On Insiders Who Can Stop at Honest Times

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- $S = (S_t)_{t \in [0,T]}$ is a 1-dimensional continuous $(\mathbb{F}, \mathbb{P})$-semimartingale representing the discounted asset price process
- We assume that S satisfies the *No Free Lunch With Vanishing Risk* (NFLVR) condition, i.e. for every sequence $\{\phi_n\}$ of simple predictable processes which are δ_n-admissible with $\delta_n \to 0$, we have $V_T(\phi_n) \to 0$ in $\mathbb{P}$, where $V_T(\phi)$ denotes the value process for the strategy ϕ at time T.

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- the insider is able to stop at a random time Λ, that is, his filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ is given by $\mathcal{G}_t = \bigcap_{\epsilon>0}(\mathcal{F}_{t+\epsilon} \vee \sigma(\Lambda \wedge (t+\epsilon)))$

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- Λ is called $\mathbb{F}$-honest time if there exists a predictable set $\Sigma \subset [0, T] \times \Omega$ such that $\Lambda(\omega) = sup\{t \geq 0 : (t, \omega) \in \Sigma\}$

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- We shall assume that Λ is an honest time, and also that the following two assumptions are satisfied:

$$\begin{aligned} &(\mathbf{C}) \quad \textit{All } \mathbb{F}-\textit{martingales are continuous}. \\ &(\mathbf{A}) \quad \Lambda \textit{ avoids } \mathbb{F}-\textit{stopping times, that is for every} \\ &\qquad\ \ \mathbb{F}-\textit{stopping time } V \textit{ we have } \mathbb{P}(\Lambda = V) = 0. \end{aligned}$$

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- NFLVR implies that S can be written as $S_t = M_t + \int_0^t \alpha_s d\langle M, M\rangle_s$ where $M = (M_t)_{t\in[0,T]}$ is an $(\mathbb{F}, \mathbb{P})$-local martingale and $\alpha = (\alpha_t)_{t\in[0,T]}$ is an $\mathbb{F}$-predictable process

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- If $\mathbb{P}(\int_0^T \alpha_t^2 d\langle M, M\rangle_t = \infty) > 0$, then S does not satisfy the NFLVR condition

- The *Azéma supermartingale* $Z = (Z_t)_{t \in [0,T]}$ associated to the honest time Λ is defined as the *càdlàg* version of $Z_t = \mathbb{P}(\Lambda > t | \mathcal{F}_t)$

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- Using Z_t we obtain the Doob-Meyer decomposition for S_t with respect to the enlarged filtration:

$$M_t = \widehat{M}_t + \int_0^{t \wedge \Lambda} \frac{d\langle M, Z\rangle_s}{Z_{s-}} + \int_\Lambda^t \frac{d\langle M, 1 - Z\rangle_s}{1 - Z_{s-}}$$

- The following multiplicative decomposition holds for Z, assuming the conditions (CA):

$$Z_t = \frac{N_t}{\overline{N}_t}$$

where $\overline{N}_t = sup_{s \leq t} N_s$, and $N = (N_t)_{t \in [0,T]}$ is a continuous, nonnegative $(\mathbb{F}, \mathbb{P})$-local martingale with $N_0 = 1$ and $lim_{t \to \infty} N_t = 0$.

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- Theorem: $\Lambda = sup\{t \geq 0 : N_t = \overline{N}_t\}$

- We shall assume that the martingale part of S, i.e. M has the predicatble representation property (PRP), that is, for every $(\mathbb{F}, \mathbb{P})$-local martingale N, there exists $c \in \mathbb{R}$ and an $\mathbb{F}$-predictable process $n = (n_t)_{t \in [0,T]}$, such that for all $t \in [0, T]$ $\int_0^t n_s^2 d\langle M, M\rangle < \infty$ and

$$N_t = c + \int_0^t n_s dM_s$$

Main Theorem

Let $S = (S_t)_{t \in [0,T]}$ be a continuous $(\mathbb{F}, \mathbb{P})$-semimartingale satisfying the (NFLVR) condition, and such that M, the martingale part of S, has the predictable representation property. Suppose that Λ is an $\mathbb{F}$-honest time such that $\mathbb{P}(\Lambda < T) = 1$, and conditions (**CA**) are fulfilled. Then, there is no equivalent local martingale measure for S in the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}$, $\mathcal{G}_t = \bigcap_{\epsilon > 0} (\mathcal{F}_{t+\epsilon} \vee \sigma(\Lambda \wedge (t + \epsilon)))$.

Sketch of the proof:
First, we find the Doob-Meyer decomposition of S with respect to the enlarged filtration using the multiplicative decomposition of Azéma supermartingale associated with Λ:

$$S_t = \widehat{M}_t + \int_0^t (\hat{\alpha}_u + \alpha_u) d\langle \widehat{M}, \widehat{M} \rangle_u$$

where

$$\hat{\alpha}_u = 1_{[0,\Lambda]} \frac{n_u}{N_u} + 1_{(\Lambda, T]} \frac{n_u}{N_u - \overline{N}_u}$$

Then, we show that it suffices to prove that

$$\int_{\Lambda}^{T} \Big(\frac{n_u}{N_u - \overline{N}_u}\Big)^2 d\langle M, M\rangle_u$$

is infinite on a set with positive probability. The rest of the proof relies on the Itô formula:

$$ln(N_T - N_\Lambda) - ln(N_{\Lambda+\epsilon} - N_\Lambda) =$$

$$= \int_{\Lambda+\epsilon}^{T} \frac{n_u}{N_u - N_\Lambda} dM_u - \frac{1}{2}\int_{\Lambda+\epsilon}^{T} \Big(\frac{n_u}{N_u - N_\Lambda}\Big)^2 d\langle M, M\rangle_u$$

and investigation of the above two integrals.

Example 1

Let $\Lambda = sup\{0 \leq t \leq 1 : B_t = a\}$ where $B = (B_t)_{t \geq 0}$ is a standard $(\mathbb{F}, \mathbb{P})$-brownian motion, $a \in \mathbb{R}$. Then, Λ is an honest time, and the drift term for the Doob-Meyer decomposition in the enlarged filtration is given by the following formula:

$$\alpha_t = -1_{[0,\Lambda]} \frac{p_{1-t}(|W_t - a|)}{1 - F_{1-t}(|W_t - a|)} sgn(W_t - a) -$$

$$-1_{]\Lambda,1]} \frac{p_{1-t}(|W_t - a|)}{F_{1-t}(|W_t - a|)} sgn(W_t - a)$$

where p_t is the density of the law of $|W_t|$, and F_t its distribution function.

Example 2

Let $\Lambda_{T_a} = sup\{u \leq T_a : \; B_u = 0\}$ where $T_a = inf\{t : \; B_t = a\}$, $a > 0$ and $B = (B_t)_{t \geq 0}$ is a standard $(\mathbb{F}, \mathbb{P})$-brownian motion. Clearly, Λ_{T_a} is an honest time. Now, if the stock price process is e.g. a geometric brownian motion with $\mu = 0$, $\sigma = 1$, that is $S_t = exp(B_t - \frac{1}{2}t)$, then buying S at Λ_{T_a} and selling them at $T_a \wedge T$ will give sure profit, and thus is an arbitrage opportunity. The nonexistence of equivalent local martingale measure in enlarged filtration can in this case also easily be deduced from the usual decomposition of B:

$$B_t = \widehat{B}_t - \int_0^{t \wedge \Lambda_{T_a}} \frac{1_{\{B_s > 0\}}}{a - B_s} ds + \int_{\Lambda_{T_a}}^{t \wedge T_a} \frac{ds}{B_s}$$

and the fact that $B_{\Lambda_{T_a}} = 0$ and therefore $\forall_{\epsilon > 0} \int_{\Lambda_{T_a}}^{\Lambda_{T_a} + \epsilon} \frac{ds}{B_s^2} = \infty$ $\mathbb{P}$-a.s.

- Initial enlargement by the value of $\overline{N}_\infty$, and progressive enlargement by honest time $\Lambda = sup\{t: \ N_t = \overline{N}_t\}$ yield the same Doob-Meyer decomposition:

$$M_t = \widehat{M}_t + \int_0^{t\wedge\Lambda} \frac{d\langle M, Z\rangle_s}{Z_{s-}} + \int_\Lambda^t \frac{d\langle M, 1-Z\rangle_s}{1-Z_{s-}}$$

- Problem: it is hard to find $N = (N_t)_{t\in[0,T]}$ such that $\overline{N}_\infty$ is one of the random variables usually considered in the initial enlargement for insiders

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Thank You