ldots \otimes g^N)(W(s)) \, ds
$$

is a $\mathbb{P}^i$-martingale, where $g^k \in D(\mathcal{A}^k)$. For simplicity we demonstrate the proof only for $N = 2$. The general case can be proved by the principle of mathematical induction. Let

$$
M^{g,g}(t) = g^k_Z(S^k(t)) - \int_0^t (\mathcal{A}^k g^k)(W^k(s)) \, ds, \quad t \geq 0.
$$

Condition $g^k \in D(\mathcal{A}^k)$ implies that $\{M^{g,g}(t), t \geq 0\}$ is a martingale. Using the formula for integration-by-parts for semimartingales (see 2.2.13) and from (6.1.8) we get

$$
g^1 \otimes g^2(W(t)) = g^1_Z(S^1(t)) g^2_Z(S^2(t)) = \int_0^t g^1_Z(S^1(s^-)) \, dM^{g^2}(s) + \int_0^t g^2_Z(S^2(s^-)) \, dM^{g^1}(s) + [g^1_Z, g^2_Z]_t,
$$

$$
\begin{align*}
&= \int_0^t g^1_Z(S^1(s^-)) \, dM^{g^2}(s) + \int_0^t g^2_Z(S^2(s^-)) \, dM^{g^1}(s) + [g^1_Z, g^2_Z]_t, \\
&\quad + \int_0^t g^1_Z(S^1(s^-)) (A^1 g^2)(W^2(s)) \, ds + \int_0^t g^2_Z(S^2(s^-)) (A^1 g^1)(W^1(s)) \, ds \\
&= \int_0^t g^1_Z(S^1(s^-)) \, dM^{g^2}(s) + \int_0^t g^2_Z(S^2(s^-)) \, dM^{g^1}(s) + [g^1_Z, g^2_Z]_t + \int_0^t (A^1 g^1)(W^1(s)) \, ds.
\end{align*}
$$

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However, by 2.2.15 the first two components are martingales and from independence of processes \( \{w^1(t), t \geq 0\} \) and \( \{w^2(t), t \geq 0\} \) by Jacod and Shiryaevev [58], Proposition 4.50 a), page 53, process
\[
[g^1_Z, g^2_Z]_t = \left[ M^{z, 1}, M^{z, 2} \right]_t, \quad t \geq 0
\]
is a martingale. This completes the proof.

Note that following lemma holds.

**Lemma 6.1.3** For functions \( h^k \) defined by (6.1.5) and (6.1.4) we have
\[
\frac{(A^1 \oplus \ldots \oplus A^N)(h^1 \odot \ldots \odot h^N)(i, x)}{(h^1 \odot \ldots \odot h^N)(i, x)} = -\eta \left( \sum_{k=1}^N r^k i^k - c \right),
\]
where \( \eta \) is a solution of BSNE given in (6.1.3).

**Proof.** If BSNE has solution, then by Lemma 4.5.2 \((A^k h^k)(i^k, x^k) = -\eta \Delta^k h^k_{x^k}(x^k)\) \((i^k \in \{0,1\}, k = 1, \ldots, N)\), where
\[
\Delta^k = \begin{pmatrix}
-c^k & 0 \\
0 & r^k - e^k
\end{pmatrix}.
\]
From (6.1.8) we have
\[
(A^1 \oplus \ldots \oplus A^N)(h^1 \odot \ldots \odot h^N)(i, x) =
\]
\[
= -\eta \sum_{k=1}^N (\Delta^k h^k)(i^k, x^k) \prod_{k \neq k} h^k_{x^k}(x^k)
\]
\[
= -\eta \left( \bigotimes_{k=1}^N \Delta^k \right) \left( \bigotimes_{k=1}^N h^k \right)(i, x),
\]
from which it follows (6.1.9).

Thus from Lemma 3.2.1 and 3.2.2 we immediately get the following corollary.

**Corollary 6.1.4** Process \( \{N(t), t \geq 0\} \) given in (5.4.3) is an exponential martingale (3.2.2) computed for extended generator \( A \) of process \( \{W(t), t \geq 0\} \). That is,
\[
N(t) = \frac{h^1 \odot \ldots \odot h^N(W(t))}{h^1 \odot \ldots \odot h^N(i, y)} \exp \left[ -\int_0^t \frac{(A^1 \oplus \ldots \oplus A^N)(h^1 \odot \ldots \odot h^N(W(s))}{h^1 \odot \ldots \odot h^N(W(s))} ds \right].
\]
6.1.3 Asymptotics

In this section we find the asymptotics of the buffer overflow probability \( \Psi(x) = \mathbb{P}(X^* > x) \) as \( x \to +\infty \) in a single on-off fluid model. Let \( \{Z(t), t \geq 0\} \) be a stationary alternating renewal process described in the previous subsection. Assume that there exists unique solution of BSNE given in (6.1.3). In particular that all moments of on-time and off-time are finite. Denote

\[
\tau(x) = \inf\{t \geq 0 : \int_0^t (r_{Z(s)} - c) \, ds > x\} ,
\]

(6.1.11)

Note that in the single fluid model \( Z(\tau(x)) = 1 \). Then \( \{(1, Z(\tau(x))), x \geq 0\} \) is a regenerative process. Where \( S(t) \) is a supplementary age component of \( \{Z(t), t \geq 0\} \).

The regenerative epochs are at times \( T_n \) such that \( \inf\{t \geq 0 : X(t) = T_n\} \neq \inf\{t \geq 0 : X(t) > T_n\} \), where \( X(t) \) is the content process at time \( t \). Thus by Smith’s Theorem there exists a stationary distribution of the process \( \{(1, S(\tau(x))), x \geq 0\} \), which be denoted by \( (1, S(\tau(+\infty))) \). Clearly, this stationary distribution does not depend on initial condition \( (1, S(\tau(0))) \). Thus without loss of generality we can assume that \( (Z(0), S(0)) = (1, S(\tau(0))) = (1, 0) \). Then by (4.4.18)

\[
\lim_{x \to +\infty} \frac{\mathbb{P}(X^* > x)}{e^{-\eta x}} = \mathbb{E}^{(1,0)} \left[ \frac{C}{h(1, S(\tau(+\infty)))} \right] ,
\]

(6.1.12)

where constant \( C \) is given in (6.1.6), function \( h(1, \cdot) \) in (6.1.5) and \( \eta \) is a solution of BSNE (6.1.3).

We now find the stationary distribution \( (1, S(\tau(+\infty))) \). Let

\[
\mathbb{P}^{(1,0)}(S(\tau(x)) > t) .
\]

Then

\[
\mathbb{P}^{(1,0)}(S(\tau(+\infty)) > t) = \lim_{x \to +\infty} b(t, x) .
\]

(6.1.13)

**Lemma 6.1.5** Function \( b(t, x) \) is the solution of the following equation:

\[
b(x, t) = \bar{F}_{on}(\frac{x}{r - c}) \mathbb{I}\{\frac{x}{r - c} \geq t\} + \int_0^\infty \int_0^{\frac{r}{r - c}} b(t, x - (r - c)y + cz) \, dF_{on}(y) \, dF_{off}(z) .
\]

(6.1.14)

**Proof.** To get level \( x \) by the content process \( \{X(t), t \geq 0\} \) we can do it at first time on. We can also go at level \( y \leq x/(r - c) \) at first time on and then go down to \( x - cz \) at first time off. This yields (6.1.14).

\( \square \)

We now get the following theorem.

**Theorem 6.1.6** Assume that there exists a solution \( \eta \) of BSNE (6.1.3). Then the probability of the buffer overflow by steady-state buffer content \( X^* \) in a single on-off fluid model has the following asymptotics:

\[
\lim_{x \to +\infty} \frac{\mathbb{P}(X^* > x)}{e^{-\eta x}} = \mathbb{E}^{(1,0)} \left[ \frac{C}{h(1, S(\tau(+\infty)))} \right] ,
\]

where \( C \) and \( h(1, \cdot) \) are given in (6.1.6) and (6.1.5) respectively and distribution of \( (1, S(\tau(+\infty))) \) is given by a limit (6.1.13) of the function \( b(t, x) \) fulfilling (6.1.14).
Remark 6.1.1 Barlow et al [9], London et al [86] and Asmussen [2] compute the generator of Markov process \( \{Z(\tau(x)), x \geq 0\} \), where \( \tau(x) \) is given in (6.1.11) and \( \{Z(t), t \geq 0\} \) is a finite state Markov process. This can be useful to determine the asymptotics of \( \Psi(x) = \mathbb{P}(X^* > x) \) in case of AMS fluid model. The asymptotics of function \( \Psi(x) \) in case of AMS fluid model can be also found directly using the representation of \( \mathbb{P}(X^* > x) \); see Mitra [90], [91] and Section 4.1. McDonald and Qian [88] apply a "state reduction" technique based on a general induced Dirichlet forms method to achieve asymptotic lower bound for the \( \Psi(x) = \mathbb{P}(X^* > x) \) in case of \( N \) sources on-off fluid model. The exact asymptotics of the buffer overflow probability for \( N > 1 \) sources is still open problem.

6.2 Markov fluid model

6.2.1 General

The results of this section are from Palmowski and Rolski [100]. Consider a source modulated by an \( \ell \)-state irreducible continuous time Markov chain (abbreviated by CTMC) \( \{Z(t), t \geq 0\} \) with infinitesimal matrix

\[
A = \{q_{ij}\}_{i,j=1,\ldots,\ell}
\]

and stationary distribution \( p \) satisfying

\[
pA = 0 \quad \text{and} \quad \sum_{i=1}^{\ell} p_i = 1 .
\]

When the CTMC is in state \( i \), the source generates fluid at rate \( r_i \). This source inputs traffic into an infinite-capacity buffer with output channel capacity \( c \). Note that \( \{Z(t), t \geq 0\} \) is a special case of an SMP with kernel

\[
F(x) = \{F_{ij}(x)\} = \begin{cases} 
\frac{q_{ij}}{q_i}(1 - e^{-q_i x}) & \text{if } i \neq j \\
0 & \text{if } i = j ,
\end{cases}
\]

where \( q_i = -q_{ii} = \sum_{j \neq i} q_{ij} \). Note that distribution function \( F_i(x) \) is exponential with parameter \( q_i \). Therefore the expected amount of time the CTMC spends in state \( i \) is

\[
\tau_i = \frac{1}{q_i} .
\]

Then definition (4.4.4) reduces to

\[
\Phi_{ij}(\delta) = \begin{cases} 
\frac{q_{ij}}{q_i - \delta(r_i - c)} & \text{if } i \neq j \\
0 & \text{if } i = j .
\end{cases}
\]

We obtain \( \eta \) and \( u \) by solving BSNE system of equations 4.4.8

\[
\Phi(\eta) u = u .
\] (6.2.1)
Let
\[ \Delta = \text{diag}\{r_1 - c, \ldots, r_\ell - c\} . \]

Then BSNE is equivalent to
\[ Au = -\eta \Delta u . \] 

(6.2.2)

In fact, (6.2.2) is equivalent to
\[ \sum_{j=1}^{\ell} q_{ij} u_j = -\eta (r_i - c) u_i . \]

Thus
\[ \sum_{j \neq i} q_{ij} u_j - q_i u_i = -\eta (r_i - c) u_i \]

which is
\[ \sum_{j \neq i} \frac{q_{ij}}{q_i} - \eta (r_i - c) u_j = u_i . \]

Above inequality is BSNE given in (6.2.1). Moreover, \( \Phi(\eta) = \Gamma(x) \) and by 4.4.12 \( h_i(x) = u_i . \)

By (4.5.27) we have:
\[ s(I - P)u = \sum_{j=1}^{\ell} \int_0^{+\infty} v_j(y) \pi_j(y) \, dy = \sum_{j=1}^{\ell} u_j p_j q_j \int_0^{+\infty} \bar{F}_j(y) \, dy = \sum_{j=1}^{\ell} u_j p_j . \]

Hence from Theorem 4.4.1 we get following theorem.

**Theorem 6.2.1** Probability of the buffer overflow in the fluid model driven by CTMC has the following exponential bounds:
\[ C_s e^{-\eta x} \leq \mathbb{P}(X^* > x) \leq C^* e^{-\eta x} , \]

where \( \eta \) and \( u \) are given by equation (6.2.2) and
\[ C_s = \frac{\sum_{j=1}^{\ell} p_j u_j}{\text{max}_{j: r_j > c} u_j} , \]
\[ C^* = \frac{\sum_{j=1}^{\ell} p_j u_j}{\text{min}_{j: r_j > c} u_j} . \]

### 6.2.2 AMS fluid model

Consider now a special but very important case of a fluid model driven by a CTMC. The following model is already classical in queueing fluid theory and plays a similar role as M|M|1 in the standard queueing theory. Anick et al [1] considered \( N \) independent homogeneous sources, each transmitting a fluid according to on-off process
with on-time exponentially distributed with parameter \( \alpha \), off-time exponentially distributed with parameter \( \beta \). The buffer is emptied with constant rate \( c > 0 \) and while a source is on it is sending fluid with rate \( r \). Note that this fluid model is a special case of the previous CTMC fluid model with environment process \( \{Z(t), t \geq 0\} \) being the number of sources that are in the state on at time \( t \). Thus the input intensities are equal to \( r_i = ir \) \( (i = 1, \ldots, N) \). We call the above fluid model the AMS fluid model.

Process \( \{Z(t), t \geq 0\} \) has following infinitesimal generator

\[
\mathcal{A} = \begin{pmatrix}
-N\beta & N\beta & 0 & 0 & 0 & 0 \\
\alpha & -\alpha - (N-1)\beta & (N-1)\beta & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(N-1)\alpha & -(N-1)\alpha - \beta & \beta & N\alpha & -N\alpha
\end{pmatrix}
\]

Thus the stationary number of on units is \( p_j = \binom{N}{j} p^j (1-p)^{N-j} \), where \( p = \beta / (\alpha + \beta) \). The stability condition is fulfilled when the drift is negative:

\[
d = Npr - c < 0
\]

and the buffer is not always empty in the steady-state:

\[
Nr > c.
\]

From Theorem 6.2.1 (see also Palmowski and Rolski [101], Theorem 2.1) the steady-state distribution of the buffer-content process \( \{X(t), t \geq 0\} \) can be estimated in the following way.

**Theorem 6.2.2** Let \( X^* \) be the steady-state in the AMS fluid model with on parameter \( \alpha \) and off parameter \( \beta \). Then

\[
C_* e^{-\eta x} \leq P(X^* > x) \leq C^* e^{-\eta x},
\]

where

\[
\eta = \frac{N (\alpha + \beta) c - \beta Nr}{N r - c}
\]

\[
C^* = \left( \frac{p}{1-p} \right)^\beta \left( \frac{N r - c}{c} \right)^\beta (1-p)^N \left( 1 - \frac{c}{N r} \right)^N,
\]

\[
C_* = \left( \frac{p N r}{c} \right)^N.
\]

**Proof.** Note that solution of (6.2.2) is

\[
\eta = \frac{N (\alpha + \beta) c - \beta Nr}{N r - c}
\]

and

\[
u_i = \theta^i,
\]

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where \[
\theta = \frac{\alpha}{\beta} \frac{c}{N r - c} .
\] (6.2.8)

From (6.2.4) we have that \( \theta > 1 \). Hence

\[
\min_{r_j > c} u_j = \theta^{1 \frac{r_j}{c}}
\]

and

\[
\max_{r_j > c} u_j = \theta^N.
\]

Moreover,

\[
\theta^{1 \frac{r_j}{c}} \geq \theta^\frac{r_j}{c}
\]

and

\[
\sum_{i=1}^N p_i \theta^i = (p \theta + 1 - p)^N = (1 - p)^N \left(1 - \frac{c}{N r}\right)^{-N}.
\]

This completes the proof in view of Theorem 6.2.1.

\[\square\]

6.2.1 Consider now the single source AMS fluid model with two possible states: \(on\) \((i = 1)\) and \(off\) \((i = 0)\) (hence \(N = 1\)). Note that in single source model buffer overflow happens when environment process is in state \(on\). Thus from Theorem 6.2.2 we have

\[\mathbb{P}(X^* > x) = C e^{-\eta x}, \]

where

\[\eta = \frac{(\alpha + \beta) c - \beta r}{c(r - c)}\]

and

\[C = \frac{p N r}{c} .\]

6.3 Tandem Buffers - single source

In this section, an exponential \(on-off\) source with \(on\)-time parameter \(\alpha\), \(off\)-time parameter \(\beta\) and rate \(r\) inputs traffic to an infinite-capacity buffer with output capacity \(c_1\). The output from the buffer acts as an input to another infinite-capacity buffer whose output capacity is \(c_2\).

Assume,

\[r \beta / (\alpha + \beta) < c_2 < c_1 < r.\]

We study the buffer content processes of the respective buffers \(\{X_1(t), t \geq 0\}\) and \(\{X_2(t), t \geq 0\}\).
From 6.2.1 for the exponential on-off source we have

\[ \mathbb{P}(X_1^* > x) = \frac{r \beta}{c_1(\alpha + \beta)} e^{-\eta_1 x}, \quad (6.3.9) \]

where

\[ \eta_1 = \frac{c_1 \alpha + c_1 \beta - \beta r}{c_1(r - c_1)}. \]

Using the analysis in Narayanan [93], we model the output process from the first buffer as an alternating renewal process. Thus the input source to the second buffer can be modeled as a general on-off source with and on-time distribution \( U(x) \) (with mean \( r/(c_1(\alpha + \beta) - r\beta) \)), off-time distribution \( D(x) \) (with mean \( 1/\beta \)) and rate \( c_1 \), where

\[ U(x) = \left( \frac{a_2}{2a_1} \right) \sum_{k=0}^{\infty} \left( \frac{a_2}{2a_1} \right)^{2k} \frac{(2k)!}{k!(k+1)!} \left( 1 - \sum_{n=0}^{2k} \left( \frac{e^{-a_2 x} (a_1 x/r)^n}{n!} \right) \right), \]

with \( a_1 = \beta r - \beta c_1 + \alpha c_1, \quad a_2 = \sqrt{4 \alpha \beta c_1 (r - c_1)}, \quad a_3 = 1/(2\beta(r - c_1)), \) and

\[ D(x) = 1 - e^{-\beta x}. \]

The moment generating function of the distribution \( U(\cdot) \) is

\[ \hat{U}(w) = \left\{ \begin{array}{ll}
\frac{-w + \beta + c_1 s_0(w)}{a_2} & \text{if } w \leq w^* \\
+\infty & \text{otherwise,}
\end{array} \right. \]

where \( w^* = -(2\sqrt{c_1 \alpha \beta (r - c_1)} - r \beta - c_1 \alpha - c_1 \beta)/r, \quad s_0(w) = \frac{-e^{-\beta w} - 4w(-w + \alpha + \beta)c_1 (r - c_1)}{2c_1(r - c_1)} \)

and \( b = -(r - 2c_1)w + (r - c_1)\beta - c_1 \alpha. \) The moment generating function of the distribution \( D(\cdot) \) is

\[ \hat{D}(w) = \left\{ \begin{array}{ll}
\frac{b}{\beta} + w & \text{if } w < \beta \\
+\infty & \text{otherwise.}
\end{array} \right. \]

Therefore the BSNE is

\[ \hat{U}(-\eta_2 c_2) \hat{D}(\eta_2 (c_1 - c_2)) = 1. \quad (6.3.11) \]

Intuitively, a random variable with the distribution \( U(t) \) (of equation (6.3.10)) is a Decreasing Failure Rate random variable (since \( U(t) \) represents the busy period distribution). The intuition can be verified (after a lot of algebra) using the expression for \( U(t) \) in the equation (6.3.10) (see Narayanan [93]).

Using Theorems 6.1.1 and 5.5.5 we can get following theorem.

**Theorem 6.3.1** Assume that BSNE given in (6.3.11) has a unique solution. Then the steady-state distribution of the buffer-content process of the second buffer \( \{X_2(t), t \geq 0\} \) is bounded as

\[ C_* e^{-\eta_2 x} \leq \mathbb{P}(X_2^* > x) \leq C^* e^{-\eta_2 x}, \]
where
\[
C^* = \frac{1}{c_1 s'(0) c_2} \hat{U}(\eta_2(c_1 - c_2)) - 1
\]
(6.3.12)
\[
C^*_* = \frac{1}{c_1 s'(0) c_2} \hat{U}(\eta_2(c_1 - c_2)) - 1 \rho_U(+\infty) - \eta_2(c_1 - c_2)
\]
(6.3.13)
where \( \rho_U \) is the hazard rate function distribution \( U \).

6.4 Tandem Buffers - Markov modulated on-off sources

Consider the tandem buffers model, in which input to the first buffer is from \( N \) independent and identical exponential on-off sources with on-time parameter \( \alpha \), off-time parameter \( \beta \) and rate \( r \). The output from buffer 1 is directly fed into buffer 2. The output capacities of buffer 1 and 2 are \( c_1 \) and \( c_2 \) respectively. We study the limiting distributions of the contents of the two buffers.

**Buffer 1:** The limiting distribution of the buffer-content process \( \{X_1(t), t \geq 0\} \) is given by Theorem 6.2.2.

**Buffer 2:** Let \( M = \lceil \frac{c_2}{\alpha} \rceil \). Define
\[
Z_2(t) = \begin{cases} 
Z_1(t) & \text{if } X_1(t) = 0 \\
M & \text{if } X_1(t) > 0,
\end{cases}
\]
(6.4.14)
where \( Z_1(t) \) is the number of sources on at time \( t \). We have also \( r_i = i r \) for \( i = 0, \ldots, M - 1 \) and \( r_M = c_1 \).

We assume that
\[
\frac{N r \beta}{\alpha + \beta} < c_2 < c_1.
\]
We can see that the \( \{Z_2(t), t \geq 0\} \) process is an SMP on state space \( \{0, 1, \ldots, M\} \) with kernel
\[
F(x) = \{F_{ij}(x)\}_{i,j=0,\ldots,M}
\]
derived below. For \( i = 0, 1, \ldots, M - 1 \) and \( j = 0, 1, \ldots, M \), we have
\[
F_{ij}(x) = \begin{cases} 
\frac{\alpha}{\alpha + (N-1)\beta} (1 - \exp\{-i(\alpha + (N - i)\beta)\}) & \text{if } j = i - 1 \\
\frac{(N-1)\beta}{\alpha + (N-1)\beta} (1 - \exp\{-i(\alpha + (N - i)\beta)\}) & \text{if } j = i + 1 \\
0 & \text{otherwise}
\end{cases}
\]
To describe distribution function \( F_{M,j}(x) \), we need to define the first passage time in \( \{X_1(t), t \geq 0\} \) process as follows:
\[
T = \min\{t > 0 : X_1(t) = 0\}.
\]
Then for \( j = 0, 1, \ldots, M - 1 \), we have
\[
F_{M,j}(x) = \mathbb{P}(T \leq x, Z_1(T) = j|X_1(0) = 0, Z_1(0) = M).
\]
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(Note that $F_{M}(x) = 0$). We require $P = F(+\infty) = \{F_{ij}(+\infty)\}$ in our analysis. It is easy to show that for $i = 0, 1, \ldots, M-1$ and $j = 0, 1, \ldots, M$,

$$
\begin{align*}
    p_{ij} &= \begin{cases} \\
        \frac{1}{\alpha + (N+1)\beta} & \text{if } j = i + 1 \\
        \frac{1}{\alpha + (N-1)\beta} & \text{if } j = i - 1 \\
        0 & \text{otherwise },
\end{cases} \\
    p_{Mj} &= \hat{F}_{M}(0) ,
\end{align*}
$$

(6.4.15)

where $\hat{F}_{M}(s)$ is the moment generating function of distribution function $F_{M}(x)$. The $\hat{F}_{M}(s)$ can be computed using the analysis in Narayanan and Kulkarni [94] (see also Gautam et al [51]). It can be shown that sojourn time $\tau_{i}$ in state $i$ is equal:

$$
\tau_{i} = \begin{cases} \\
    \frac{1}{\sum_{j=1}^{M-1} \hat{F}_{M}(0)} & \text{if } i = 0, 1, \ldots, M-1 \\
    \frac{\pi_{i} \tau_{i}}{\sum_{k=0}^{M} \pi_{k} \tau_{k}} ,
\end{cases}
$$

(6.4.16)

where

$$
\pi = \pi F(+\infty).
$$

Now, $\Phi(\delta)$ is given by

$$
\Phi_{ij}(\delta) = \hat{F}_{ij}(\delta(i \tau_{2} - c_{2}))
$$

for $i = 1, \ldots, M-1$, $j = 1, \ldots, M$ and

$$
\Phi_{Mj}(\delta) = \hat{F}_{Mj}(\delta(c_{1} - c_{2}))
$$

Then we solve BSNE (we assume that there exists solution of this equation):

$$
\kappa(\Phi(\eta), e) = 1.
$$

Finally, by Theorem 4.4.1 we get exponential bounds of probability of the buffer overflow at the second buffer:

$$
C_{\ast} e^{-\eta x} \leq \lim_{t \to +\infty} \mathbb{P}(X_{2}(t) > x) \leq C_{\ast} e^{-\eta x}.
$$
Chapter 7

Other models

In this chapter we find an upper bound for a ruin probability of a risk process with Coxian arrival process generated by a diffusion process.

7.1 Risk process

In this section we consider a canonical risk reserve process \( \{R(t), t \geq 0\} \). Point process \( \{P(t), t \geq 0\} \) is an input process and constant premium rate is equal to \( p \). That is, the arrival epochs of claims are given by process \( \{P(t), t \geq 0\} \). The claim sizes \( U_1, U_2, \ldots \), are i.i.d. and independent of process \( \{P(t), t \geq 0\} \). We assume that common distribution function \( F_U(x) \) has continuous density \( f_U(x) \). Between jumps, \( \{R(t), t \geq 0\} \) moves according to differential equation

\[
\frac{d}{dt} R(t) = p .
\]

Putting things together we see that evolution of the reserve process may be described by the equation

\[
R(t) = u + pt - \sum_{i=1}^{P(t)} U_i ;
\]

see Figure 7.
Assume that $R(t) \to +\infty$ a.e. In this chapter we are interested in the infinite horizon ruin probability

$$\psi(u) = \mathbb{P}(\inf_t R(t) < 0)$$

with initial reserve $R(0) = u$. In particular, we find the upper exponential bound for $\psi(u)$:

$$\psi(u) \leq e^{-\tau u}.$$  \hfill (7.1.2)

called Cramér-Lundberg upper bound, when arrival process $\{P(t), t \geq 0\}$ is a Cox process generated by a diffusion process. Define

$$\tau(u) = \inf\{t \geq 0 : R(t) < 0\}.$$  

Then

$$\psi(u) = \mathbb{P}(\tau(u) < +\infty).$$

### 7.2 General Björk-Grandell risk model

Björk and Grandell [16] derived by a martingale approach an exponential upper bound for the ruin probability $\psi(u)$ when the occurrence of the claims is described by the Cox process with an intensity process having Markovian piecewise constant realizations. In this section we combine the theory of PDMPs and diffusion processes described in Chapter 3 to get Cramér-Lundberg inequality, when occurrence
of the claims is described by the Cox process with intensity process \( \{ \lambda(t), t \geq 0 \} \) generated by a Markov diffusion process \( \{ X(t), t \geq 0 \} \). We mainly focus on finding the extended generator \( \mathcal{A} \) of the risk process and its domain \( D(\mathcal{A}) \). The result seems to be new in literature and can be applied, using some results from Björk and Grandell [17], to get Crâmer-Lundberg inequality.

7.2.1 On the stochastic basis \( (C[0, +\infty), \mathcal{F}^X, \{ \mathcal{F}^X_t \}, \mathbb{P}^d) \) we define canonical Markov diffusion process \( \{ X(t), t \geq 0 \} \) with given initial state \( X(0) = y \). By Section 3.4, process \( \{ X(t), t \geq 0 \} \) has the following extended generator

\[
(\mathcal{A}^d f)(z) = \frac{1}{2} a(z) \frac{\partial^2}{\partial z^2} f(z) + b(z) \frac{\partial}{\partial z} f(z),
\]

where \( \inf_z a(z) > 0 \) and measurable functions \( a, b \in \mathcal{C}(\mathbb{R}) \) fulfill following conditions

\[
a(z) \leq L(1 + z^2), \quad |b(z)| \leq L(1 + |z|) \tag{7.2.1}
\]

for some constant \( L \). Then the family of functions \( \mathcal{C}^2(\mathbb{R}) \) are included in the domain \( D(\mathcal{A}^d) \).

If realizations of the process \( \{ X(t), t \geq 0 \} \) is \( x(t) \in \mathcal{C}(\mathbb{R}_+) \), then for nonnegative continuous function \( \lambda : \mathbb{R} \to \mathbb{R}_+ \cup \{0\} \), we define a non-homogeneous Poisson process \( \{ P^{(x)}(t), t \geq 0 \} \) with intensity function \( \lambda(t) = \lambda(x(t)) \). We now consider canonical risk process

\[
R^{(x)}(t) = u + pt - \sum_{i=1}^{P^{(x)}(t)} U_i. \tag{7.2.2}
\]

For a given realization \( x(t) \) of the process \( \{ X(t), t \geq 0 \} \) we consider now PDMP \( \{ R^{(x)}(t) = (t, R^{(x)}(t)), t \geq 0 \} \) with intensity function \( \lambda(t) = \lambda(x(t)) \). Let \( \mathbb{P}^{(x)} \) be the law of the canonical process \( \{ R^{(x)}(t), t \geq 0 \} \), that is the probability measure on \( (D, \mathcal{F}, [0, +\infty), \mathcal{F}^{(x)}, \{ \mathcal{F}^{(x)}_t \}) \) under which \( \{ R^{(x)}(t), t \geq 0 \} \) is the risk process described in (7.2.2). Then by Remark 3.3.3 and Theorem 3.3.1 we have the following lemma.

**Lemma 7.2.1** Process \( \{ \lambda^{(x)}(t) = (t, R^{(x)}(t)), t \geq 0 \} \) has the extended generator

\[
(\mathcal{A}^{(x)} g)(t, w) = \frac{\partial}{\partial t} g(t, w) + \frac{\partial}{\partial w} g(t, w) + \lambda(t) \left( \int_0^{+\infty} (g(t, w-y) - g(t, w)) f_U(y) dy \right) \]

with domain \( D(\mathcal{A}^{(x)}) \) consisting of functions \( g(t, w) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) fulfilling the following conditions:

1) function \( t \to g(t, pt) \) is absolutely continuous;

ii) \[
\mathbb{E}^{(x)} \left[ \sum_{i=1}^{n} |g(\sigma_i, R^{(x)}(\sigma_i)) - g(\sigma_i-, R^{(x)}(\sigma_i-))| \right] < +\infty \tag{7.2.3}
\]

for all \( n \geq 1 \), where \( \{ \sigma_i \} \) are moments of jumps of the process \( \{ R^{(x)}(t), t \geq 0 \} \) and the expectation is with respect to \( \mathbb{P}^{(x)} \).
For each \( x \in C(\mathbb{R}_+) \) we restrict domain \( D(\mathcal{A}^{(x)}) \) to family of function \( g(t, r) \in C^1(\mathbb{R}^2) \) fulfilling (7.2.3).

Let \( \Omega = D_{\mathbb{R}^2}[0, +\infty) \times C[0, +\infty) \). By \( \mathcal{F}_t^{(\tau, X)} \) we denote the \( \sigma \)-field which consists of sets \( \{(\tau, x) \in \Omega : x_{[0, \varepsilon]} \in A \} = \{(\tau, x) \in \Omega : x_{[0, t]} \in A, \tau_{[0, t]} \in D_{\mathbb{R}^2}[0, t]\} \), where \( x_{[0, t]} \) and \( \tau_{[0, t]} \) are restrictions of functions \( x \) and \( \tau \) respectively to \( [0, t] \), where \( A \in \mathcal{B}(C[0, t]) \) and \( \tau(t) = (t, r(t)) \) for \( r \in D[0, +\infty) \). Similarly, by \( \mathcal{F}_t^{(\tau)} \) we mean the \( \sigma \)-field which consists of sets \( \{(\tau, x) \in \Omega : \tau_{[0, t]} \in A \} \), where \( A \in \mathcal{B}(D_{\mathbb{R}^2}[0, t]) \). Let \( \mathcal{F}(\tau, x) = \bigvee_{t \geq 0} \mathcal{F}_t^{(\tau, x)} \) and \( \mathcal{F} = \mathcal{F}(\tau, x) \), and \( \mathcal{F} = \mathcal{F}(\tau, x) \). On probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) we define canonical Markov process \( \{(t, R(t), X(t)), t \geq 0\} = \{(\tau(t), X(t)), t \geq 0\} \), where \( R(t) \) is a risk process with Coxian arrival process driven by the diffusion \( X(t) \); see Brémaud [23], Grandell [53] and Grandell [54]. Note that \( \mathcal{F}_t^{(\tau, x)} = \mathcal{F}_t^{(\tau, x)} \) and \( \mathcal{F} = \mathcal{F}(\tau, x) \). Consider measurable mapping \( \pi \) of \( \Omega \) onto \( C[0, +\infty) \) given by \( \pi(\tau, x) = x \). By Parasarathy [102], Theorem 8.1, page 147, there exists a version of conditional probability \( \mathbb{P} \) given \( \pi \), that is a probability measure \( \mathbb{P}^{\pi, x} \) on \( (\Omega, \mathcal{F}) \) such that mapping \( x \to \mathbb{P}^{\pi, x}(A) \) is \( \mathcal{F} \)-measurable for all \( A \in \mathcal{F} \) and the following holds

\[
\mathbb{P}(A \cap \pi^{-1}(B)) = \int_B \mathbb{P}^{\pi, x}(A) d\mathbb{P}^d(x), \tag{7.2.4}
\]

where \( A \in \mathcal{F} \) and \( B \in \mathcal{F}^X \). The \( \sigma \)-field \( \mathcal{F}(\tau, x) \) consists of sets \( A \times B = \{(\tau, x) \in \Omega : \tau \in A, x \in B\} = A^\tau \times B^x \), where \( A \in \mathcal{B}(D_{\mathbb{R}^2}[0, +\infty]) = \mathcal{F}(\tau, x) = \mathcal{F}(\tau), B \in \mathcal{B}(C[0, +\infty]) = \mathcal{F}(\tau) \) and \( A^\tau = \{(\tau, x) \in \Omega : \tau \in A\}, B^x = \{(\tau, x) \in \Omega : x \in B\} = \pi^{-1}(B) \). Then

\[
\mathbb{P}(A \times B) = \int_B \mathbb{P}^{\pi, x}(A^\tau) d\mathbb{P}^d(x) = \int_A \int_B \mathbb{P}^{(\tau, x)}(\tau) d\mathbb{P}^d(x) \]

and

\[
d\mathbb{P}(\pi, x) = d\mathbb{P}(x)(\pi) d\mathbb{P}^d(x); \tag{7.2.5}
\]

see Grandell [53]. In analysis of extended generator the most convenient filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is right-continuous natural filtration \( \{\mathcal{F}_t^{(\tau, x)}\} \) of Markov process \( \{(\tau(t), X(t)), t \geq 0\} \); see Grandell [54], page 114. The \( \sigma \)-field \( \mathcal{F}(\tau, x) \) consists of sets

\[
\{(\tau, x) \in \Omega : r_{[0, \varepsilon]} \in A_1, x_{[0, \varepsilon]} \in A_2\} = \{\tau \in D_{\mathbb{R}^2}[0, +\infty) : \tau_{[0, t]} \in A_1\} \times \{x \in C[0, +\infty) : x_{[0, t]} \in A\},
\]

where \( A_1 \in \mathcal{B}(D_{\mathbb{R}^2}[0, t]) \) and \( A_2 \in \mathcal{B}(C[0, t]) \). Thus

\[
\mathcal{F}_t^{(\tau, x)} = \mathcal{F}_t^{(\tau)} \times \mathcal{F}_t^X = \mathcal{F}_t^{(\tau)} \times \mathcal{F}_t^X.
\]

We now find the extended generator of process \( \{(\tau(t), X(t)), t \geq 0\} \).

Define operator \( (\mathcal{A}^{(X)} g)(t, w) \) in the following way. If the realization of the process \( \{X(t), t \geq 0\} \) is \( x(t) \in C(\mathbb{R}_+) \), then

\[
(\mathcal{A}^{(X)} g)(t, w) = (\mathcal{A}^{(x)} g)(t, w).
\]

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That is
\[(A^{(X)} g)(t, w) = \frac{\partial}{\partial t} g(t, w) + p \frac{\partial}{\partial w} g(t, w) + \lambda(X(t)) \int_0^{+\infty} (g(t, w-y) - g(t, w)) f_Y(y) \, dy .\]

By $D(A^{(X)})$ we denote a collection of all functions $g(t, w) \in C^1(\mathbb{R}^2)$ fulfilling
\[
E^{[x]} \left[ \sum_{i=1}^n \left| g(\sigma_i, R^{(x)}(\sigma_i)) - g(\sigma_i-, R^{(x)}(\sigma_i-)) \right| \right] < +\infty \quad (7.2.7)
\]
for all $n \geq 1$ and $x \in C[0, +\infty)$, where $\{\sigma_i\}$ are moments of jumps of the process $\{R(t), t \geq 0\}$.

**Lemma 7.2.2** If $g(t, r) \in D(A^{(X)})$, then
\[
M^{X, \beta}(t) = g(t, R(t)) - \int_0^t (A^{(X)} g)(s, R(s)) \, ds
\]
\[= g(R(t)) - \int_0^t (A^{(X)} g)(R(s)) \, ds
\]
is a local $\{F_{t\uparrow}^{(X)} \}, \mathbb{P}\}$-martingale.

**Proof.** Let $\{\tau_n^{(x)}\}$ be a fundamental sequence of $F_{t\uparrow}^{(x)}$-stopping times for local martingale $\{M^{X, \beta}(t) = g(R^{(x)}(t)) - \int_0^t (A^{(x)} g)(R^{(x)}(s)) \, ds\}$; see Lemma 7.2.1. It can be for instance the jump epochs of process $\{R^{(x)}(t), t \geq 0\}$, that is $\tau_n^{(x)}(\overline{t}) = \sigma_n = \inf\{t \geq \tau_n^{(x)} : \overline{t}(t) \neq \overline{t}(\tau(t))\}$. Define $\tau_n^{(R)}(\overline{t}, x) = \tau_n^{(x)}(\overline{t})$. Then
\[
\{(\overline{t}, x) \in \Omega : \tau_n^{(R)}(\overline{t}, x) \leq t\} = \{(\overline{t}, x) \in \Omega : \tau_n^{(x)}(\overline{t}) \leq t\} \in F_{t\uparrow}^{(R, x)} \subset F_{t\uparrow}^{(R, X)} .
\]
It means that $\tau_n^{(R)}$ is an $F_{t\uparrow}^{(R, X)}$-stopping time. The sequence of stopping times $\{\tau_n^{(R)}\}$ will be also the fundamental sequence for $\{M^{X, \beta}(t), t \geq 0\}$. Note that by Lemma 7.2.1 we have
\[
E M^{X, \beta}(t \wedge \tau_n^{(R)}) = E^d E^{[x]} M^{X, \beta}(t \wedge \tau_n^{(R)})
\]
\[= E^d g(R^{(x)}(0)) = g(u) < +\infty ,
\]
where $E^d$ is a expectation with respect to $\mathbb{P}^d$. Thus it suffices to prove that
\[
E \left[ M^{X, \beta}(t \wedge \tau_n^{(R)}) | F_{s\uparrow}^{(R, X)} \right] = M^{X, \beta}(s \wedge \tau_n^{(R)}) \quad \text{a.e.}
\]
for $g \in D(A^{(X)})$ and $s \leq t$. That is, that for $A \in F_{s\uparrow}^{(R, X)}$ we have
\[
\int_A M^{X, \beta}(t \wedge \tau_n^{(R)}) \, d\mathbb{P} = \int_A M^{X, \beta}(s \wedge \tau_n^{(R)}) \, d\mathbb{P} . \quad (7.2.8)
\]
By (7.2.6) it suffices to prove (7.2.8) for the set $A = A_1 \times A_2$, where $A_1 \in F_{s\uparrow}^{(R)}$ and $A_2 \in F_{t\uparrow}^{(X)}$. Let
\[
m^{X, \beta}(t) = g(\overline{t}(t)) - \int_0^t (A^{(x)} g)(\overline{t}(s)) \, ds .
\]

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By Lemma 7.2.1 we have

$$\int_A M^{X,\beta}(t \wedge \tau_n^{(R)}) \, d\mathbb{P} = \int_A m^{x,\beta}(t \wedge \tau_n^{(R)}) \, d\mathbb{P}(\bar{\tau}, x) =$$

$$= \int_{A_2} \int_{A_1} m^{x,\beta}(t \wedge \tau_n^{(R)}) \, d\mathbb{P}(x)(\bar{\tau}) \, d\mathbb{P}^d(x)$$

$$= \int_{A_2} \int_{A_1} m^{x,\beta}(s \wedge \tau_n^{(R)}) \, d\mathbb{P}(x)(\bar{\tau}) \, d\mathbb{P}^d(x)$$

$$= \int_A m^{x,\beta}(s \wedge \tau_n^{(R)}) \, d\mathbb{P}(\bar{\tau}, x) = \int_A M^{X,\beta}(s \wedge \tau_n^{(R)}) \, d\mathbb{P}$$,

which completes the proof.

\[\square\]

7.2.2 By 7.2.1 for \( f \in C^2(\mathbb{R}) \)

$$M^{d,f}(t) = f(X(t)) - \int_0^t (\mathcal{A}^d f)(X(s)) \, ds \quad (7.2.9)$$

is a local \((\mathcal{F}^X_t, \mathbb{P}^d)\)-martingale. Hence it is also a local \((\mathcal{F}^{(n,X)}_t, \mathbb{P})\)-martingale and finally a local \((\mathcal{F}^X_t, \mathbb{P})\)-martingale. By \( \{\tau_n^{(X)}\} \) we denote its fundamental sequence. Let

$$m^{d,f}(t) = f(x(t)) - \int_0^t (\mathcal{A}^d f)(x(s)) \, ds .$$

**Lemma 7.2.3** Process \( \{M^{X,\beta}(t)M^{d,f}(t), t \geq 0\} \) is a local \((\mathcal{F}^{(n,X)}_t, \mathbb{P})\)-martingale for \( g \in D(\mathcal{A}^{(X)}) \) and \( f \in C^2(\mathbb{R}) \).

**Proof.** Consider sequence of stopping times \( \tau_n = \tau_n^{(X)} \wedge \tau_n^{(R)} \); see 2.1.18. We prove that process \( \{M^{X,\beta}(t \wedge \tau_n)M^{d,f}(t \wedge \tau_n), t \geq 0\} \) is a martingale. Note that by 7.2.2 and Lemma 7.2.1 we have

$$\mathbb{E} M^{X,\beta}(t \wedge \tau_n)M^{d,f}(t \wedge \tau_n) = \mathbb{E}^d M^{d,f}(t \wedge \tau_n) \mathbb{E}^{(x)} M^{x,\beta}(t \wedge \tau_n) =$$

$$= \int_{C[0, +\infty]} m^{d,f}(t \wedge \tau_n) \int_{D_{R^d}[0, +\infty]} m^{x,\beta}(t \wedge \tau_n) \, d\mathbb{P}(x)(\bar{\tau}) \, d\mathbb{P}^d(x) =$$

$$= \int_{C[0, +\infty]} m^{d,f}(t \wedge \tau_n) \int_{D_{R^d}[0, +\infty]} m^{x,\beta}(0) \, d\mathbb{P}(x)(\bar{\tau}) \, d\mathbb{P}^d(x) =$$

$$= g(u) \mathbb{E}^d M^{d,f}(t \wedge \tau_n) = g(u) f(X(0)) = g(u) f(y) < +\infty .$$

We now prove the martingale property, that is that

$$\mathbb{E} \left[ M^{X,\beta}(t \wedge \tau_n)M^{d,f}(t \wedge \tau_n) | \mathcal{F}^{(\bar{\tau})}_s \right] = M^{X,\beta}(s \wedge \tau_n)M^{d,f}(s \wedge \tau_n), \quad \text{a.e.}$$
Note that for $A = A_1 \times A_2 \in \mathcal{F}_{s+}^{(\mathbb{F}, X)}$, where $A_2 \in \mathcal{F}^X_{s+}$ and $A_1 \in \mathcal{F}_{s+}^\mathbb{R}$, by Lemma 7.2.1 we have

$$
\int_A M^{X, \varphi}(t \wedge \tau_n) M^{d, f}(t \wedge \tau_n) \, d\mathbb{P} = \int_{A_1} m^{d, f}(t \wedge \tau_n) \int_{A_2} m^{x, \varphi}(s \wedge \tau_n) \, d\mathbb{P}^{(x)}(\mathbb{F}) \, d\mathbb{P}^{d}(x)
$$

$$
= \int_{A_2} m^{d, f}(t \wedge \tau_n) \int_{A_1} m^{x, \varphi}(s \wedge \tau_n) \, d\mathbb{P}^{(x)}(\mathbb{F}) \, d\mathbb{P}^{d}(x) = \int_A M^{X, \varphi}(s \wedge \tau_n) M^{d, f}(t \wedge \tau_n) \, d\mathbb{P}.
$$

Thus

$$
\mathbb{E} \left[ M^{X, \varphi}(t \wedge \tau_n) M^{d, f}(t \wedge \tau_n) \big| \mathcal{F}^{(\mathbb{F}, X)}_{s+} \right] = \mathbb{E} \left[ M^{X, \varphi}(s \wedge \tau_n) M^{d, f}(t \wedge \tau_n) \big| \mathcal{F}^{(\mathbb{F}, X)}_{s+} \right]
$$

$$
= M^{X, \varphi}(s \wedge \tau_n) \mathbb{E} \left[ M^{d, f}(t \wedge \tau_n) \big| \mathcal{F}^{(\mathbb{F}, X)}_{s+} \right] = M^{X, \varphi}(s \wedge \tau_n) M^{d, f}(s \wedge \tau_n), \quad \text{a.e.}
$$

which completes the proof.

□

Using notations from Section 6.1.2 for family of functions $A$ and $B$ we define new family $A \otimes B = \{ f(x)g(y) : f \in A \text{ and } g \in B \}$.

**Theorem 7.2.4** The Markov process $\{(t, R(t), X(t)), t \geq 0\}$ has the following extended generator:

$$(A_k)(t, w, z) =$$

$$= \frac{1}{2} a(z) \frac{\partial^2}{\partial z^2} k(t, w, z) + b(z) \frac{\partial}{\partial z} k(t, w, z)$$

$$+ \frac{\partial}{\partial t} k(t, w, z) + p \frac{\partial}{\partial w} k(t, w, z) + \lambda(z) \int_0^{+\infty} (k(t, w - y, z) - k(t, w, z)) f(y) \, dy
$$

and $D(A^{(X)}(\mathbb{R})) \subset D(A)$.

**Remark 7.2.1** In notations of Section 6.1.2 the assertion of Theorem 7.2.4 can be formulated in the following way:

$$
A = A^{(X)} \oplus A^d
$$

on $D(A^{(X)}(\mathbb{R})) \subset D(A)$.

**Proof.** Note that it suffices to prove that for $g(t, w) \in D(A^{(X)})$ and $f(x) \in C^2(\mathbb{R})$, process

$$
M^{f, g}(t) = g(t, R(t)) f(X(t)) - \int_0^t (Af)(g(s, R(s), X(s))) \, ds
$$

(7.2.10)

is a local $(\mathcal{F}^{(\mathbb{F}, X)}_{s+}, \mathbb{P})$-martingale. From integration-by-parts formula for semimartingales (see 2.2.13) we have

$$
g(t, R(t)) f(X(t)) =$$

$$= \int_0^t f(X(s-)) g(s, R(s)) \, ds + \int_0^t g(s-, R(s-)) \, df(X(s)) - [f(X), g(t, R)]_t.
$$
Note $\mathcal{A}^{(X)}g$ and $\mathcal{A}^d f$ are continuous functions on domains $D(\mathcal{A}^{(X)})$ and $C^2(\mathbb{R})$ respectively. Thus by 3.2.1 and Remark 3.2.1 processes \( \{ \int_0^t \mathcal{A}^{(X)}g(R(s)) \, ds, t \geq 0 \} \) and \( \{ \int_0^t \mathcal{A}^d f(X(s)) \, ds, t \geq 0 \} \) are continuous processes of finite variation. Hence
\[
H(t) = [f(X), g(t, R)]_t = [M^{d,f}, M^{X,g}]_t .
\]

By Jacod and Shiryaev [58], Theorem 4.50 a), page 53, and Lemma 7.2.3 this process is the local martingale. Moreover,
\[
g(t, R(t)) f(X(t)) = \\
= \int_0^t f(X(s-))dg(s, R(s)) + \int_0^t g(s-, R(s-)) df(X(s)) + H(t) \\
= \int_0^t f(X(s-))dM^{X,g}(s) + \int_0^t g(s-, R(s-)) dM^{d,f}(s) \\
+ \int_0^t f(X(s))(\mathcal{A}^{(X)}g)(s, R(s)) \, ds + \int_0^t g(s, R(s))(\mathcal{A}^d f)(X(s)) \, ds + H(t) \\
= \int_0^t f(X(s-))dM^{X,g}(s) + \int_0^t g(s-, R(s-)) dM^{d,f}(s) + H(t) \\
+ \int_0^t (\mathcal{A}f)(g, R(s), X(s)) \, ds .
\]

(7.2.11)

By (7.2.2), Lemma 7.2.2 and 2.2.15 processes \( \{ \int_0^t f(X(s-))dM^{X,g}(s), t \geq 0 \} \) and \( \{ \int_0^t g(s-, R(s-)) dM^{d,f}(s), t \geq 0 \} \) similarly like \( \{ H(t), t \geq 0 \} \) are local martingales. This completes the proof in view of (7.2.11).

\[\square\]

Let \( h(t, r, x) = f(x)e^{-\delta r} \), where \( f \in C^2(\mathbb{R}) \) and \( f(x) > 0 \). Similar derivations like in Rolski et al [108], page 459, show that if
\[
\hat{F}_U(\delta) < +\infty ,
\]
then
\[
\mathbb{E} \left[ \sum_{i=1}^n \left| \exp\{R(\sigma_i)\} - \exp\{R(\sigma_i^-)\} \right| \right] < +\infty .
\]

(7.2.13)

In that case \( h \in D(\mathcal{A}^{(X)}) \otimes C^2(\mathbb{R}) \). From Theorem 7.2.4 and Lemma 3.2.1 process
\[
N(t) = \frac{h(R(t))}{h(R(0))} \exp\left\{ -\int_0^t \frac{\mathcal{A}h(R(s))}{h(R(s))} \, ds \right\}
\]
is the local martingale. Assume that
\[
\mathcal{A}h = 0 ,
\]
that is that function \( f(x) > 0 \) is a solution of
\[
\frac{1}{2}a(z) \frac{\partial^2}{\partial z^2} f(z) + b(z) \frac{\partial}{\partial z} f(z) - (p\delta - \lambda(z)(\hat{F}_U(\delta) - 1)) f(z) = 0 .
\]

(7.2.15)
Then process
\[
\{ N(t) = \frac{h(t, R(t), X(t))}{h(0, R(0), X(0))}, t \geq 0 \}
\] (7.2.16)
is a mean-one positive local martingale. Thus from Dellacherie and Meyer [33], page 88, \( \{ N(t), t \geq 0 \} \) is a supermartingale; see 2.2.1. Let \( \mathbb{P}^{(0, u, y)} \) be a probability under which \( X(0) = y \) and \( \mathbb{E}^{(0, u, y)} \) expectation with respect to \( \mathbb{P}^{(0, u, y)} \). Choose \( \overline{u} \) such that \( \overline{u} < +\infty \). Consider \( \tau(u) \wedge \overline{u} \), which is bounded stopping time. By Optional Sampling Theorem 2.2.1 we get
\[
\mathbb{E}^{(0, u, y)} N(0) \geq \mathbb{E}^{(0, u, y)} [N(\tau(u)) \wedge \overline{u}] = \mathbb{E}^{(0, u, y)} [N(\tau(u)) \mid \tau(u) \leq \overline{u}] \mathbb{P}^{(0, u, y)}(\tau(u) \leq \overline{u}).
\]
Provided that
\[
\mathbb{E}^{(0, u, y)} [N(\tau(u)) \mid \tau(u) < +\infty] > 0
\] (7.2.17)
and letting \( \overline{u} \to +\infty \) we get
\[
\mathbb{P}^{(0, u, y)}(\tau(u) < +\infty) \leq \frac{\mathbb{E}^{(0, u, y)} N(0)}{\mathbb{E}^{(0, u, y)} [N(\tau(u)) \mid \tau(u) < +\infty]}.
\] (7.2.18)

Note that \( N(0) = e^{-\delta u} f(X(0)) \) and \( R(\tau(u)) < 0 \) on \( \{ \tau(u) < +\infty \} \). Hence assuming (7.2.17) inequality (7.2.18) yields Cramér-Lundberg inequality:
\[
\psi(u) \leq Ce^{-\delta u},
\]
where
\[
C = \int_{-\infty}^{+\infty} \frac{f(y)}{\mathbb{E}^{(0, u, y)} [f(X(\tau(u))) \mid \tau(u) < +\infty]} dF^0(y)
\]
and \( F^0(x) \) is a distribution function of random variable \( X(0) \).

In general equation (7.2.15) is difficult to solve. However, Björk and Grandell [17] solve it for a few examples.

**Example 7.2.1** Consider a stationary Ornstein-Uhlenbeck process \( \{ X(t), t \geq 0 \} \) with generator
\[
(\mathcal{A}f)(z) = \frac{1}{2} a \frac{\partial^2}{\partial z^2} f(z) - \alpha z \frac{\partial}{\partial z} f(z)
\]
for some constants \( a > 0 \) and \( \alpha \in \mathbb{R} \); see Example 3.4.2. That is, \( \{ X(t), t \geq 0 \} \) is defined by equation
\[
dX(t) = \sqrt{a} dB(t) - \alpha X(t) \; dt,
\]
where \( \{ B(t), t \geq 0 \} \) is a Brownian motion. In case of stationary Ornstein-Uhlenbeck process \( X(0) \) has a normal distribution function \( F^0(x) \) with mean zero and variance \( \frac{a}{2\alpha} \). Assume that \( \lambda(z) = z^2 \) and \( f(z) = \exp \{ K z^2 \} \), for some
\[
0 < K < \frac{\alpha}{a}.
\] (7.2.19)

Note that in this case \( \mathbb{E} N(0) < +\infty \). Under assumption (7.2.19) we have
\[
\int_{-\infty}^{+\infty} f(y) dF^0(y) = \frac{\sqrt{2}}{\sqrt{1 - K\frac{a}{\alpha}}} < +\infty.
\] (7.2.20)
Note also that \( f(y) \geq 1 \). Thus condition \((7.2.17)\) is fulfilled and

\[
C \leq \frac{\sqrt{2}}{\sqrt{1 - K_{\alpha}^2}}. \tag{7.2.21}
\]

Equation \((7.2.15)\) is reduced to

\[
\frac{1}{2}a(4x^2K^2 + 2K) - 2\alpha x^2K - p\delta + x^2\tilde{F}_U(\delta) = 0,
\]

which implies that

\[
K = \frac{\alpha}{2a} - \sqrt{\frac{\alpha^2}{4a^2} - \frac{\tilde{F}_U(\delta)}{2a}} < \frac{\alpha}{a}\tag{7.2.22}
\]

and \( \delta \) is a solution of following equation

\[
p\delta = \frac{\alpha - \sqrt{\alpha^2 - 2a\tilde{F}_U(\delta)}}{2}. \tag{7.2.23}
\]

Note that we must have

\[
\delta \leq \overline{\delta}, \tag{7.2.24}
\]

where \( \overline{\delta} \) is the solution of \( \tilde{F}_U(\overline{\delta}) = \frac{\alpha^2}{2a} \). In this case assumption \((7.2.12)\) is obviously fulfilled. For exponentially distributed claims with mean value \( \mu \) we have that \( \overline{\delta} = \frac{\alpha^2}{(\alpha^2 + 2\alpha)\mu} \) and for

\[
p < \frac{\alpha}{2\alpha}(\mu + 1) + \frac{\alpha}{2}\mu \tag{7.2.25}
\]

there exists solution of \((7.2.23)\)

\[
\delta = \frac{\alpha\mu + p}{2p\mu} \left( 1 - \sqrt{\frac{2\mu^2(2p\alpha - a)}{(\alpha\mu + p)^2}} \right). \tag{7.2.26}
\]

Note that if \((7.2.25)\) holds, then assumption \((7.2.24)\) and hence also \((7.2.12)\) is fulfilled. Thus, if occurrence of claims are described by the Cox process with the intensity process being the function of stationary Ornstein-Uhlenbeck process and claims are exponentially distributed, then under assumption \((7.2.25)\) by \((7.2.21)\) we have the following Cramér-Lundberg inequality

\[
\psi(u) \leq \frac{\sqrt{2}}{\sqrt{1 - K_{\alpha}^2}} e^{-\delta u},
\]

where \( K \) is given in \((7.2.22)\) and \( \delta \) in \((7.2.26)\).
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