Geometric constructions and elements of Galois' theory
List 7
Constructible complex numbers and construcitibility of regular polygons

Wam-up exercise

1. Prove that the following complex numbers are algebraic, by indicating for each of them a polynomial with integer coefficients for which this number is a root: $2+i$, $1+i\sqrt{2}$, $\sqrt{2}+i$.

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1. Show that, given a quadratic polynomial $W(x) = ax^2 + bx + c$ with real coefficients a, b, c such that the discriminant $\Delta = b^2 - 4ac$ is negative, then its roots are indeed given as $(-b \pm i\sqrt{-\Delta})/2a$. Proof also that if the coefficients a, b, c are rational then both roots of W are constructible complex numbers.
2. Prove directly from the definition (not referring to the characterization in terms of complex quadratic extensions) that if z is a constructible complex number, then any of its roots of degree 4 is also constructible. HINT: express z in the trigonometric form, $z = r(\cos\theta + i\sin\theta)$, express also its roots of degree 4 in trigonometric form, and apply the definition of constructibility of a complex number which refers to the trigonimetric form.
3. Find all four complex roots of the polynomial $x^4 + 2x^2 + 4$ and check that they are all constructible. HINT: solve first the induced quadratic equation with the unknown $y = x^2$; express solutions of this quadratic equation in trigonometric form, so that you can easily compute then their square roots.
4. Prove that for any polynomial $x^4 + ax^2 + b \in Q[x]$ its all complex roots are constructible complex numbers.
5. Describe the set of all algebraic numbers of degree 2 (find general form of such numbers).
6. Prove that if p is an odd prime number, and if $\varepsilon_{2p} = \cos\frac{2\pi}{2p} + i \cdot \sin\frac{2\pi}{2p}$ is the principal root of degree $2p$ of the number 1, then the degree of ε_{2p} is $p-1$. Proceed along the following steps of the argument:
   (a) ε_{2p} is the root of the polynoimial $x^p + 1$;
   (b) ε_{2p} is the root of the polynomial $Z(p) = \frac{x^p+1}{x+1}$;
   (c) polynomial $Z(x)$ is irreducible, since the related polynomial
   $$\widetilde{Z}(x) := Z(x-1) = \frac{(x-1)^p + 1}{(x-1)+1} = \frac{(x-1)^p + 1}{x}$$
   is irreducuible;
   (d) final conclusions.
7. Assuming we know how to construct the regular 17-gon, describe constructions of the regular 34-gon, 51-gon and 85-gon.
8. For all natural numbers $3 \le n \le 100$, decide which regular n-gons are constructible, and which are not. Calculate the numbers of constructible and non-constructible regular n-gons with this restriction for n.
9. For which natural n is the number $\cos\frac{2\pi}{n}$ constructible? And how about the number $\sin\frac{2\pi}{n}$?
10. Verify whether the angles $12°$, $3°$, $5°$, $2°$ are constructible or not. Is the angle $75°$ constructible? Which of the angles $n°$, where n is a natural number, are constructible, and which are not?